About identifiability and observability of SIRQ models

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14 october 2021

The SIRQ model

$$(\Sigma): \left\{ \begin{array}{lcl} \dot{S} & = & -\frac{\beta}{N-Q} SI \\ \dot{I} & = & \beta \frac{S}{N-Q} I - (\rho+\alpha) I \\ \dot{Q} & = & \alpha I - \rho Q \\ \dot{R} & = & \rho I + \rho Q \end{array} \right. \qquad y(t) = \left[\begin{array}{c} \alpha I(t) \\ Q(t) \end{array} \right]$$

Assumption. Only $N = S + I + Q + R$ is known.

Hypothesis. $\mathcal{R}_0 := \frac{\beta}{\rho+\alpha} > 1$

Simplified model when $N >> 1$ and $Q << N$:

$$(\Sigma_R): \left\{ \begin{array}{lcl} \dot{S} & = & -\frac{\beta}{N} SI \\ \dot{I} & = & \frac{\beta}{N} SI - (\rho+\alpha) I \\ \dot{Q} & = & \alpha I - \rho Q \\ \dot{R} & = & \rho I + \rho Q \end{array} \right. \qquad y(t) = \left[\begin{array}{c} \alpha I(t) \\ Q(t) \end{array} \right]$$

Identifiability and observability

Definition. Given $N > 0$ and $\varepsilon \in (0, N)$, (Σ) resp. (Σ_R) is identifiable if there exists $t > 0$ such that

$$\begin{bmatrix} \alpha \\ \beta \\ \rho \end{bmatrix} \in \left(\mathbb{R}_+^\star\right)^3 \quad \longmapsto \quad y(\cdot) \in \mathcal{C}^\infty([0,t], \mathbb{R}_+^2) \text{ is injective}$$

where $(S(\cdot), I(\cdot), Q(\cdot), R(\cdot))$ is solution for $S(0) = N - \varepsilon$, $I(0) = \varepsilon$, $Q(0) = 0$, $R(0) = 0$. If moreover

$$\begin{bmatrix} \alpha \\ \beta \\ \rho \\ \varepsilon \end{bmatrix} \in \left(\mathbb{R}_+^\star\right)^3 \times (0, N) \longmapsto \quad y(\cdot) \in \mathcal{C}^\infty([0,t], \mathbb{R}_+^2) \text{ is injective}$$

then (Σ) resp. (Σ_R) is identifiable and observable.

Identifiability and observability

Proposition. (Σ) resp. (Σ_R) is identifiable and observable.

Proof. not straightforward...

see *F. Hamelin, A. Iggidr, A. Rapaport, G. Sallet, M. Souza Identifiability and observability of the SIR model with quarantine (2021) preprint hal-03160829.*

Another SIRQ model

$$\left\{\begin{array}{rcl} \dot S &=& -\frac{\beta}{N-Q}S(I+J) \\ \dot I &=& p\frac{\beta}{N-Q}S(I+J)-\rho I \\ \dot J &=& (1-p)\frac{\beta}{N-Q}S(I+J)-(\rho+\alpha))J \\ \dot Q &=& \alpha J-\rho Q \\ \dot R &=& \rho(I+J)+\rho Q \end{array}\right. \qquad y(t)=\begin{bmatrix}\alpha J(t)\\ Q(t)\end{bmatrix}$$

1. $\rho=\frac{y_1(t)-\dot y_2(t)}{y_2(t)}, \quad t>0 \;\Rightarrow\; \rho$ is identifiable
2. $I(0)+J(0)+Q(0)+R(0) << N \;\Rightarrow\; \frac{S}{N-Q}\simeq 1$ is slow variable

   $\Rightarrow$ reduced dynamics :

$$\left\{\begin{array}{rcl} \dot I &=& p\beta(I+J)-\rho I \\ \dot J &=& (1-p)\beta(I+J)-(\rho+\alpha))J \end{array}\right. \qquad y_1(t)=\alpha J(t)$$

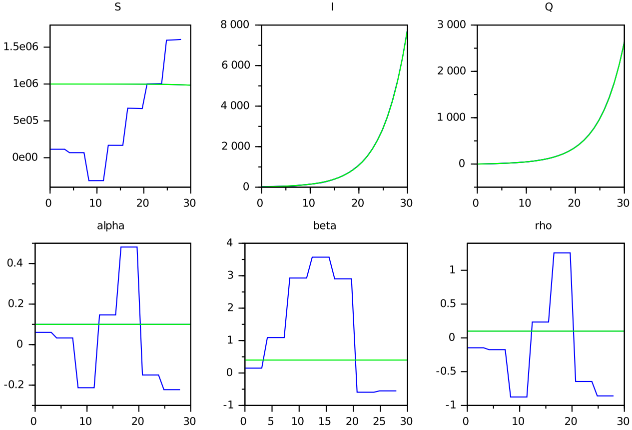
$$z=\frac{1-p}{p}I-J \Rightarrow \left\{\begin{array}{rcl} \dot z &=& -\rho z+y_1 \\ \dot y_1 &=& \rho\beta p z+(\beta-\rho-\alpha)y_1 \end{array}\right.$$ non identifiable !

