Curves and surfaces

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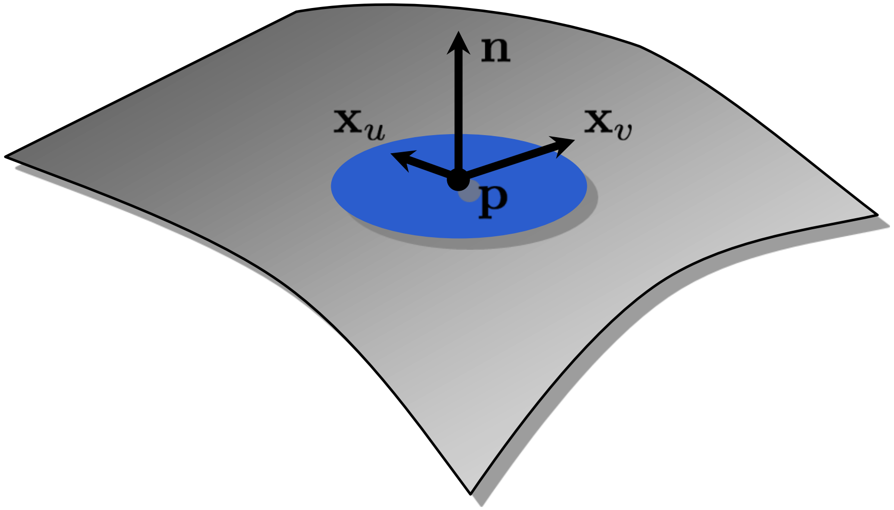
Parametric surfaces

- Continuous surface

$$\mathbf{x}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$$



- Normal vector

$$\mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}$$

- Assume *regular* parameterization

$$\mathbf{x}_u \times \mathbf{x}_v \neq \mathbf{0}$$

Parametric surfaces

Angles on surface

- Curve $[u(t), v(t)]$ in *uv*-plane defines curve on the surface x(u,v)

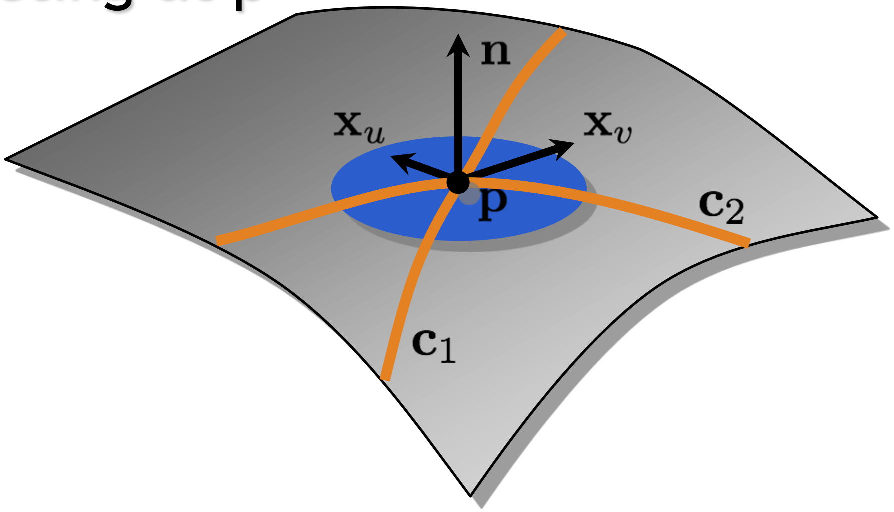
$$\mathbf{c}(t) = \mathbf{x}(u(t), v(t))$$

- Two curves c_1 and c_2 intersecting at p
  - Angle of intersection?
  - Two tangents t_1 and t_2

$$\mathbf{t}_i = \alpha_i \mathbf{x}_u + \beta_i \mathbf{x}_v$$

  - Compute inner product

$$\mathbf{t}_1^T \mathbf{t}_2 = \cos\theta \, \|\mathbf{t}_1\| \, \|\mathbf{t}_2\|$$



Parametric surfaces

Angles on surface

- Curve $[u(t), v(t)]$ in *uv*-plane defines curve on the surface x(u,v)

$$\mathbf{c}(t) = \mathbf{x}(u(t), v(t))$$

- Two curves c_1 and c_2 intersecting at p

$$\begin{aligned}
\mathbf{t}_1^T \mathbf{t}_2 &= (\alpha_1 \mathbf{x}_u + \beta_1 \mathbf{x}_v)^T (\alpha_2 \mathbf{x}_u + \beta_2 \mathbf{x}_v) \\
&= \alpha_1 \alpha_2 \mathbf{x}_u^T \mathbf{x}_u + (\alpha_1 \beta_2 + \alpha_2 \beta_1) \mathbf{x}_u^T \mathbf{x}_v + \beta_1 \beta_2 \mathbf{x}_v^T \mathbf{x}_v \\
&= (\alpha_1, \beta_1) \begin{pmatrix} \mathbf{x}_u^T \mathbf{x}_u & \mathbf{x}_u^T \mathbf{x}_v \\ \mathbf{x}_u^T \mathbf{x}_v & \mathbf{x}_v^T \mathbf{x}_v \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}
\end{aligned}$$

Parametric surfaces

First fundamental form

- First fundamental form
- Defines inner product on tangent space

$$\mathbf{I} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} := \begin{pmatrix} \mathbf{x}_u^T \mathbf{x}_u & \mathbf{x}_u^T \mathbf{x}_v \\ \mathbf{x}_u^T \mathbf{x}_v & \mathbf{x}_v^T \mathbf{x}_v \end{pmatrix}$$

$$\left\langle \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}, \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} \right\rangle := \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}^T \mathbf{I} \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}$$

Parametric surfaces

First fundamental form

$$\mathbf{t}_1^T \mathbf{t}_2 = \langle (\alpha_1, \beta_1) \ , \ (\alpha_1, \beta_1) \rangle$$

$$\begin{aligned} \mathrm{d}s^2 &= \langle (\mathrm{d}u, \mathrm{d}v), (\mathrm{d}u, \mathrm{d}v) \rangle \\ &= E\mathrm{d}u^2 + 2F\mathrm{d}u\mathrm{d}v + G\mathrm{d}v^2 \end{aligned}$$

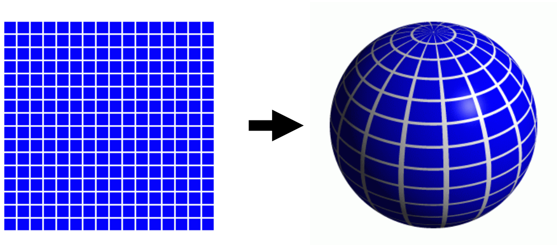
$$\begin{aligned} \mathrm{d}A &= \|\mathbf{x}_u \times \mathbf{x}_v\| \ \mathrm{d}u\,\mathrm{d}v \\ &= \sqrt{\mathbf{x}_u^T\mathbf{x}_u \cdot \mathbf{x}_v^T\mathbf{x}_v - (\mathbf{x}_u^T\mathbf{x}_v)^2}\,\mathrm{d}u\,\mathrm{d}v \\ &= \sqrt{EG - F^2}\mathrm{d}u\,\mathrm{d}v \end{aligned}$$

Parametric surfaces: exercices

- Sphere centrée en (0,0,0) de rayon 1
  - Parametrage de la surface
  - Longueur à l'équateur
  - Aire de la sphere

Parametric surfaces: exercices

Sphere example



- Spherical parameterization

$$\mathbf{x}(u,v) = \begin{pmatrix} \cos u \sin v \\ \sin u \sin v \\ \cos v \end{pmatrix}, \quad (u,v) \in [0,2\pi) \times [0,\pi)$$

