Minimizing epidemic final size through social distancing

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- The current outbreak of Covid-19 led to renewed interest in modelling/analysis of **"social distancing"** strategies to control infectious diseases.
- This refers to attempts to directly **reduce the infecting contacts** within the population (in contrast to vaccination or quarantine). In absence of vaccine or therapy, such strategies are probably the only mid-term option.

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- This refers to attempts to directly **reduce the infecting contacts** within the population (in contrast to vaccination or quarantine). In absence of vaccine or therapy, such strategies are probably the only mid-term option.
- We study here, **on the SIR model, optimal use of confinement to minimize the total number of infected** along the outbreak.

Stationary SIR model

Optimal confinement on $[0, T]$

Optimal confinement on interval of duration T

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We provide here the **main properties of the classical SIR model**

SIR model

S, I, R: proportions of **susceptible, infected** and **removed** individuals.

$$\dot{S} = -\beta SI, \qquad S(0) = S_0, \tag{1a}$$

$$\dot{I} = \beta SI - \gamma I, \qquad I(0) = I_0, \tag{1b}$$

$$\dot{R} = \gamma I, \qquad R(0) = R_0. \tag{1c}$$

with $S_0 + I_0 + R_0 = 1$.

$\beta, \gamma > 0$: **infection, recovery rates**

Analysis of the dynamical behaviour

- All state variables remain nonnegative
- $\dot{S} = -\beta SI \leqslant 0$, so there exists $\lim_{t\to+\infty} S(t) := S_\infty \in [0, S_0]$
- $\dot{S} + \dot{I} = -\gamma I \leqslant 0$, so that $\lim_{t\to+\infty}(S+I)(t)$ exists, and $\lim_{t\to+\infty} I(t)$ too
- Necessarily

$$\lim_{t\to+\infty} I(t) = 0$$

(otherwise $\lim_{t\to+\infty} S(t) = 0$, and $\lim_{t\to+\infty} I(t) = 0$ too : contradiction)

- Necessarily $\int_0^{+\infty} I(t)\cdot dt = -\frac{1}{\gamma}\int_0^{+\infty}(\dot{S}(t)+\dot{I}(t))\cdot dt < \frac{1}{\gamma}$, so that

$$S_\infty = e^{-\beta \int_0^{+\infty} I(t)\cdot dt} S_0, \qquad \text{and} \qquad S_\infty \in (0, S_0)$$

Basic reproduction number

As $\dot{I} = (\beta S - \gamma) I$ and $S \leqslant 1$, no outbreak may occur if

$$\boxed{\mathcal{R}_0 := \frac{\beta}{\gamma} \leqslant 1}$$

This is the so-called **basic reproduction number**.

Herd immunity

If $\mathcal{R}_0 > 1$, as $\dot{I} = (\beta S - \gamma) I$, the outbreak stops spreading (i.e. $\dot{I} \leqslant 0$) if

$$S(t) \leqslant \boxed{S_{\text{herd}} := \frac{\gamma}{\beta}} = \frac{1}{\mathcal{R}_0}$$

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Proposition 1 (Equilibrium stability)

- *The set of equilibrium points of* (1) *is* $\{(S_{\mathrm{equi}}, 0) \ : \ S_{\mathrm{equi}} \geqslant 0\}$
- $(S_{\mathrm{equi}}, 0)$ *is* **unstable** *iff*

$$\mathcal{R}_0 S_{\mathrm{equi}} > 1, \quad \textit{i.e.} \quad S_{\mathrm{equi}} > S_{\mathrm{herd}}$$

Epidemic final size

One has

$$\frac{d}{dt}(S + I) = -\gamma I = \frac{\gamma}{\beta}\frac{\dot{S}}{S}$$

so that

$$t \mapsto S(t) + I(t) - S_{\mathrm{herd}} \ln S(t) \qquad \textbf{is constant}$$

In particular,

$$\boxed{S_\infty - S_{\mathrm{herd}} \ln S_\infty = S_0 + I_0 - S_{\mathrm{herd}} \ln S_0} \qquad (2)$$

$S_0 - S_\infty$ is the **epidemic final size**

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Theorem 2 (Final size equation)

S_∞ **is the unique root of** (2) **in the interval** $[0, S_{\text{herd}}]$

Numerical values of $S_{\rm herd}$ and S_∞ for various $\mathcal{R}_0$

The outbreak does not stop when immunity is reached!