Different approaches for parameters estimation

- Usual least-squares
- An alternative approach
- Observers in cascade
- High-gain observers
- Asymptotic observers

Use of the classical least square method

$N = 10^6$ and (unkwnown) $\alpha = 0.1$, $\beta = 0.4$, $\rho = 0.1$, $I(0) = 20$



30 data points without noise... already long to compute...

An alternative approach

$$\text{Posit } \eta := \frac{\beta}{N\alpha} \quad \Rightarrow \quad S(t) = S(0) e^{-\eta \int_0^t y_1(\tau) d\tau}, \quad t > 0$$

$$\Rightarrow \dot{I}(t) = \left(\beta \frac{S(0)}{N} e^{-\eta \int_0^t y_1(\tau) d\tau} - (\rho + \alpha) \right) I(t), \quad t > 0.$$

$$\frac{S(0)}{N} \simeq 1 \Rightarrow \dot{y}_1(t) \approx \left(\beta e^{-\eta \int_0^t y_1(\tau) d\tau} - (\rho + \alpha) \right) y_1(t)$$

or equivalently

$$\underbrace{\log \left(\frac{d}{dt} \log y_1(t) + \rho + \alpha \right)}_{Y(t, \rho+\alpha)} \approx \log(\beta) - \eta \underbrace{\int_0^t y_1(\tau) d\tau}_{X(t)}$$

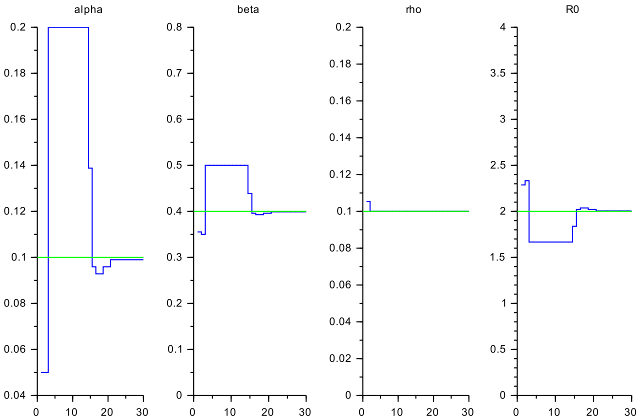
$\Rightarrow$ $\rho + \alpha$ is sought to minimizing the residual sum of least squares in the linear regression of the $\{Y(\cdot), X(\cdot)\}$ data

An alternative approach

$$\dot{Q} = \alpha I - \rho Q \Rightarrow \dot{y}_2(t) = y_1(t) - \rho y_2(t)$$

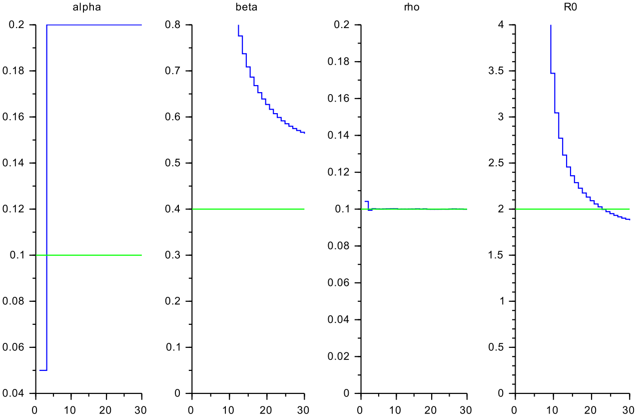
$$\Rightarrow \rho = \frac{\displaystyle\int_0^t y_1(\tau)d\tau - y_2(t) + y_2(0)}{\displaystyle\int_0^t y_2(\tau)d\tau}, \quad t > 0$$