- Tangent vectors

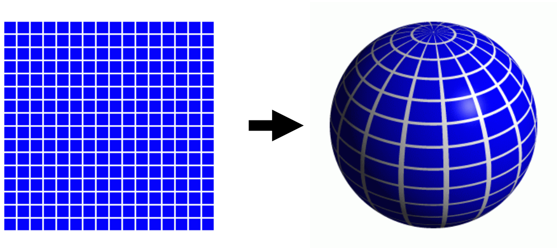
$$\mathbf{x}_u(u,v) = \begin{pmatrix} -\sin u \sin v \\ \cos u \sin v \\ 0 \end{pmatrix} \quad \mathbf{x}_v(u,v) = \begin{pmatrix} \cos u \cos v \\ \sin u \cos v \\ -\sin v \end{pmatrix}$$

- First fundamental form

$$\mathbf{I} = \begin{pmatrix} \sin^2 v & 0 \\ 0 & 1 \end{pmatrix}$$

Parametric surfaces: exercices

Sphere example



- Length of equator $\mathbf{x}(t, \pi / 2)$

$$\int_0^{2\pi} 1\,\mathrm{d}s \;=\; \int_0^{2\pi} \sqrt{E\left(u_t\right)^2 + 2F u_t v_t + G\left(v_t\right)^2}\,\mathrm{d}t$$

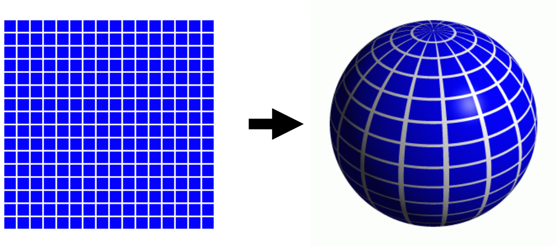
$$= \int_0^{2\pi} \sin v\,\mathrm{d}t$$

$$= 2\pi \sin v \;=\; 2\pi$$

Parametric surfaces: exercices

Sphere example

- Area of a sphere



$$\int_0^{\pi}\int_0^{2\pi} 1\,\mathrm{d}A = \int_0^{\pi}\int_0^{2\pi}\sqrt{EG-F^2}\,\mathrm{d}u\,\mathrm{d}v$$

$$= \int_0^{\pi}\int_0^{2\pi}\sin v\,\mathrm{d}u\,\mathrm{d}v$$

$$= 4\pi$$

Parametric surfaces: exercices

- cylindre centrée en (0,0,0), de normal (0,0,1), de rayon 1 et de hauteur 2h
  - Parametrage de la surface
  - Longueur à l’équateur
  - Aire du cylindre

Parametric surfaces: exercices

- tore centrée en (0,0,0), de normal (0,0,1), de grand rayon 10 et de petit rayon 1
  - Parametrage de la surface
  - Aire du tore

Parametric surfaces: exercices

Exercice 2 (Une surface réglée). On considère deux arcs d'ellipse fournis pour $0 \leq u \leq \pi$ par les paramétrisations suivantes :

$$\alpha(u) := (0, a_1 \cos u, b_1 \sin u), \quad \beta(u) := (1, a_2 \cos u, b_2 \sin u),$$

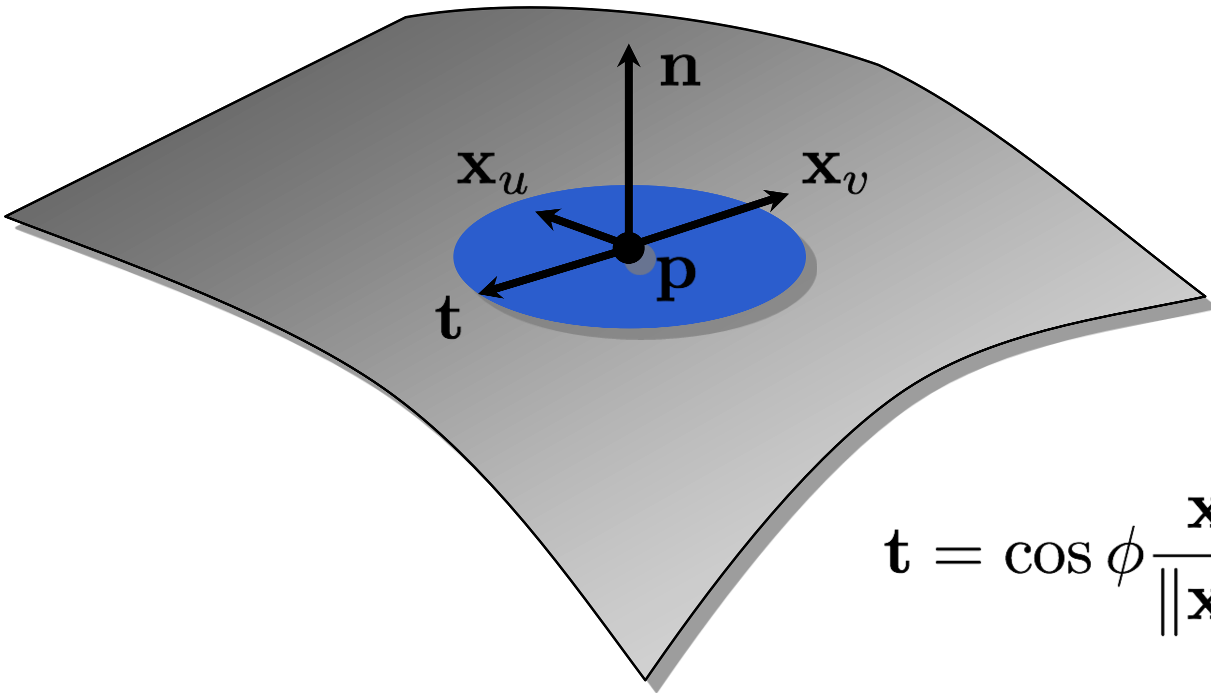
$a_1, b_1, a_2, b_2 > 0$. Pour chaque valeur de u on trace la droite qui joint le point $\alpha(u)$ au point $\beta(u)$, et on considère la surface S obtenue comme la réunion de toutes ces droites.

1. Tracer qualitativement la surface.
2. Donner un paramétrage de cette surface en rajoutant un paramètre v.
3. Préciser la nature de la section de cette surface par des plans d'équation $x = \mathit{constante}$.
4. Calculer en tout point le vecteur normal unitaire.
5. On considère la courbe obtenue en coupant la surface par le plan d'équation $x = \frac{1}{2}$. Calculer le repère de Frenet de cette courbe.
6. Quel est le lien entre ce repère de Frenet et le vecteur normal à la surface calculé plus haut ?
7. Pour quelles valeurs des paramètres du modèle la surface est-elle une portion de cylindre, ou de cône ?

Parametric surfaces

Normal curvature

- Tangent vector $\mathbf{t}$...