$\mathcal{R}_0$	1.5	2	2.5	**2.9**	3	3.5
$S_{\rm herd}$	0.67	0.50	0.40	**0.34**	0.33	0.29
S_∞ (for $S_0 \simeq 1$)	0.42	0.20	0.11	**0.067**	0.059	0.034
$\frac{S_{\rm herd} - S_\infty}{1 - S_\infty}$	43%	37%	33%	**30%**	29%	27%

$\frac{S_{\rm herd} - S_\infty}{1 - S_\infty}$: proportion of susceptible infected after passing $S_{\rm herd}$.

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$\frac{S_{\mathrm{herd}} - S_\infty}{1 - S_\infty}$: proportion of susceptible infected after passing S_{herd}.

How can one minimize S_∞? Or equivalently:

How best can one stop close to herd immunity?

Setting

For initial data $(S, I)(0) = (S_0, I_0) \geqslant 0$, $S_0 + I_0 \leqslant 1$, we consider the **controlled SIR model**

$$\dot{S}(t) = -u(t)\beta S(t)I(t), \quad t \geqslant 0 \tag{3a}$$

$$\dot{I}(t) = u(t)\beta S(t)I(t) - \gamma I(t), \quad t \geqslant 0 \tag{3b}$$

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$$\dot{I}(t) = u(t)\beta S(t)I(t) - \gamma I(t), \quad t \geqslant 0 \tag{3b}$$

and define the **admissible control set** $\mathcal{U}_{\alpha,T,T'}$, $0 \leqslant T < T'$, $\alpha \in [0, 1)$

$$\mathcal{U}_{\alpha,T,T'} := \{u \in L^\infty,\ \alpha \leqslant u(t) \leqslant 1 \text{ if } t \in [T, T'], u(t) = 1 \text{ otherwise}\}.$$

α is the **maximal lockdown intensity**

A key property of the final size $S_\infty(u)$

For any **admissible input** u, write $S_\infty(u)$ the corresponding final size.

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Lemma 3

- *For any* $u \in \mathcal{U}_{\alpha,T,T'}$, $S_\infty(u)$ *is* **the unique** $S_\infty \in (0, S_{\mathrm{herd}})$ **s.t.**

$$S_\infty - S_{\mathrm{herd}} \ln S_\infty = S(T') + I(T') - S_{\mathrm{herd}} \ln S(T')$$

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- *For any* $\mathcal{V} \subset \mathcal{U}_{\alpha,T,T'}$, **maximizing** $S_\infty(u)$ **over** $\mathcal{V}$ *amounts to* **minimizing** $S(T') + I(T') - S_{\text{herd}} \ln S(T')$ **among the solutions** $(S(u), I(u))$ **of** (3)**,** $u \in \mathcal{V}$

Stationary SIR model

Optimal confinement on $[0, T]$

Optimal confinement on interval of duration T

We study here how to **minimize the total number of infected through application of lockdown during interval $[0, T]$, with maximal intensity α**

Optimal immunity control (1/2)

$$\textbf{Find} \quad \sup_{u \in \mathcal{U}_{\alpha,0,T}} S_\infty(u) := S_\infty^* \qquad (\mathcal{P}_{\alpha,T})$$

For any $T_0 \in [0, T]$, define $u_{T_0} \in \mathcal{U}_{\alpha,0,T}$ by

$$u_{T_0} = \mathbb{1}_{[0,T_0]} + \alpha \mathbb{1}_{[T_0,T]} + \mathbb{1}_{[T,+\infty)} \qquad (4)$$

and (S^{T_0}, I^{T_0}) the solution of (3) with $u = u_{T_0}$.

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and (S^{T_0}, I^{T_0}) the solution of (3) with $u = u_{T_0}$.

