A nonlinear observer for a multi-group SIS model

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Outline

1. Multi-group SIS model
2. Background
3. Preliminaries
4. Problem statement
5. Design of observer
6. Application to SIS model
7. Conclusion

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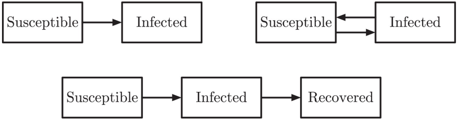
Susceptible-infected-susceptible (SIS) model



$$\dot{I}(t) = \beta S(t)I(t) - \gamma I(t),$$
$$I(t) + S(t) = 1$$

$\beta, \gamma > 0$ are infection/recovery rates

A multi-group SIS model Mei et al., 2017; Niazi et al., 2021:

$$\dot{x}(t) = \mathrm{diag}(1_n - x(t))(\beta A x(t) + d(t)) - \gamma x(t), \tag{1}$$

- $x(t) \in [0,1]^n$ represents infected populations in n groups
- $1_n \in \mathbb{R}^n$ is the vector of ones
- $A \in [0,1]^{n\times n}$ is the adjacency matrix of infection transmission
- $d(t) \in [0,1]^n$ corresponds to unmodelled cumulative infection receipt

Infected populations are measured in $0 < p < n$ groups:

$$y(t) = Cx(t),\ C = [I_p\ \mathrm{O}_{p\times n-p}]. \tag{2}$$

Goal: estimator for SIS model

- Structure (for state estimate $\hat{x}(t) \in \mathbb{R}^n$ and observer gain $L \in \mathbb{R}^{n\times p}$):
  - copy of the system with output injection (is $\hat{x}(t) \in [0,1]^n$?):

$$\dot{\hat{x}}(t) = \mathrm{diag}(1_n - \hat{x}(t))\beta A\hat{x}(t) - \gamma\hat{x}(t) + L(y(t) - C\hat{x}(t))$$

  - a nonlinear observer for $m_1, m_2 \geq 0$:

$$\dot{\hat{x}}(t) = \mathrm{diag}(1_n - \hat{x}(t))[(L + m_2 C^\top \mathrm{diag}(1_p - y(t))^{-1})(y(t) - C\hat{x}(t)) + \beta A\hat{x}(t)] - \gamma\hat{x}(t) + m_1 C^\top (y(t) - C\hat{x}(t)),$$

- Analysis of estimation error $e(t) = x(t) - \hat{x}(t)$ dynamics:
  - autonomous $\Leftrightarrow$ ISS/SIIOS case:

$$\dot{e}(t) = \ell(e(t), d(t))$$

  - non-autonomous $\Leftrightarrow$ IOS case:

$$\dot{e}(t) = \tilde{\ell}(e(t), x(t), d(t))$$

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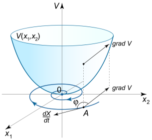
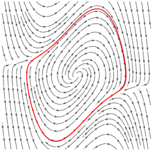
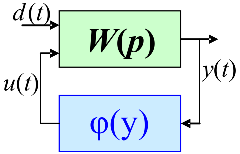
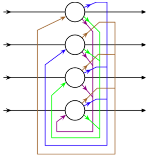
Background





- Stability analysis ⇔ Lyapunov function (LF) method
  - linear: $V(x) = x^\top Px$
  - nonlinear: $V(x) = ?$
- Canonical forms of nonlinear systems:
  - Lurie systems, Lipschitz dynamics
    - close to linear
  - Persidskii systems
  - homogeneous systems...