Simulations without noise



$N = 10^6$, 30 data points

Simulations with small noise



$N = 10^6$, 30 data points, smoothing and Lagrange interpolation

A first observer to estimate ρ

$$\left\{\begin{array}{l}\dot{\hat{Q}} = y_1(t) - \hat{\rho}\, y_2(t) - K_1\, y_2(t)(\hat{Q} - Q)\\ \dot{\hat{\rho}} = -K_2\, y_2(t)(\hat{Q} - Q)\end{array}\right. \quad \text{with} \quad \left\{\begin{array}{l}K_1 = \lambda_1 + \lambda_2\\ K_2 = -\lambda_1\, \lambda_2\end{array}\right.$$

$$\Rightarrow \dot{e} = y_2(t) \underbrace{\begin{pmatrix} -K_1 & -1 \\ -K_2 & 0 \end{pmatrix}}_{M} e$$

$$V = e^T P e \text{ with } P \text{ SDP s.t. } PM + M^T P = -\underbrace{diag(\lambda_1, \lambda_2)}_{D}$$

$$\Rightarrow \dot{V} = -y_2 e^T D e \leq -\min_t(y_2(t)) e^T D\, e < 0.$$

An high-gain observer to reconstruct α and β

Assumption. $\alpha \in [\alpha_m, \alpha_M], \quad \beta \in [\beta_m, \beta_M], \quad \rho \in [\rho_m, \rho_M]$

$$\dot{\hat{z}} = \begin{bmatrix} \hat{z}_2 \\ \hat{z}_3 \\ \hat{z}_4 \\ \tilde{z}_3^2 + \tilde{z}_2\tilde{z}_4 + (\tilde{z}_4 - \tilde{z}_2\tilde{z}_3)\left(\tilde{z}_2 + \frac{\tilde{z}_4}{\tilde{z}_3}\right) \end{bmatrix} - \begin{bmatrix} 4\theta \\ 6\theta^2 \\ 4\theta^3 \\ \theta^4 \end{bmatrix} (\hat{z}_1 - \log(y_1(t))),$$

$$\tilde{z}_2 = \mathrm{sat}_{-\rho_M-\alpha_M}^{\beta_M-\rho_m-\alpha_m}(\hat{z}_2),\ \tilde{z}_3 = \mathrm{sat}_{-\beta_M}^{-\epsilon}(\hat{z}_3),\ \tilde{z}_4 = \mathrm{sat}_{-\beta_M(\beta_M-\rho_m-\alpha_m)}^{\beta_M(\beta_M+\rho_M+\alpha_M)}(\hat{z}_4)$$

An high-gain observer to reconstruct α and β

Proposition. For $\epsilon > 0$ small enough, there exists $T_\epsilon > 0$ with $T_\epsilon \to +\infty$ when $\epsilon \to 0$ such that for $\theta > 0$ large enough, there exists $K > 0$ such that the error of the estimations

$$\hat{\alpha}(t) = \frac{\hat{z}_3(t)^2}{\max\left(\hat{z}_4(t) - \hat{z}_2(t)\hat{z}_3(t)\,,\ \epsilon\frac{\beta_m}{\alpha_M N}y_1(t)\right)} - \hat{z}_2(t) - \rho$$

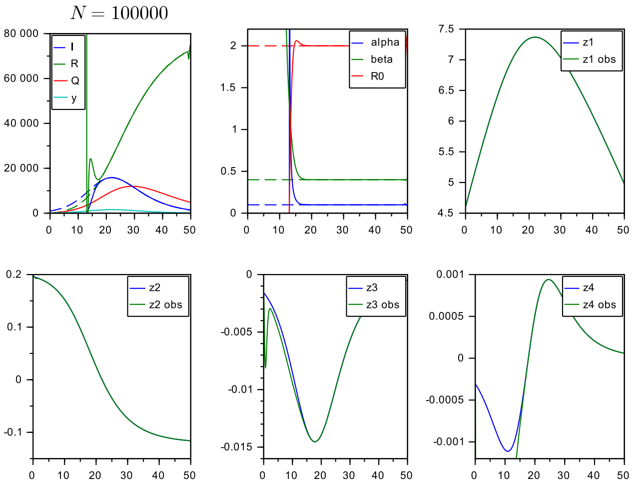
$$\hat{\beta}(t) = Ne^{-\hat{z}_1(t)}\left(\frac{(\hat{z}_2(t)-\rho)(\hat{z}_4(t)-\hat{z}_2(t)\hat{z}_3(t))}{\min(\hat{z}_3(t), -\epsilon)} - \hat{z}_3(t)\right)$$