Theorem 4 (Optimal control is unique and bang-bang)

- **Problem $(\mathcal{P}_{\alpha,T})$ admits a unique solution u^*.**
- **There exists a unique $T_0^* \in [0, T)$ such that $u^* = u_{T_0^*}$, so that $(\mathcal{P}_{\alpha,T})$ is equivalent to the 1D optimization problem**

$$\sup_{T_0 \in [0,T)} S_\infty(u_{T_0}) \qquad (\widetilde{\mathcal{P}}_{\alpha,T})$$

Optimal immunity control (2/2)

For $T_0 \in [0, T]$, and (S^{T_0}, I^{T_0}) solution to (3) with $u = u_{T_0}$, let

$$\psi(T_0) := (1-\alpha)\beta I^{T_0}(T) \int_{T_0}^{T} \frac{S^{T_0}(t)}{I^{T_0}(t)}\, dt - 1 \tag{5}$$

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Theorem 5 (Characterization of T_0^*)

Let $u_{T_0^*}$ be the optimal control, then

- **if $\psi(0) \leqslant 0$, then $T_0^* = 0$.**
- **if $\psi(0) > 0$, then T_0^* is the unique $T_0 \in (0, T)$ s.t. $\psi(T_0) = 0$.**

If $T_0^* > 0$, then $S^{T_0^*}(T_0^*) \geqslant S_{\mathrm{herd}}$.
If $\alpha = 0$, $T_0^* > 0$ iff $T > \frac{1}{\gamma} \ln \frac{S_0}{S_0 - S_{\mathrm{herd}}}$, and then T_0^* is the unique T_0 s.t. $S^{T_0}(T_0) = \frac{S_{\mathrm{herd}}}{1 - e^{\gamma(T_0 - T)}}$.

Hints of proof

1. **Existence of optimal control:** by studying minimizing sequence
2. **Optimality conditions:** Pontryagin's maximum principle yields optimum at some $u^* = u_{T_0},\ T_0 \in [0, T)$ **(bang-bang)**.
3. **Optimality conditions** in terms of T_0: ψ defined in (5) is such that

$$\frac{\partial S_\infty(u_{T_0})}{\partial T_0} \propto \psi(T_0),$$

   so that $S_\infty(u_{T_0})$ reaches optimum in $(0, T)$ iff $\psi(T_0) = 0$.
4. ψ **is decreasing,** thus admits at most one zero: **uniqueness**.

Properties of the value function

Theorem 6
$S_\infty^* = \sup\{S_\infty(u) : \; u \in \mathcal{U}_{\alpha,0,T}\}$ **is non-increasing wrt** α **and non-decreasing wrt** T**.**

Theorem 7 (How close from herd immunity can we stop?)

$$\overline{\alpha} := \frac{S_{\text{herd}}}{S_0 + I_0 - S_{\text{herd}}}(\ln S_0 - \ln S_{\text{herd}}) \in (0,1) \tag{6}$$

- **If** $\boxed{0 \leqslant \alpha \leqslant \overline{\alpha}}$ **then** $\forall \varepsilon > 0, \exists T > 0, (1-\varepsilon)S_{\text{herd}} \leqslant S_\infty^* < S_{\text{herd}}$
- **If** $\boxed{\overline{\alpha} < \alpha < 1}$ **then** $\forall T > 0, S_\infty^* < S_\infty(\alpha \mathbf{1}) < S_{\text{herd}}$

Numerical illustrations – 1 (1/5)

Parameter	*Name*	*Value*
β	Infection rate	0.29 day^{-1}
γ	Recovery rate	0.1 day^{-1}
α_{lock}	Lockdown level (France, 03-05/2020)	0.231
S_0	Initial proportion of susceptible	$1 - I_0$
I_0	Initial proportion of infected (March 8th)	$\frac{1 \times 10^3}{6.7 \times 10^7} \approx 1.49 \times 10^{-5}$
R_0	Initial proportion of removed	0