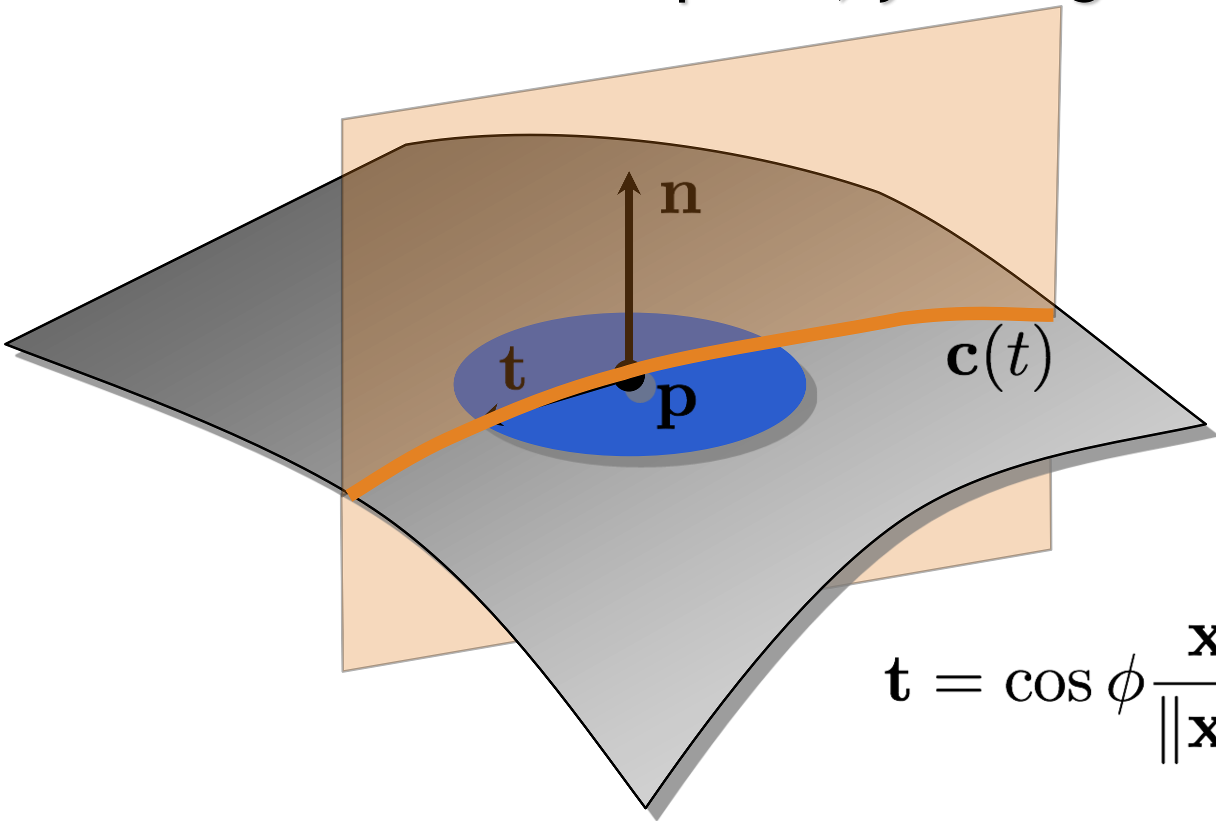


$$\mathbf{t} = \cos\phi \frac{\mathbf{x}_u}{\|\mathbf{x}_u\|} + \sin\phi \frac{\mathbf{x}_v}{\|\mathbf{x}_v\|}$$

Parametric surfaces

Normal curvature

- .. defines intersection plane, yielding curve $\mathbf{c}(t)$



$$\mathbf{t} = \cos\phi \frac{\mathbf{x}_u}{\|\mathbf{x}_u\|} + \sin\phi \frac{\mathbf{x}_v}{\|\mathbf{x}_v\|}$$

Parametric surfaces

Normal curvature

- Normal curvature $\kappa_n(\mathbf{t})$ is defined as curvature of the normal curve c(t) at point p = x(u, v).

- With second fundamental form

$$\mathbf{II} = \begin{pmatrix} e & f \\ f & g \end{pmatrix} := \begin{pmatrix} \mathbf{x}_{uu}^T\mathbf{n} & \mathbf{x}_{uv}^T\mathbf{n} \\ \mathbf{x}_{uv}^T\mathbf{n} & \mathbf{x}_{vv}^T\mathbf{n} \end{pmatrix}$$

- normal curvature can be computed as

$$\kappa_n(\bar{\mathbf{t}}) = \frac{\bar{\mathbf{t}}^T\,\mathbf{II}\,\bar{\mathbf{t}}}{\bar{\mathbf{t}}^T\,\mathbf{I}\,\bar{\mathbf{t}}} = \frac{ea^2 + 2fab + gb^2}{Ea^2 + 2Fab + Gb^2}$$

$$\boxed{\begin{aligned} \mathbf{t} &= a\mathbf{x}_u + b\mathbf{x}_v \\ \bar{\mathbf{t}} &= (a, b) \end{aligned}}$$

Parametric surfaces

Surface curvatures

- *Principal curvatures*
  - Maximum curvature $\kappa_1 = \max_{\phi} \kappa_n(\phi)$
  - Minimum curvature $\kappa_2 = \min_{\phi} \kappa_n(\phi)$
  - Euler theorem: $\kappa_n(\phi) \;=\; \kappa_1 \cos^2 \phi + \kappa_2 \sin^2 \phi$
  - Corresponding *principal directions* e_1, e_2 are orthogonal

Parametric surfaces

Surface curvatures

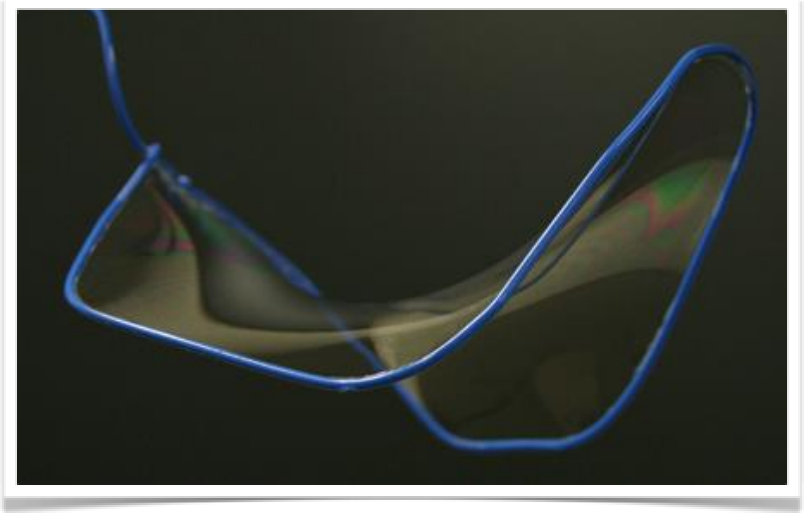
- *Principal curvatures*
  - Maximum curvature $\kappa_1 = \max_{\phi} \kappa_n(\phi)$
  - Minimum curvature $\kappa_2 = \min_{\phi} \kappa_n(\phi)$
  - Euler theorem: $\kappa_n(\phi) = \kappa_1 \cos^2 \phi + \kappa_2 \sin^2 \phi$
  - Corresponding *principal directions* e_1, e_2 are orthogonal

- Special curvatures
  - Mean curvature $H = \frac{\kappa_1 + \kappa_2}{2}$
  - Gaussian curvature $K = \kappa_1 \cdot \kappa_2$

Parametric surfaces

Curvature of surfaces

- Mean curvature $H = \dfrac{\kappa_1 + \kappa_2}{2}$
  - H = 0 everywhere → minimal surface



soap films

Parametric surfaces

Curvature of surfaces

- Gaussian curvature $K = \kappa_1 \cdot \kappa_2$
  - K = 0 everywhere → developable surface