$$\hat{S}(t) = N\frac{\hat{z}_2(t)+\rho+\hat{\alpha}(t)}{\max(\hat{\beta}(t), \beta_m)}$$

$$\hat{I}(t) = \frac{e^{\hat{z}_1(t)}}{\max(\hat{\alpha}(t), \alpha_m)}$$

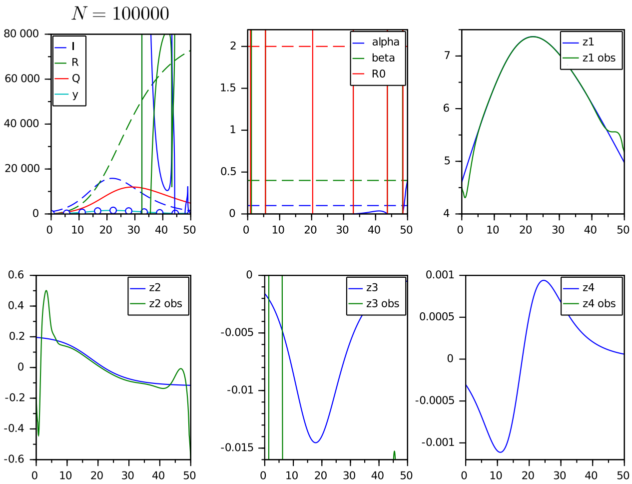
verify $||E(t)|| \leq K||E(0)||\exp\left(-\frac{\theta}{3}\,t\right),\ t \in [0, T_\epsilon]$

Simulations of the high-gain observer



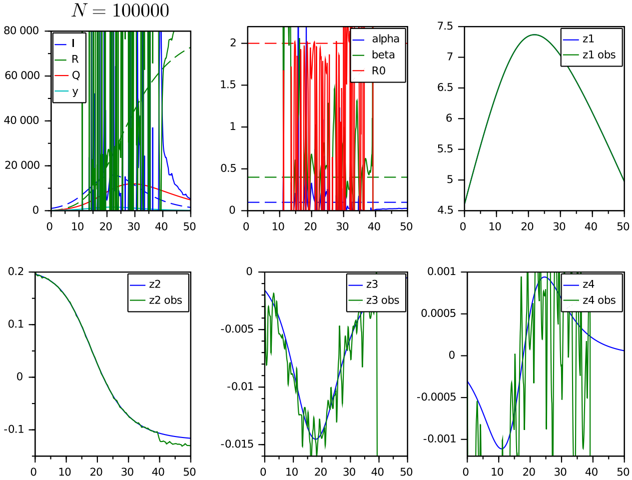
ρ assumed to be known, $N = 10^5$, $\theta = 2.5$, no noise

Simulations with discrete measurements



ρ assumed to be known, $N = 10^5$, $\theta = 2.5$, no noise, 30 points, Lagrange interpolation

Simulations with small noise



ρ assumed to be known, $N = 10^5$, $\theta = 2.5$

A reduced-order high-gain observer

$$\left\{\begin{array}{l} v = \beta\frac{S}{N} - \rho - \alpha \\ k = \frac{\beta^2}{\alpha} \end{array}\right. \underset{S/N\sim 1}{\Rightarrow} \left\{\begin{array}{l} \dot{y}_1 = v y_1 \\ \dot{v} = -\frac{k}{N} y_1 \\ \dot{k} = 0 \end{array}\right.$$

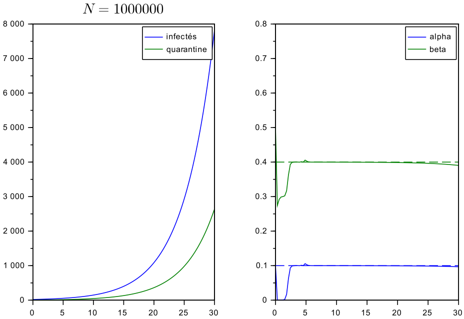
$$\hat{\beta}(t) = \frac{1}{2}\left(-\hat{z}_3(t)N - \sqrt{\max\left((\hat{z}_3(t)N)^2 + 4\hat{z}_3(t)N(\rho + \hat{z}_2(t))e^{\hat{z}_1(t)}, 0\right)}\right)e^{-\hat{z}_1(t)}$$

$$\hat{\alpha}(t) = \hat{\beta}(t) - \rho - \hat{z}_2(t)$$

$$\text{with} \quad \dot{\hat{z}} = \begin{bmatrix} \hat{z}_2 \\ \hat{z}_3 \\ \tilde{z}_2\tilde{z}_3 \end{bmatrix} - \begin{bmatrix} 3\theta \\ 3\theta^2 \\ \theta^3 \end{bmatrix} (\hat{z}_1 - \log(y_1(t)))$$