Table: Parameters used in simulation

One has

$$\boxed{\mathcal{R}_0 \approx 2.9, \qquad S_{\text{herd}} \approx 0.34,}$$

and with initial conditions (S_0, I_0), the **critical lockdown intensity** is

$$\boxed{\overline{\alpha} \approx 0.56}$$

Numerical illustrations – 1 (2/5)

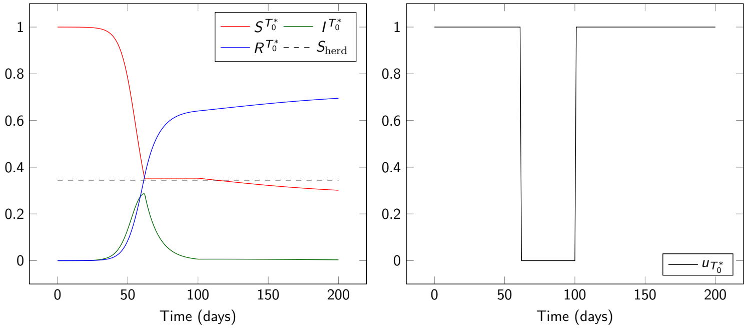


Figure: Optimal solution of $(\mathcal{P}_{\alpha,T})$ for $\alpha = 0.0$, $T = 100$ days, shown on $[0, 200]$. Here $S_\infty^* = 0.282 < 0.34 = S_{\text{herd}}$, $T_0^* = 61.9$ days.

Numerical illustrations – 1 (3/5)

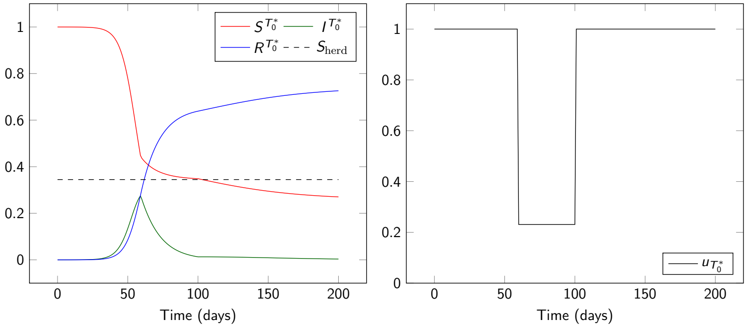


Figure: Optimal solution of $(\mathcal{P}_{\alpha,T})$ for $\alpha = \alpha_{\text{lock}}$, $T = 100$ days shown on $[0, 200]$. Here $S_\infty^* = 0.259 < 0.34 = S_{\text{herd}}$, $T_0^* = 59.2$ days.

Numerical illustrations – 1 (4/5)

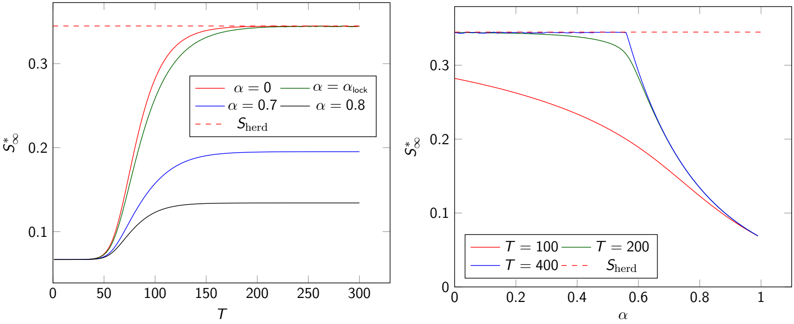
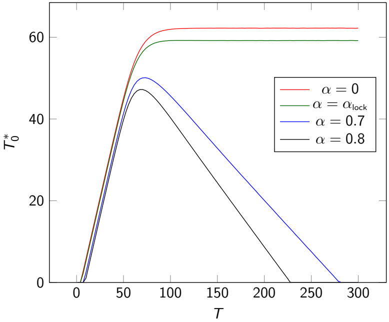


Figure: Graph of S_∞^* wrt T for $\alpha \in \{0, \alpha_{\text{lock}}, 0.7, 0.8\}$ (left), and wrt α for $T \in \{100, 200, 400\}$ (right).