Disney Concert Hall, L.A.
Architects: Gehry Partners

Timber Fabric
IBOIS, EPFL

Parametric surfaces

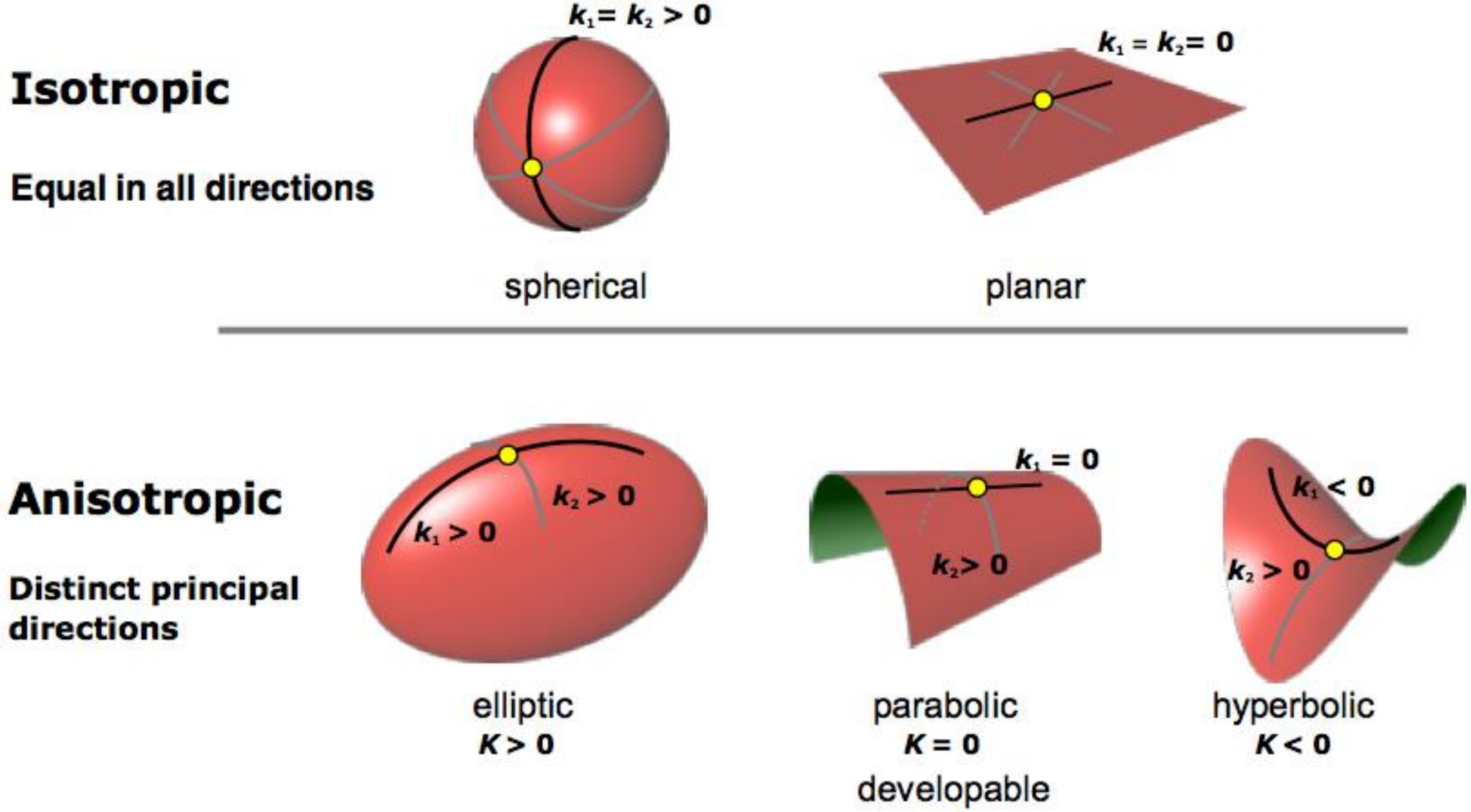
Classification

- A point $\mathbf{x}$ on the surface is called
  - *elliptic*, if $K > 0$
  - *hyperbolic*, if $K < 0$
  - *parabolic*, if $K = 0$
  - *umbilic*, if $\kappa_1 = \kappa_2$

Parametric surfaces

Classification

- A point $\mathbf{x}$ on the surface is called



Parametric surfaces: exercices

Exercice 3 (Aire des surfaces de révolution). On fait tourner autour de l'axe z une courbe γ définie dans le plan (x, z), paramétrée à vitesse unité, et ne touchant pas l'axe z.

1. Écrire le paramétrage de cette surface.
2. Calculer ses coefficients métriques.
3. Pour chaque valeur du paramètre t, on note $\rho(t)$ la distance du point à l'axe vertical. Montrer que la surface totale est donnée par :

$$S = 2\pi \int \rho(u)\, du.$$

Examiner le cas particulier de la sphère et du tore.

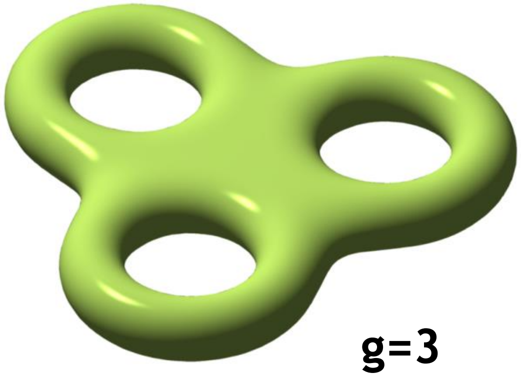
Genus of a surface

- largest number of nonintersecting simple closed curves that can be drawn on the surface without separating it

- It is equal to the number of holes in a surface

g=0

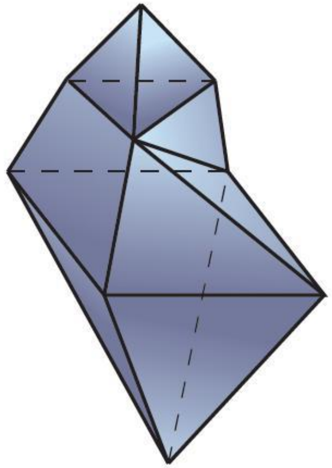
g=1



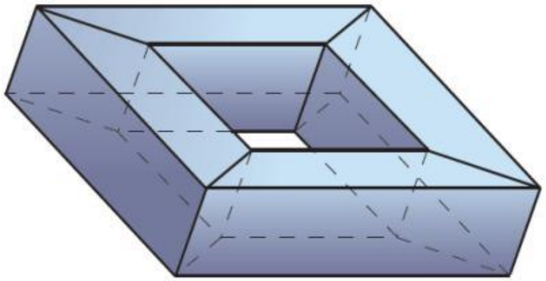
g=3

Euler characteristic

$$\chi = V - E + F$$



V – E + F = 2

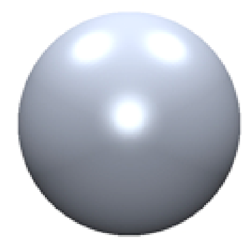


V – E + F = 0

Gauss-Bonnet theorem

- For **any** closed manifold surface with Euler characteristic χ = 2-2g

$$\int K = 2\pi\chi$$

$\int K($  $) = \int K($  $) = \int K($  $) = 4\pi$

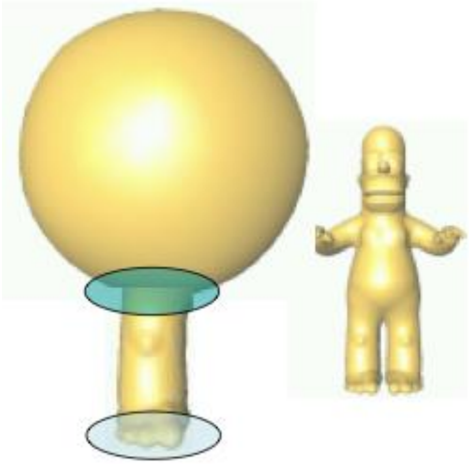
Gauss-Bonnet theorem

- Sphere

$$\kappa_1 = \kappa_2 = 1/r$$

$$K = \kappa_1 \kappa_2 = 1/r^2$$

$$\int K = 4\pi r^2 \cdot \frac{1}{r^2} = 4\pi$$



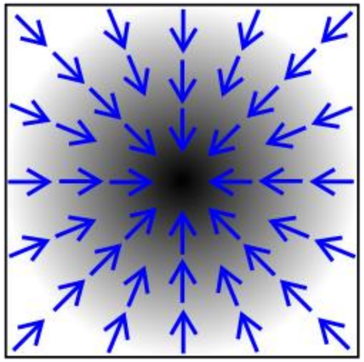
- when sphere is deformed new positive and negative curvature cancel out!