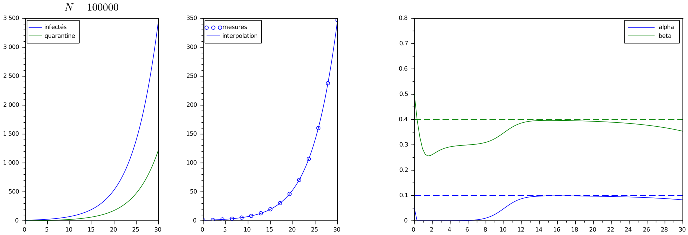
$$\text{and} \quad \tilde{z}_2 = \text{sat}_{-\sqrt{\overline{v}_0^2 + \frac{2\overline{k}}{N} y_{10}}}^{\;\overline{v}_0}(\hat{z}_2), \quad \tilde{z}_3 = \text{sat}_{-\frac{\overline{v}_0^2}{2} - \frac{\overline{k}}{N} y_{10}}^{\;0}(\hat{z}_3)$$

Simulations of the reduced high-gain observer



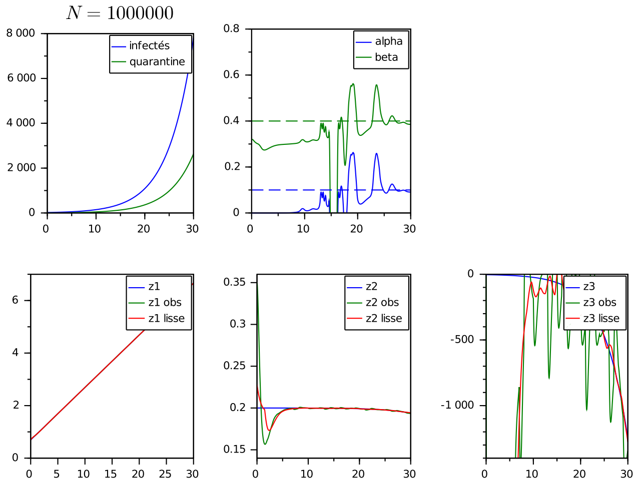
ρ assumed to be known, $\theta = 2$, no noise

Simulations with discrete measurements



ρ assumed to be known, $\theta = 2$, no noise, 30 points, Lagrange interpolation

Simulations with noise



ρ assumed to be known, $\theta = 2$, moving average smoothing

An asymptotic observer to estimate $\delta = \beta - \alpha$ and ρ

Approximation : $\frac{S}{N} \simeq 1$

$$\Rightarrow \begin{cases} \dot{y}_1 = \delta\, y_1 - \rho y_1 \\ \dot{y}_2 = y_1 - \rho\, y_2 \end{cases} \underset{z_i = \log(y_i)}{\Rightarrow} \begin{cases} \dot{z}_1 = \delta - \rho \\ \dot{z}_2 = \exp\left(z_1 - z_2\right) - \rho \end{cases}$$

Observer :

$$\begin{cases} \dot{\hat{z}}_1 = \hat{\delta} - \hat{\rho} - K_1(\hat{z}_1 - z_1) - K_{12}(\hat{z}_2 - z_2), \\ \dot{\hat{z}}_2 = \exp\left(z_1 - z_2\right) - \hat{\rho} - K_2(\hat{z}_1 - z_1) - K_{22}(\hat{z}_2 - z_2), \\ \dot{\hat{\delta}} = -K_3(\hat{z}_1 - z_1) - K_{32}(\hat{z}_2 - z_2), \\ \dot{\hat{\rho}} = -K_4(\hat{z}_1 - z_1) - K_{42}(\hat{z}_2 - z_2), \end{cases}$$

An asymptotic observer to estimate $\delta = \beta - \alpha$ and ρ

$$\dot{e} = \underbrace{\begin{pmatrix} -K_1 & -K_{12} & 1 & -1 \\ -K_2 & -K_{22} & 0 & -1 \\ -K_3 & -K_{32} & 0 & 0 \\ -K_2 & -K_{42} & 0 & 0 \end{pmatrix}}_{M} e$$

$$\begin{cases} K_{12} = 0, K_{22} = 0, K_{32} = 0, K_{42} = 1 \\ K_1 = \sum \lambda_i \\ K_2 = K_1 + \sum \lambda_i \lambda_j \lambda_k \\ K_3 = -\prod \lambda_i \\ K_4 = -\sum \lambda_i \lambda_j + K_3 - 1 \end{cases} \quad \Rightarrow \mathrm{Spec} M = \{-\lambda_1, -\lambda_2, -\lambda_3, -\lambda_4\}$$