Numerical illustrations – 1 (5/5)



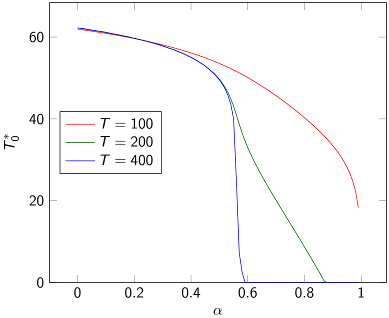


Figure: Graph of T_0^* wrt T for $\alpha \in \{0.0, \alpha_{\text{lock}}, 0.7, 0.8\}$ (left), and wrt α for $T \in \{100, 200, 400\}$ (right).

Stationary SIR model

Optimal confinement on $[0, T]$

Optimal confinement on interval of duration T

Distancing enforcement cannot last for long time. But there is no reason why it should be restricted to start at a given date — typically “right now”.

Based on the previous results, we study now how to **minimize the total number of infected through application of lockdown of maximal intensity α during maximal time duration T, but without prescribing the onset of this measure**

Optimal immunity control (1/2)

$$\textbf{Find} \quad \sup_{T_0 \geqslant 0} \sup_{u \in \mathcal{U}_{\alpha, T_0, T_0+T}} S_\infty(u) := S_\infty^{**} \qquad (\mathcal{P}'_{\alpha,T})$$

For any $T_0 \geqslant 0$, define $u'_{T_0} \in \mathcal{U}_{\alpha, T_0, T_0+T}$ by

$$u'_{T_0} = \mathbb{1}_{[0, T_0]} + \alpha \mathbb{1}_{[T_0, T_0+T]} + \mathbb{1}_{[T_0+T, +\infty)}, \qquad (7)$$

and (S^{T_0}, I^{T_0}) the solution of (3) with $u = u'_{T_0}$.

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Theorem 8 (Optimal control is unique and bang-bang)

- **Problem $(\mathcal{P}'_{\alpha,T})$ admits a unique solution u^{**}.**
- **There exists a unique $T_0^{**} \geqslant 0$ such that $u^{**} = u'_{T_0^{**}}$, so that $(\mathcal{P}'_{\alpha,T})$ is equivalent to the 1D optimization problem**

$$\sup_{T_0 \geqslant 0} S_\infty(u'_{T_0}) \qquad (\widetilde{\mathcal{P}}'_{\alpha,T})$$

Optimal immunity control (2/2)

For $T_0 \geqslant 0$, and (S^{T_0}, I^{T_0}) solution to (3) with $u = u'_{T_0}$, let

$$\psi'(T_0) := \frac{I^{T_0}(T_0 + T)}{I^{T_0}(T_0)} + (1-\alpha)\gamma \int_{T_0}^{T_0+T} \frac{I^{T_0}(T_0+T)}{I^{T_0}(t)} dt - 1 \quad (8)$$

Theorem 9 (Characterization of T_0^{**})

Let $u'_{T_0^{**}}$ be the optimal control, then

- **if $\psi'(0) \leqslant 0$, then $T_0^{**} = 0$;**
- **if $\psi'(0) > 0$, then T_0^{**} is the unique $T_0 \geqslant 0$ s.t. $\psi'(T_0) = 0$.**

Moreover, if $T_0^{**} > 0$, then $S^{T_0^{**}}(T_0^{**}) \geqslant S_{\text{herd}}$, with equality iff $\alpha = 0$.