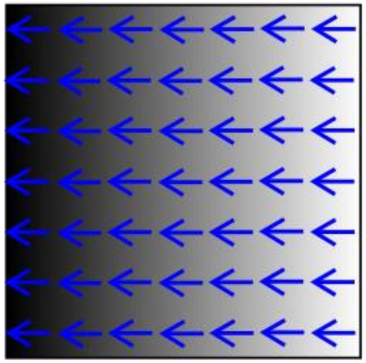
Differential operators

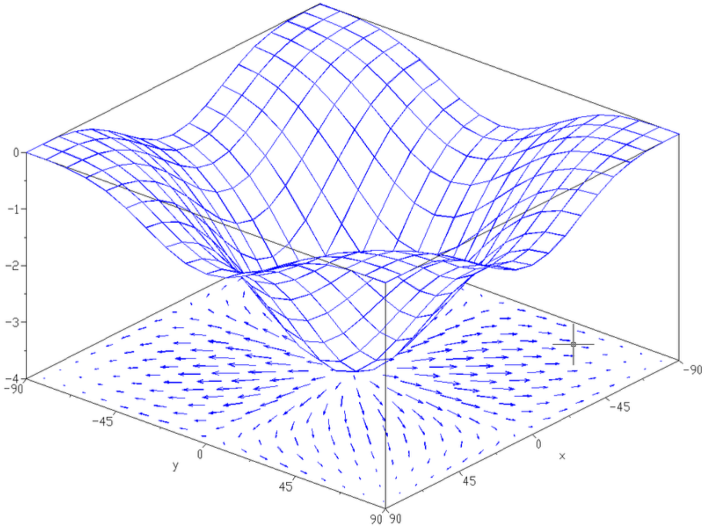
- Gradient

$$\nabla f := \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right)$$

  - points in the direction of steepest ascent







Differential operators

- Divergence

$$\mathrm{div} F = \nabla \cdot F := \frac{\partial F_1}{\partial x_1} + \ldots + \frac{\partial F_n}{\partial x_n}$$

Laplace operator

Laplace operator

gradient operator

2nd partial derivatives

$$\Delta f = \mathrm{div} \nabla f = \sum_i \frac{\partial^2 f}{\partial x_i^2}$$

function in Euclidean space

divergence operator

Cartesian coordinates

Laplace-Beltramy operator

- Extension of Laplace to functions on manifolds

Laplace-Beltrami

gradient operator

$$\Delta_{\mathcal{S}} f = \mathrm{div}_{\mathcal{S}} \, \nabla_{\mathcal{S}} f$$

function on manifold $\mathcal{S}$

divergence operator

Laplace-Beltramy operator

- Extension of Laplace to functions on manifolds

Laplace-Beltrami

gradient operator

mean curvature

$$\Delta_{\mathcal{S}} \mathbf{x} = \mathrm{div}_{\mathcal{S}} \, \nabla_{\mathcal{S}} \mathbf{x} = -2H\mathbf{n}$$

coordinate function

divergence operator

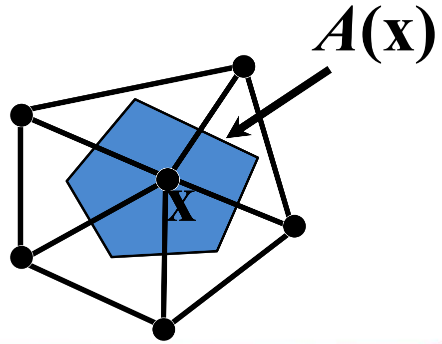
surface normal

Contents

- Differential Geometry
- Discrete Differential Geometry
- Mesh Quality Measures

Discrete curvatures

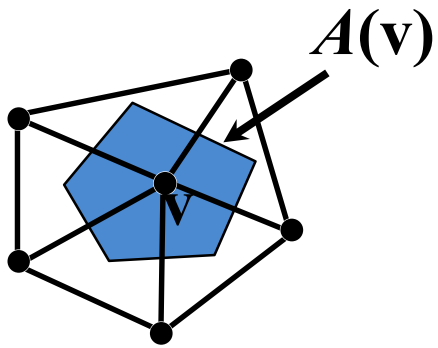
- How to discretize curvatures on a mesh?
- Approximate differential properties at point $\mathbf{x}$ as average over local neighborhood $A(\mathbf{x})$
  - $\mathbf{x}$ is a mesh vertex
  - $A(\mathbf{x})$ within one-ring neighborhood



Discrete curvatures

- How to discretize curvatures on a mesh?
- Approximate differential properties at point $\mathbf{x}$ as average over local neighborhood $A(\mathbf{x})$

$$K(v) \approx \frac{1}{A(v)} \int_{A(v)} K(\mathbf{x}) \, \mathrm{d}A$$

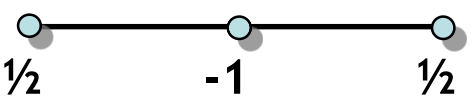


Discrete curvatures

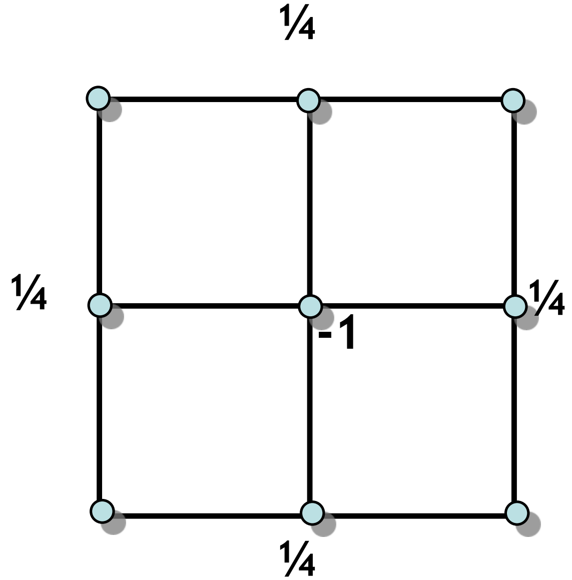
- Which curvatures to discretize?
  - Discretize Laplace-Beltrami operator
  - Laplace-Beltrami gives us mean curvature H
  - Discretize Gaussian curvature K
  - From H and K we can compute κ_1 and κ_2

Laplace operator on mesh?

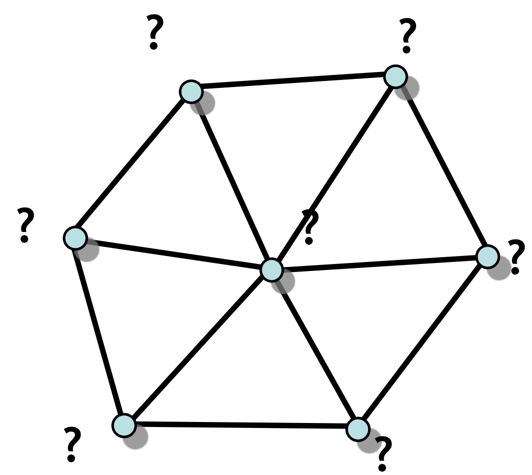
- Extend finite differences to meshes?
  - What weights per vertex / edge?



