Master 2 Internship proposal

Title: Non-linear Perron-Frobenius approach to Matrix Multiplication Games

Place and supervision: This internship will be supervised by Marianne Akian, e-mail: marianne.akian@inria.fr, Web page: `http://www.cmap.polytechnique.fr/~akian/`
and co-supervised by Stéphane Gaubert (DR INRIA, Prof. associé à l'École Polytechnique), e-mail: stephane.gaubert@inria.fr, Web page: `http://www.cmap.polytechnique.fr/~gaubert/`.
It will be done inside the Inria Project team "Tropical", common to Inria Saclay Île-de-France and CMAP, École polytechnique, CNRS, IPP. Address: CMAP, Ecole Polytechnique, Route de Saclay, 91128 Palaiseau Cedex. The intern will receive a gratification from INRIA.
Schedule: at least 4 months, starting at Spring 2023.

Subject: In [9], matrix multiplication games and entropy games were introduced, in which two players are optimizing the growth rate, or mean spectral radius of a product of nonnegative matrices: one player wishes to maximize it, while the other player wishes to minimize it. Entropy games consist in the particular case in which the set of possible matrices is rectangular, meaning that each row of the matrix is chosen independently of the other rows. Then, one player, called "Tribune" wishes to maximize the freedom of a half-player, "People", measured by a topological entropy, whereas the oponent, called "Despot", wishes to minimize it.

In [6], we studied entropy games with a dynamic programming (a.k.a. operator) approach. Entropy games are reformulated, after a log-exp transformation, as zero-sum games on a finite state space, with a cardinality n equal to the dimension of matrices, an infinite number of actions (action spaces are essentially simplices), and an instantaneous payment given by a Kullback-Leibler divergence (relative entropy). This allowed us to obtain the existence of optimal policies, using techniques from o-minimal or "tame" geometry, and to obtain a strong duality theorem, using a non-linear eigenvalue characterization of the value of the game (Collatz-Wielandt property). In the one-player case, in which the player wishes to maximize a growth rate, this leads in particular to a convex programming formulation, which entails a polynomial complexity result. This also allowed us to apply algorithms arising from dynamic programming, like value or policy iteration.

In this internship, we will study general matrix multiplication games (without the assumption of rectangularity of the set of matrices). In the one player case, this problem boils down to computing the joint spectral radius of a collection of nonnegative matrices, a problem which has been extensively studied, see e.g. [1], and with numerous applications, including population dynamics (see e.g. [10] for an application to growth fragmentation processes arising in prion proliferation modelling). Ergodic eigenproblems of a somehow similar nature also arise in mean-field control and optimal pricing, see [13].

Our goal is to develop an operator approach, using a non-linear spectral eigenproblem in infinite dimension, in which the state space is now a subset of $\mathbb{R}^n$, representing an invariant set of population profiles. We expect the value of the matrix multiplication game to be given by a non-linear eigenvalue. We plan to address the issue of the existence and regularity of the eigenvector, by combining techniques of non-linear Perron-Frobenius theory [15, 8, 3], and of discrete weak-KAM theory. Indeed, the matrix multiplication game can be reformulated as a two player discrete-time weak-KAM problem. Weak-KAM theory has been developed essentially in the one player case, mainly in the continuous time setting [11], but also in the discrete time setting [16]. The extension to the two player case is of current interest. As a first step, we plan to establish a general non-linear strong duality result, characterizing the value of the game.

In the perspective of a PhD thesis, the next steps include sufficient conditions of ergodicity extending [7] to the infinite dimensional setting, and also the algorithmic issues. One may in particular be interested to develop efficient algorithms inspired by the tropical or max-plus numerical method introduced in [14], and specialized in [12] to joint spectral radius problems. One may also develop a combined tropical-SDDP method for ergodic (mean-payoff) and matrix multiplication games, inspired by the tropical SDDP method developped for finite horizon one-player deterministic and stochastic games in [2, 4, 5].

References

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