Persidskii systems

- Stability analysis of

$$\dot{x}(t) = A_1 f(x(t)),\ x(t) \in \mathbb{R}^n,\ f(x) = [f_1(x_1), \dots, f_n(x_n)]^\top$$

in (Barbashin, 1961), with a LF

$$V(x) = \sum_{i=1}^{n} \lambda_i \int_0^{x_i} f_i(s) ds,\ \lambda_i > 0\ \forall i = \overline{1, n}$$

- In (Persidskii, 1969):

$$V(x) = \sum_{i=1}^{n} \lambda_i |x_i|,\ \lambda_i > 0\ \forall i = \overline{1, n}$$

- Further

$$\dot{x}(t) = A_0 x(t) + A_1 f(x(t)),\ x(t) \in \mathbb{R}^n$$

  - diagonal stability (Kazkurewicz & Bhaya,1999; Ferreira et al., 2005)
  - neural networks (Hopfield & Tank, 1986; Sontag, 1993)
  - interval observers (Leurent et al., 2019)

Results of this work

- Robust observer design for a class of generalized Persidskii systems
- Possibility of IOS or SIIOS analysis of the estimation error dynamics
- Stability conditions and gain selection using LMIs
- Application to a multi-group SIS model

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System under consideration

A nonlinear system:

$$\dot{x}(t) = f(x(t), d(t)),\ t \geq 0, \qquad (3)$$
$$y(t) = h(x(t)),$$

- $x(t) \in \mathbb{R}^n$ is the state
- $d(t) \in \mathbb{R}^m$ is the external input, $d \in \mathcal{L}_\infty^m$
- $y(t) \in \mathbb{R}^p$ is the output of interest
- $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$ is locally Lipschitz continuous, $f(0,0) = 0$
- $h : \mathbb{R}^n \to \mathbb{R}^p$ is continuously differentiable, $h(0) = 0$

For $x_0 \in \mathbb{R}^n$ and $d \in \mathcal{L}_\infty^m$, define the solution of (3) by $x(t, x_0, d)$, and $y(t, x_0, d) = h(x(t, x_0, d))$

Input-to-output stability (Sontag&Wang, 2000)

Definition 1 (3) is IOS, if $\exists\ \beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$ s.t. $\forall d \in \mathcal{L}_\infty^m$ and $\forall x_0 \in \mathbb{R}^n$

$$\|y(t,x_0,d)\| \leq \beta(\|x_0\|,t) + \gamma(||d||_{[0,t)}) \quad \forall t \geq 0.$$

(3) is SIIOS, if $\exists\ \beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$ s.t. $\forall d \in \mathcal{L}_\infty^m$ and $\forall x_0 \in \mathbb{R}^n$

$$\|y(t,x_0,d)\| \leq \beta(\|h(x_0)\|,t) + \gamma(||d||_{[0,t)}) \quad \forall t \geq 0.$$

Definition 2 A C^∞ function $V : \mathbb{R}^n \to \mathbb{R}_+$ is IOS-LF for (3) if $\exists\ \alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\alpha_3 \in \mathcal{KL}$, $\theta \in \mathcal{K}$ s.t. $\forall x \in \mathbb{R}^n$, $\forall d \in \mathbb{R}^m$

$$\alpha_1(\|h(x)\|) \leq V(x) \leq \alpha_2(\|x\|),\ V(x) \geq \theta(\|d\|) \Rightarrow DV(x)f(x,d) \leq -\alpha_3(V(x),\|x\|).$$

A C^∞ function $V : \mathbb{R}^n \to \mathbb{R}_+$ is SIIOS-LF for (3) if $\exists\ \alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$, $\theta \in \mathcal{K}$ s.t. $\forall x \in \mathbb{R}^n$, $\forall d \in \mathbb{R}^m$

$$\alpha_1(\|y\|) \leq V(x) \leq \alpha_2(\|y\|),\ \|y\| \geq \theta(\|d\|) \Rightarrow DV(x)f(x,d) \leq -\alpha_3(\|y\|).$$

Theorem 1 (3) is IOS (SIIOS) $\Leftarrow \exists$ IOS (SIIOS)-LF.