1D grid



2D grid

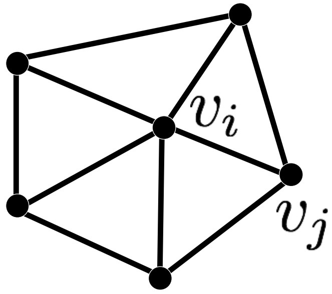


2D/3D mesh

Uniform Laplace

- Uniform discretization

$$\Delta_{\text{uni}} f(v_i) \; := \; \frac{1}{|\mathcal{N}_1(v_i)|} \sum_{v_j \in \mathcal{N}_1(v_i)} (f(v_j) - f(v_i))$$

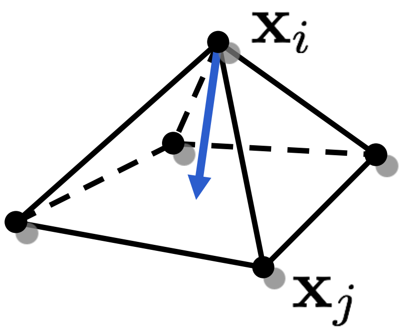


Uniform Laplace

- Uniform discretization

$$\Delta_{\text{uni}} \mathbf{x}_i \; := \; \frac{1}{|\mathcal{N}_1(v_i)|} \sum_{v_j \in \mathcal{N}_1(v_i)} (\mathbf{x}_j - \mathbf{x}_i) \; \approx \; -2H\mathbf{n}$$

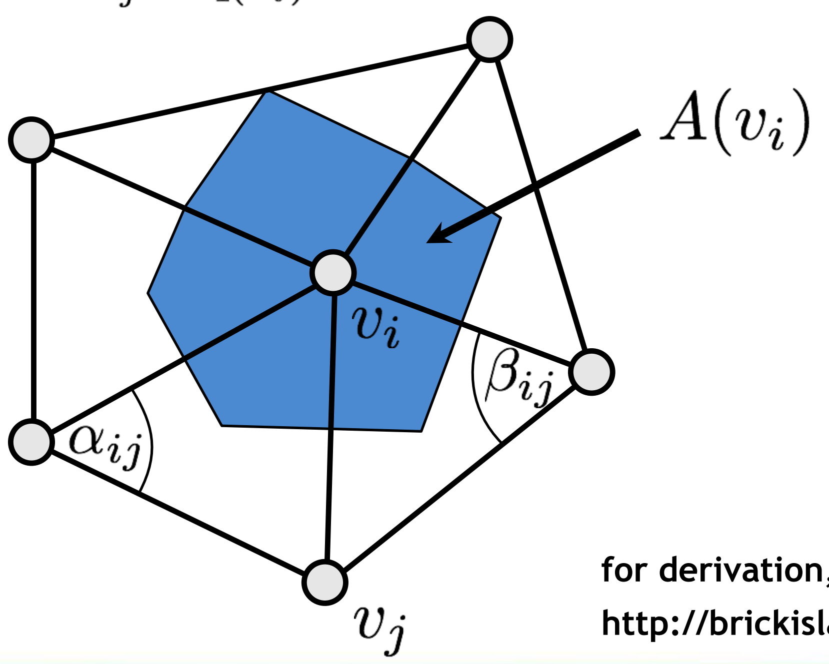
- Properties
  - depends only on connectivity
  - simple and efficient
  - bad approximation for irregular triangulations



Discrete Laplace-Beltrami

- Cotangent discretization

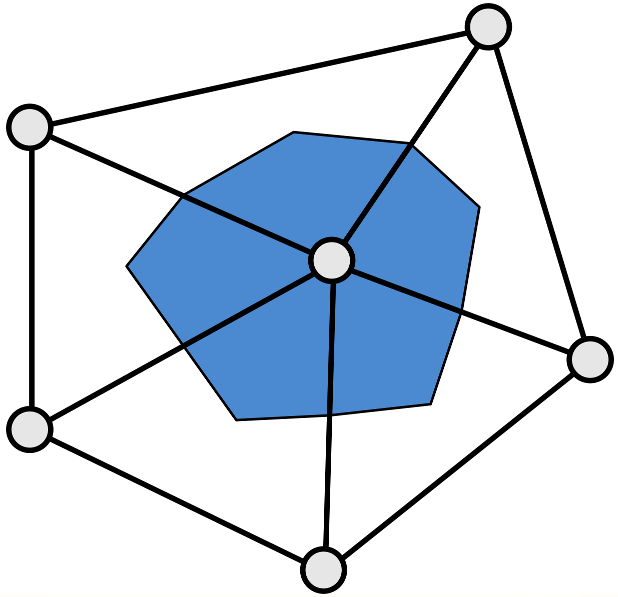
$$\Delta_{\mathcal{S}} f(v_i) \;:=\; \frac{1}{2A(v_i)} \sum_{v_j \in \mathcal{N}_1(v_i)} \left(\cot\alpha_{ij} + \cot\beta_{ij}\right)\left(f(v_j) - f(v_i)\right)$$



for derivation, check out: http://brickisland.net/cs177/

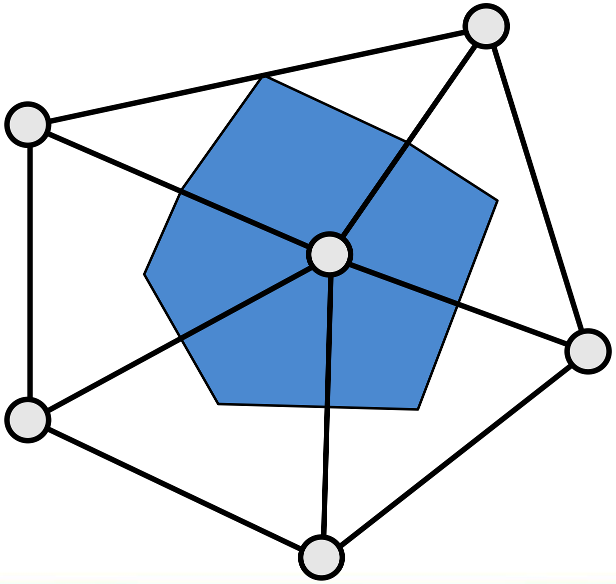
Barycentric cells

- Connect edge midpoints and triangle barycenters
  - Simple to compute
  - Area is 1/3 of triangle areas
  - Slightly wrong for obtuse triangles



Mixed cells

- Connect edge midpoints and
  - Circumcenters for non-obtuse triangles
  - Midpoint of opposite edge for obtuse triangles
  - Better approximation, more complex to compute...



Discrete Laplace-Beltrami

- Cotangent discretization

$$\Delta_{\mathcal{S}} f(v) := \frac{1}{2A(v)} \sum_{v_i \in \mathcal{N}_1(v)} (\cot \alpha_i + \cot \beta_i) \left(f(v_i) - f(v) \right)$$

- Problems
  - weights can become negative (when?)
  - depends on triangulation
- Still the most widely used discretization

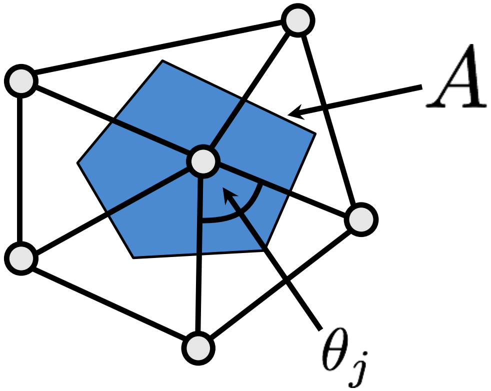
Discrete curvatures

- Mean curvature (absolute value)

$$H = \frac{1}{2}\|\Delta_{\mathcal{S}}\mathbf{x}\|$$

- Gaussian curvature

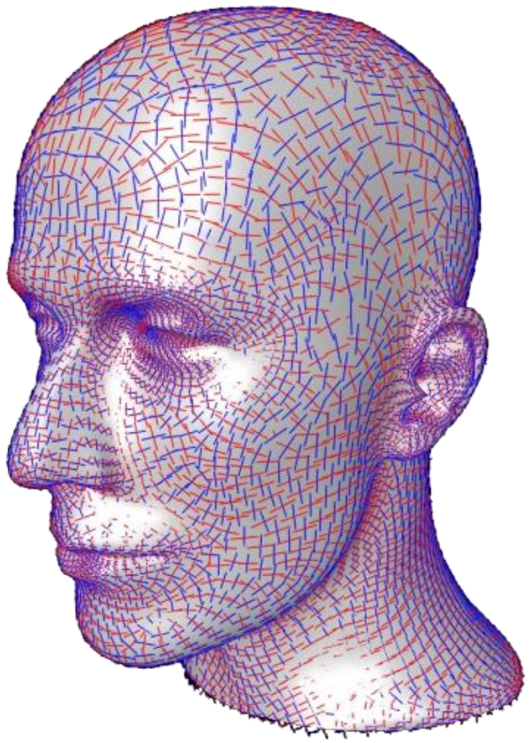
$$K = (2\pi - \sum_j \theta_j)/A$$



- Principal curvatures

$$\kappa_1 = H + \sqrt{H^2 - K} \qquad \kappa_2 = H - \sqrt{H^2 - K}$$

- P. Alliez: *Estimating Curvature Tensors on Triangle Meshes* (source code)
  - http://www-sop.inria.fr/geometrica/team/Pierre.Alliez/demos/curvature/
- CGAL package