An asymptotic observer to estimate $k = \frac{\beta^2}{\alpha}$

$$v = \beta \frac{S}{N} - \rho - \alpha \underset{S/N \sim 1}{\Rightarrow} \begin{cases} \dot{y}_1 = v y_1 \\ \dot{v} = -\frac{k}{N} y_1 \end{cases}$$

$$\begin{cases} \dot{\hat{y}}_1 & = \hat{v} y_1 - y_1 K_1 (\hat{y}_1 - y_1) \\ \dot{\hat{v}} & = -\hat{k} \frac{y_1}{N} - y_1 K_2 (\hat{y}_1 - y_1) \\ \dot{\hat{k}} & = -y_1 K_3 (\hat{y}_1 - y_1) \end{cases} \Rightarrow \dot{e} = y_1 \underbrace{\begin{pmatrix} -K_1 & 1 & 0 \\ -K_2 & 0 & -\frac{1}{N} \\ -K_3 & 0 & 0 \end{pmatrix}}_{A} e$$

with $K_1 = \lambda_1 + \lambda_2 + \lambda_3, \; K_2 = \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_3 \lambda_2 \; K_3 = -N \lambda_1 \lambda_1 \lambda_3$

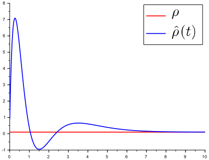
$V = e^T P e$ with P SDP s.t. $PA + A^T P = -\underbrace{diag(\lambda_1, \lambda_2, \lambda_3)}_{D}$

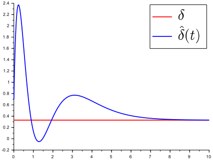
$\Rightarrow \dot{V} = -y_1 e^T D e \leq -\min_t (y_1(t)) e^T D \, e < 0.$

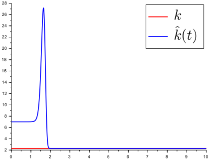
Coupling the two observers

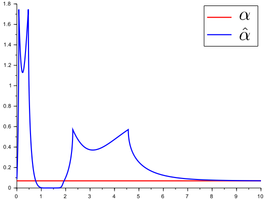
$$\begin{pmatrix} \hat{\rho} \\ \hat{\delta} \\ \hat{k} \end{pmatrix} \Rightarrow \begin{pmatrix} \hat{\rho} \\ \hat{\beta} = \dfrac{\hat{k} - \sqrt{\max(\hat{k}^2 - 4\hat{k}\hat{\delta}, 0)}}{2} \\ \hat{\alpha} = \hat{\beta} - \hat{\delta} \end{pmatrix}$$

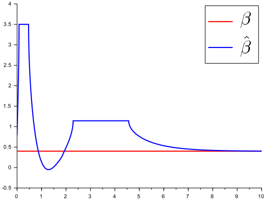
Simulations without noise



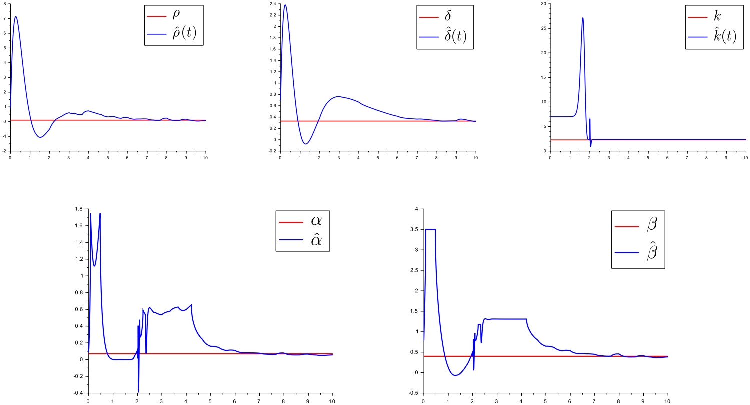








Simulations with noise



good ! ... but slow...

Conclusions

- Parameters and state estimations of epidemiological models is not robust at the beginning of the epidemic, being closed from a non-observable state (disease free equilibrium)...
- One may look for a speed-accuracy compromise, with the help of asymptotic observers in cascade.