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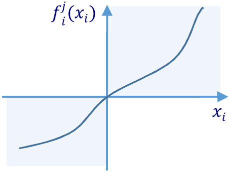
Generalized Persidskii system

$$\dot{x}(t) = A_0 x(t) + \sum_{j=1}^{M} A_j f^j(x(t)) + d(t),\ y(t) = \begin{bmatrix} C_0 x(t) \\ C_1 f^1(x(t)) \\ \vdots \\ C_M f^M(x(t)) \end{bmatrix} + v(t) \qquad (4)$$



- $x(t) = [x_1(t) \dots x_n(t)]^\top \in \mathbb{R}^n$ is the state
- $d(t) \in \mathbb{R}^n$, $v(t) \in \mathbb{R}^p$ are the perturbations, $d \in \mathcal{L}_\infty^n$, $v \in \mathcal{L}_\infty^p$
- $f^j(x) = [f_1^j(x_1) \dots f_n^j(x_n)]^\top$, $j = \overline{1, M}$

Assumption 1 $\forall\ i = \overline{1, n},\ j = \overline{1, M}$: $x_i f_i^j(x_i) > 0 \quad \forall x_i \in \mathbb{R} \setminus \{0\}$;

$\exists\ m \in \{0, \dots, M\}$: $\lim_{x_i \to \pm\infty} f_i^z(x_i) = \pm\infty\ \forall\ i = \overline{1, n},\ z = \overline{1, m}$;

$\exists\ \mu \in \{m, \dots, M\}$: $\lim_{x_i \to \pm\infty} \int_0^{x_i} f_i^z(\sigma) d\sigma = +\infty\ \forall\ i = \overline{1, n},\ z = \overline{1, \mu}$.

Observer form

$$\dot{\hat{x}}(t) = A_0\hat{x}(t) + \sum_{j=1}^{M} A_j f^j(\hat{x}(t)) + L(y(t) - \hat{y}(t)),\ \hat{y}(t) = \begin{bmatrix} C_0\hat{x}(t) \\ C_1 f^1(\hat{x}(t)) \\ \vdots \\ C_M f^M(\hat{x}(t)) \end{bmatrix}, \quad (5)$$

$\hat{x}(t) \in \mathbb{R}^n$ is estimate of $x(t)$; $L = [L_0\ L_1 \dots L_M] \in \mathbb{R}^{n\times p}$, estimation error $e(t) = x(t) - \hat{x}(t)$.

- IOS case:

$$\dot{X} = \tilde{A}_0 X + \sum_{j=1}^{M} \tilde{A}_j F^j(X) + \mathcal{D},\ \mathcal{D} = \begin{bmatrix} d \\ Lv \end{bmatrix},\ e = \Gamma X := \begin{bmatrix} I_n & -I_n \end{bmatrix} X, \quad (6)$$

$X = [x^\top\ \hat{x}^\top]^\top \in \mathbb{R}^{2n}$ is extended state, $\mathcal{D} \in \mathbb{R}^{2n}$ is augmented disturbance,

$$\tilde{A}_s = \begin{bmatrix} A_s & 0_{n\times n} \\ L_s C_s & A_s - L_s C_s \end{bmatrix},\ s \in \overline{0, M},\ F^j(X) = \begin{bmatrix} f^j(x) \\ f^j(\hat{x}) \end{bmatrix},\ j \in \overline{1, M}.$$

- SIIOS case:

$$\dot{e} = \mathscr{A}_0 e + \sum_{j=1}^{M} \mathscr{A}_j \delta f^j + \mathscr{D}, \qquad \delta f^j(x, \hat{x}) := f^j(x) - f^j(\hat{x}),\ j \in \overline{1, M}, \quad (7)$$

$\mathscr{D} := d - Lv$ is another auxiliary input, $\mathscr{A}_s = A_s - L_s C_s$, $s \in \overline{0, M}$.

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Assumptions for SIIOS case

Denote by $\mathbb{D}_+^n$ the set of *nonnegative diagonal* matrices in $\mathbb{R}^{n\times n}$.