principal directions

Contents

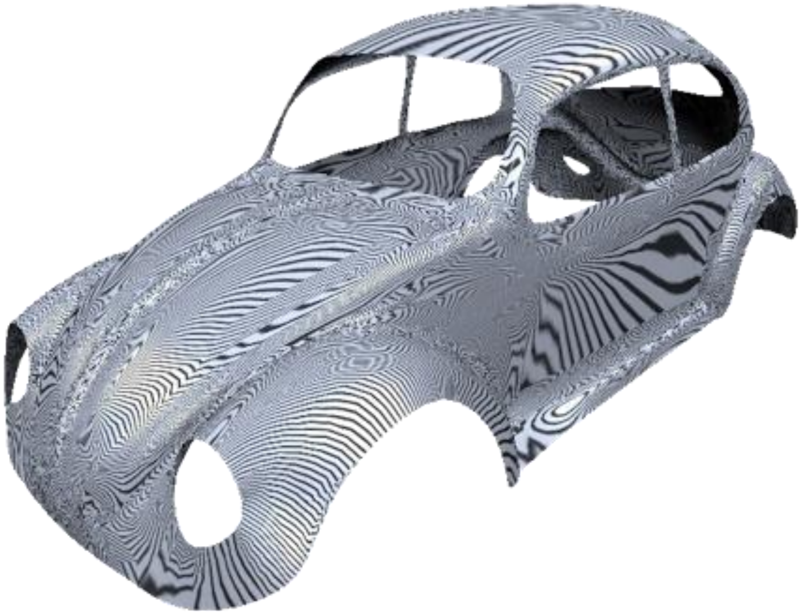
- Differential Geometry
- Discrete Differential Geometry
- Mesh Quality Measures

Mesh quality

- Visual inspection of “sensitive” attributes
  - Specular shading

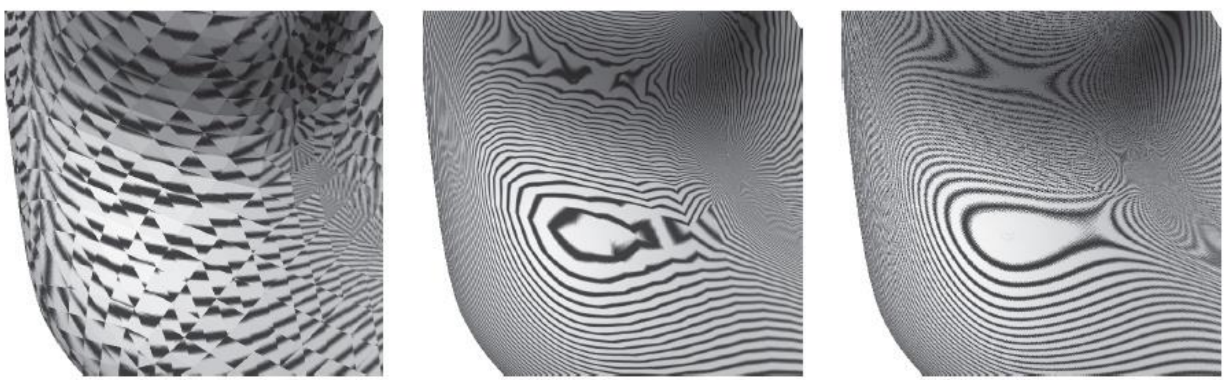
Mesh quality

- Visual inspection of “sensitive” attributes
  - Specular shading
  - Reflection lines



Mesh quality

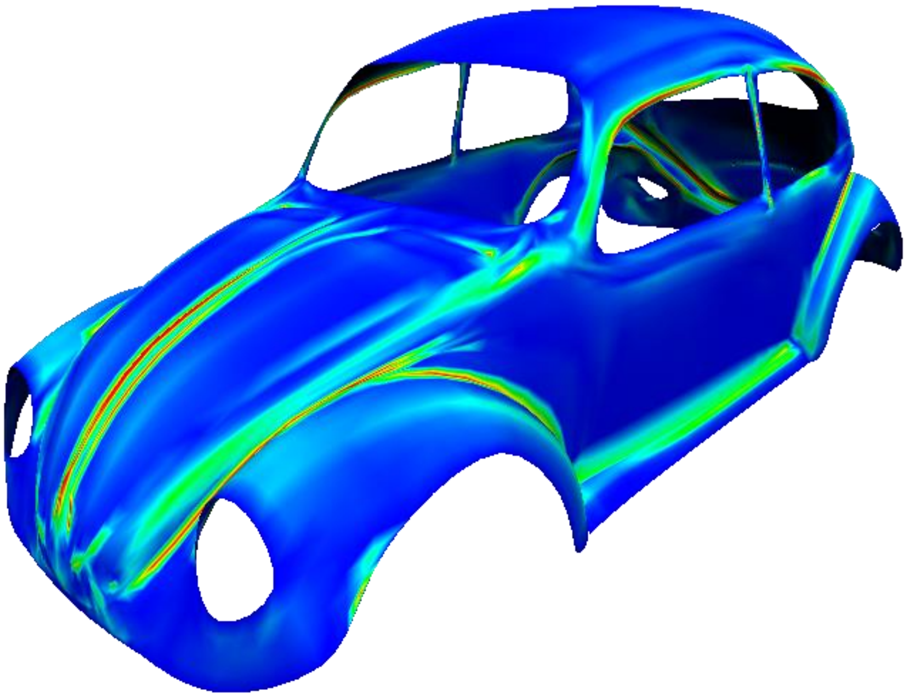
- Visual inspection of “sensitive” attributes
  - Specular shading
  - Reflection lines
    - differentiability one order lower than surface
    - can be efficiently computed using graphics hardware



C^0 C^1 C^2

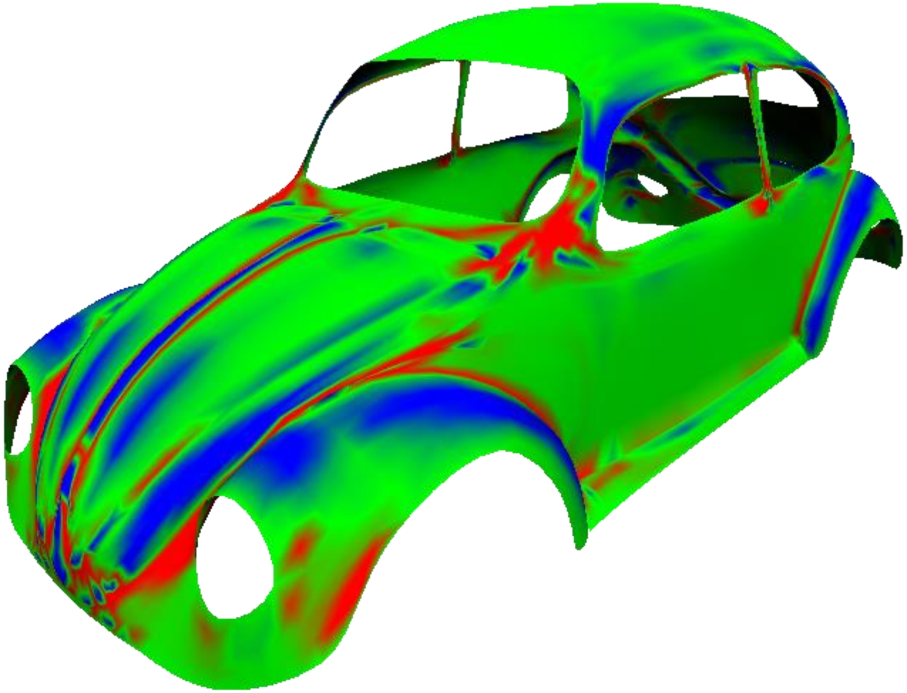
Mesh quality

- Visual inspection of “sensitive” attributes
  - Specular shading
  - Reflection lines
  - Curvature
    - Mean curvature