Hints of proof

1. **Semigroup property:** $\forall u \in \mathcal{U}_{\alpha, T_0, T_0+T}$, $S_\infty(u) = S_\infty(v)$ with:
   - $v \in \mathcal{U}_{\alpha,0,T}$, $v(\cdot) := u(\cdot + T_0)$
   - initial condition $(S(u)(T_0), I(u)(T_0))$

   This allows to use $(\mathcal{P}_{\alpha,T})$, and thus $(\widetilde{\mathcal{P}}_{\alpha,T})$, to solve $(\mathcal{P}'_{\alpha,T})$.
2. **Optimality conditions:** using this reduction yields
   $$S_\infty^{**} = \sup\{S_\infty(u_{T_0,T_0+\tau}) : T_0 \geqslant 0, \tau \in [0, T]\}$$
   where $u_{T_0,T_0+\tau} \equiv \alpha$ on $[T_0, T_0+\tau]$, 1 otherwise.
3. **2D$\to$ 1D:** one is led to $(\widetilde{\mathcal{P}}'_{\alpha,T})$ by showing that
   $$\sup\{S_\infty(u_{T_0,T_0+\tau}) : T_0 \geqslant 0, \tau \in [0, T]\} = \sup\{S_\infty(u'_{T_0}) : T_0 \geqslant 0\}$$
4. **Optimality conditions in terms of T_0**: $S_\infty(u'_{T_0})$ reaches optimum in $(0, +\infty)$ iff $\psi'(T_0) = 0$ because
   $$\frac{\partial S_\infty(u'_{T_0})}{\partial T_0} \propto \psi'(T_0)$$
5. ψ' **is decreasing,** thus admits at most one zero: **uniqueness**.

Properties of the value function

Theorem 10

$S_\infty^{**} = \sup\{S_\infty(u) : u \in \mathcal{U}_{\alpha, T_0, T_0+T}, T_0 \geqslant 0\}$ **is increasing wrt** $T > 0$ **and decreasing wrt** $\alpha \in [0, 1)$**.**

Theorem 11 (How close from herd immunity can we stop?)

- **If** $\boxed{0 \leqslant \alpha \leqslant \overline{\alpha}}$ **then** $\lim_{T \to +\infty} S_\infty^{**} = S_{\mathrm{herd}}$
- **If** $\boxed{\overline{\alpha} < \alpha < 1}$ **then** $\lim_{T \to +\infty} S_\infty^{**} = S_\infty(\alpha \mathbf{1}) < S_{\mathrm{herd}}$

Numerical illustrations – 2 (1/5)

T	T_0^{**}	S_∞^{**}	$S_\infty^{**}/S_{\mathrm{herd}}$	$\max_{t\geqslant 0} I(t)$
No lockdown	—	0.0668	0.194	0.288
30 days	74.3 days (May 21st)	0.255	0.739	0.288
60 days	74.3 days (May 21st)	0.323	0.937	0.288
90 days	74.3 days (May 21st)	0.340	0.985	0.288

Table: Lockdown intensity $\alpha = 0$, starting dates computed from March 8th.

T	T_0^{**}	S_∞^{**}	$S_\infty^{**}/S_{\mathrm{herd}}$	$\max_{t\geqslant 0} I(t)$
No lockdown	—	0.0668	0.194	0.288
30 days	72.1 days (May 19th)	0.222	0.644	0.282
60 days	71.5 days (May 18th)	0.302	0.875	0.278
90 days	71.3 days (May 18th)	0.331	0.959	0.277

Table: Lockdown intensity $\alpha = \alpha_{\mathrm{lock}} \approx 0.231$.

Numerical illustrations – 2 (2/5)

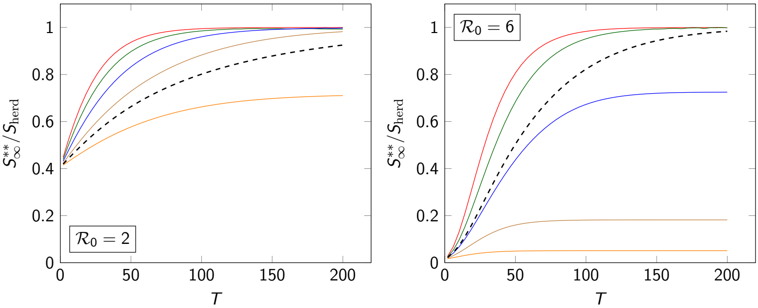


Figure: Graph of $S_\infty^{**}/S_{\text{herd}}$ for Problem $(\mathcal{P}'_{\alpha,T})$ as a function of T, for $\alpha \in \{0.0$ (—), 0.2 (—), 0.4 (—), 0.6 (—), 0.8 (—), $\overline{\alpha}$ (- -)$\}$ and $\mathcal{R}_0 \in \{2, 6\}$.