Assumption 2 For any $j \in \overline{1, M}$:

$$e^\top \delta f^j > 0 \quad \forall x, \hat{x} \in \mathbb{R}^n,\ e = x - \hat{x} \neq 0.$$

Assumption 3 $\exists\ S_0^j, S_1^j, S_2^j, S_3^{j,q}, \Sigma_0^j, \Sigma_1^j, \Sigma_2^j, \Sigma_3^{j,q} \in \mathbb{D}_+^n$ for $j, q \in \overline{1, M}$ s.t.

$$(\delta f^j)^\top \delta f^j \leq e^\top S_0^j e + 2e^\top S_1^j(\delta f^j) + 2e^\top S_2^j f^j(e) + 2\sum_{q=1}^{M}(\delta f^j)^\top S_3^{j,q} f^q(e),$$

$$f^j(e)^\top f^j(e) \leq e^\top \Sigma_0^j e + 2e^\top \Sigma_1^j(\delta f^j) + 2e^\top \Sigma_2^j f^j(e) + 2\sum_{q=1}^{M}(\delta f^j)^\top \Sigma_3^{j,q} f^q(e),$$

$\forall x, \hat{x} \in \mathbb{R}^n$ with $e = x - \hat{x}$.

SIIOS-LF candidate:

$$V(x) = x^\top P x + 2\sum_{j=1}^{M}\sum_{i=1}^{n}\lambda_i^j \int_0^{x_i} f_i^j(s)ds,$$

$\Lambda^j = \mathrm{diag}\{\lambda^j\},\ \lambda^j = [\lambda_1^j \dots \lambda_n^j]^\top \in \mathbb{R}^n.$

SIIOS conditions

Theorem 2 Let assumptions 1, 2 and 3 be satisfied. If $\exists\ 0 \leq P = P^\top \in \mathbb{R}^{n\times n}$, $\Xi, \Lambda^j, \Gamma_j, \Omega_j, \Upsilon_{j,k} \in \mathbb{D}^n_+$ with $j,k = \overline{1,M}$, $0 < \Phi = \Phi^\top \in \mathbb{R}^{n\times n}$, $\varrho \in \mathbb{R}$ and $\gamma, \eta > 0$ s.t.

$$P + \varrho \sum_{z=1}^{\mu} \Lambda^z > 0;\ Q \leq 0; \tag{8}$$

$$\Xi^0 - \gamma \sum_{j=1}^{M} S_0^j - \eta \sum_{j=1}^{M} \Sigma_0^j > 0,\ \Gamma_j - \gamma S_1^j - \eta \Sigma_1^j \geq 0,$$

$$\Omega_j - \gamma S_2^j - \eta \Sigma_2^j \geq 0,\ \Upsilon_{j,k} - \gamma S_3^{j,k} - \eta \Sigma_3^{j,k} \geq 0,$$

then the system (4), (5) is SIIOS w.r.t. e, where

$$Q = \begin{bmatrix} \mathscr{A}_0^\top P + P\mathscr{A}_0 + \Xi & P\mathscr{A} + \bar{\Gamma} & \mathscr{A}_0^\top \Lambda + \Omega & P \\ * & -\gamma I_{nM} & \mathscr{A}^\top \Lambda + \Upsilon & 0_{nM\times n} \\ * & * & -\eta I_{nM} & \Lambda^\top \\ * & * & * & -\Phi \end{bmatrix}$$

$$\mathscr{A} = \begin{bmatrix} \mathscr{A}_1 & \dots & \mathscr{A}_M \end{bmatrix},\ \bar{\Gamma} = \begin{bmatrix} \Gamma_1 & \dots & \Gamma_M \end{bmatrix},\ \Upsilon = (\Upsilon_{j,k})_{j,k=1}^{M},$$

$$\Lambda = \begin{bmatrix} \Lambda^1 & \dots & \Lambda^M \end{bmatrix},\ \Omega = \begin{bmatrix} \Omega_1 & \dots & \Omega_M \end{bmatrix}.$$

Extension

What if a part of the argument of a nonlinearity f^j is measured by the linear components of the output $y_0(t) = C_0 x(t)$?

If $\exists\ \Delta_j = \mathrm{diag}(\Delta_j^1, \ldots, \Delta_j^n) \in \mathbb{D}_+^n$, $\Delta_j^i \in \{0, 1\}$ for $i \in \overline{1, n}$ and $\Pi_j \in \mathbb{R}^{n \times p_0}$ s.t. $\Delta_j = \Pi_j C_0$ for $j \in \overline{1, M}$, then the observer (5) can be extended:

$$\dot{\hat{x}}(t) = A_0 \hat{x}(t) + \sum_{j=1}^{M} A_j f^j(\hat{x}(t)) + L(y(t) - \hat{y}(t)) \qquad (9)$$

$$+ \sum_{j=1}^{M} \delta_j \Delta_j f^j(\Pi_j y_0(t) - \Pi_j C_0 \hat{x}(t)),$$

$\delta_j > 0$ are tuning parameters, and the error dynamics (7) takes the form:

$$\dot{e} = \mathscr{A}_0 e + \sum_{j=1}^{M} \mathscr{A}_j \delta f^j - \sum_{j=1}^{M} \delta_j \Delta_j f^j(e) + \mathscr{D}.$$

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Presentation of SIS model in Persidskii form

- Multi-group SIS model (1) Mei et al., 2017; Niazi et al., 2021:

$$\dot{x}(t) = \mathrm{diag}(1_n - x(t))(\beta A x(t) + d(t)) - \gamma x(t)$$

- Change of variables:

$$z = \ln(1_n - x),\ x = 1_n - e^z$$

- Persidskii representation of SIS model:

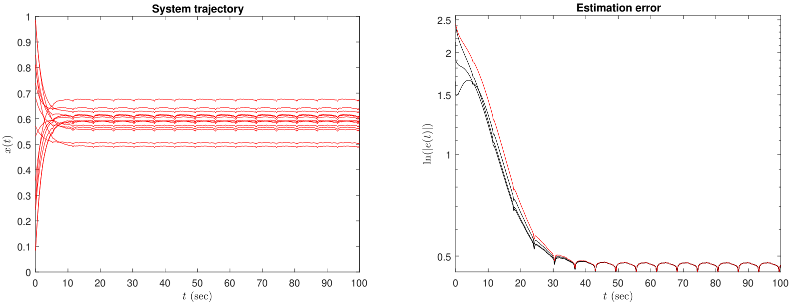
$$\dot{z}(t) = \beta A f_1(z(t)) - \gamma f_2(z(t)) - d(t),\ y(t) = -C f_1(z(t)),$$
$$f_1(z) = e^z - 1_n,\ f_2(z) = 1_n - e^{-z}$$

  - assumptions 1 and 2 are satisfied
  - assumption 3 holds locally
- The observer in the form (9):

$$\dot{\hat{z}}(t) = \beta A f_1(\hat{z}(t)) - \gamma f_2(\hat{z}(t)) + L(y(t) + C f_1(\hat{z}(t)))$$
$$+ \sum_{j=1}^{2} m_j C^\top C f_j(z(t) - \hat{z}(t)),$$

Observer simulation

$$\dot{\hat{x}}(t) = \mathrm{diag}(1_n - \hat{x}(t))[(L + m_2 C^\top \mathrm{diag}(1_p - y(t))^{-1})(y(t) - C\hat{x}(t)) + \beta A\hat{x}(t)] - \gamma\hat{x}(t) + m_1 C^\top (y(t) - C\hat{x}(t)),$$



$n = 15$ and a harmonic $d(t)$

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- Class of **generalized Persidskii systems**
- SIIOS using LMIs
- Observer design

- Application:
  - a multi-group SIS model with disturbances
  - global observer in state and local in estimation error