Numerical illustrations – 2 (3/5)

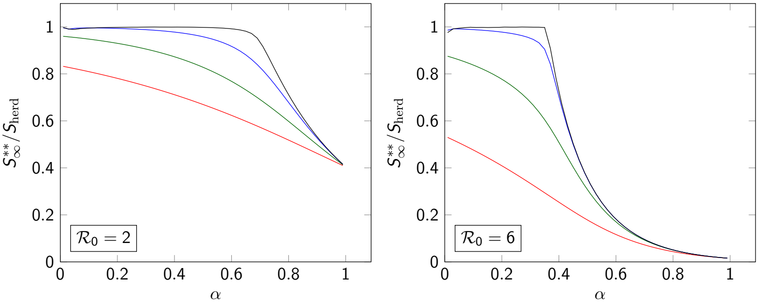


Figure: Graph of $S_\infty^{**}/S_{\text{herd}}$ for Problem $(\mathcal{P}'_{\alpha,T})$ as a function of α, for $T \in \{30$ (−), 60 (−), 120 (−), 240 (−)$\}$ and $\mathcal{R}_0 \in \{2, 6\}$.

Numerical illustrations – 2 (4/5)

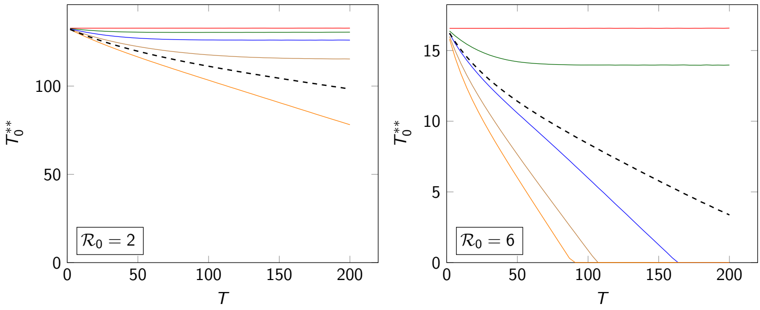
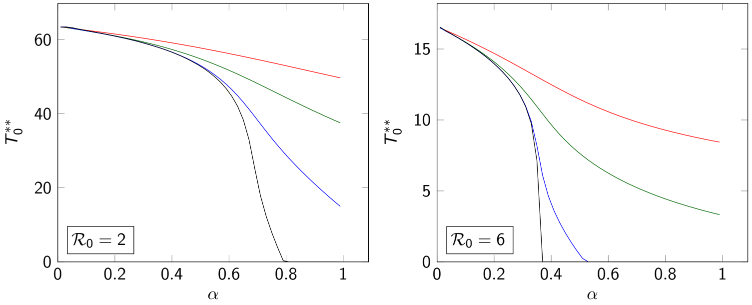


Figure: Graph of T_0^{**} for Problem $(\mathcal{P}'_{\alpha,T})$ as a function of T, for $\alpha \in \{0.0$ (—), 0.2 (—), 0.4 (—), 0.6 (—), 0.8 (—), $\overline{\alpha}$ (- -)$\}$ and $\mathcal{R}_0 \in \{2, 6\}$.

Numerical illustrations – 2 (5/5)



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Conclusion – "Don't try this at home!"

- **Best policy achieved through lockdown of possibly time-varying, but limited, intensity on time interval of restricted duration** has been studied. It is **unique**, and reached by **enforcing the strictest distancing during the whole time interval**.

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- But the strategy consisting in **reaching herd immunity without considering other factors appears unsustainable** (peak of infected cases almost equal to 30% of the population !)

Thank you for your attention !

For more details (and **references!**):

- Bliman, P.-A., Duprez, M., Privat, Y. & Vauchelet, N. (2021) "Optimal immunity control by social distancing for the SIR epidemic model", *Journal of Optimization Theory and Applications* v. 189, 408–436
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