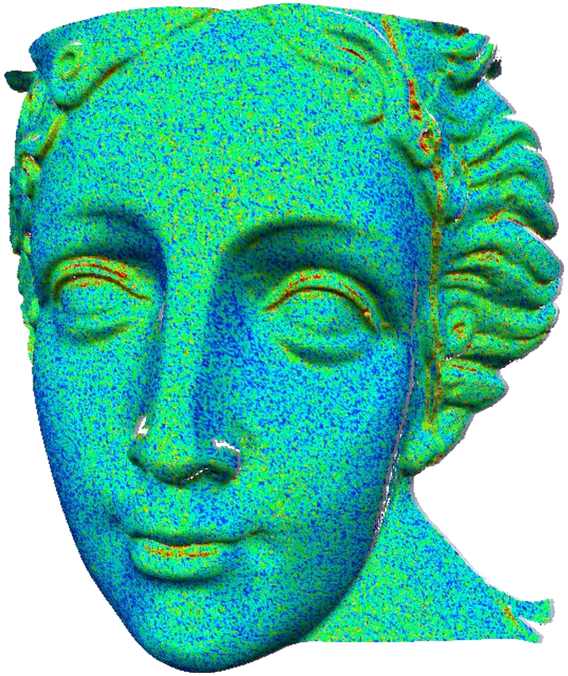
Mesh quality

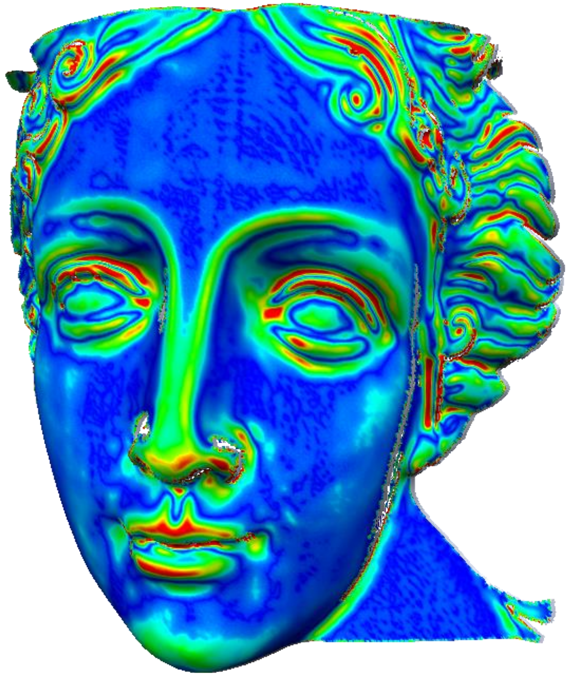
- Visual inspection of "sensitive" attributes
  - Specular shading
  - Reflection lines
  - Curvature
    - Mean curvature
    - Gauss curvature



Mesh quality criteria

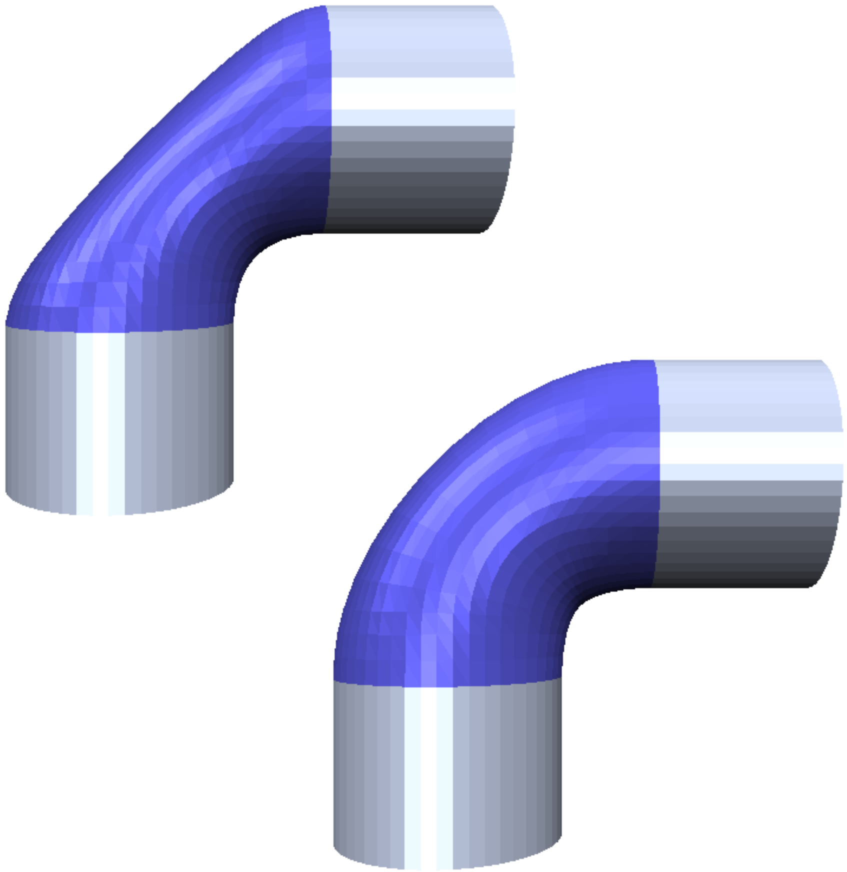
- Smoothness
    - Low geometric noise





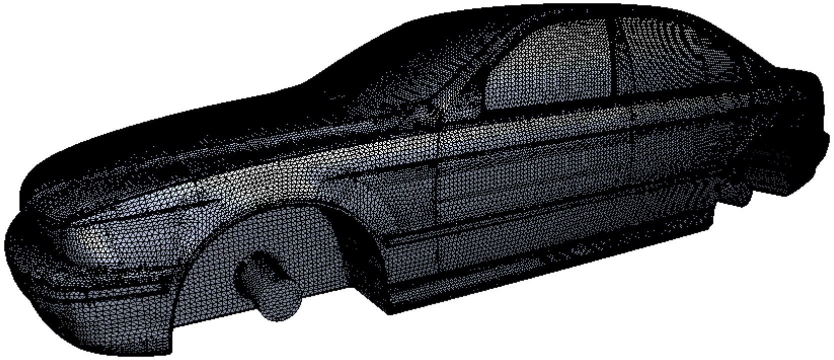
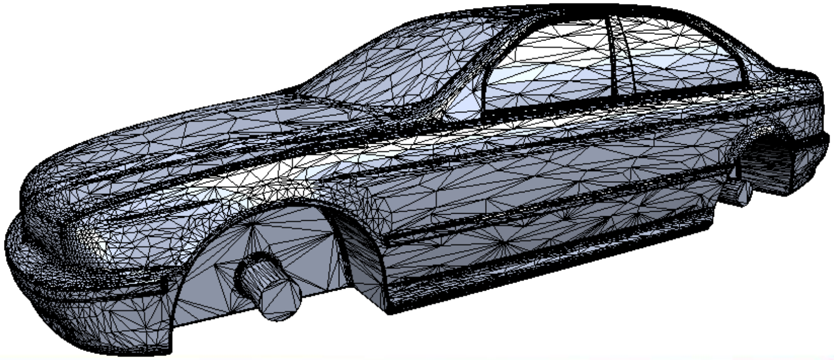
Mesh quality criteria

- Smoothness
  - Low geometric noise
- Fairness
  - Simplest shape



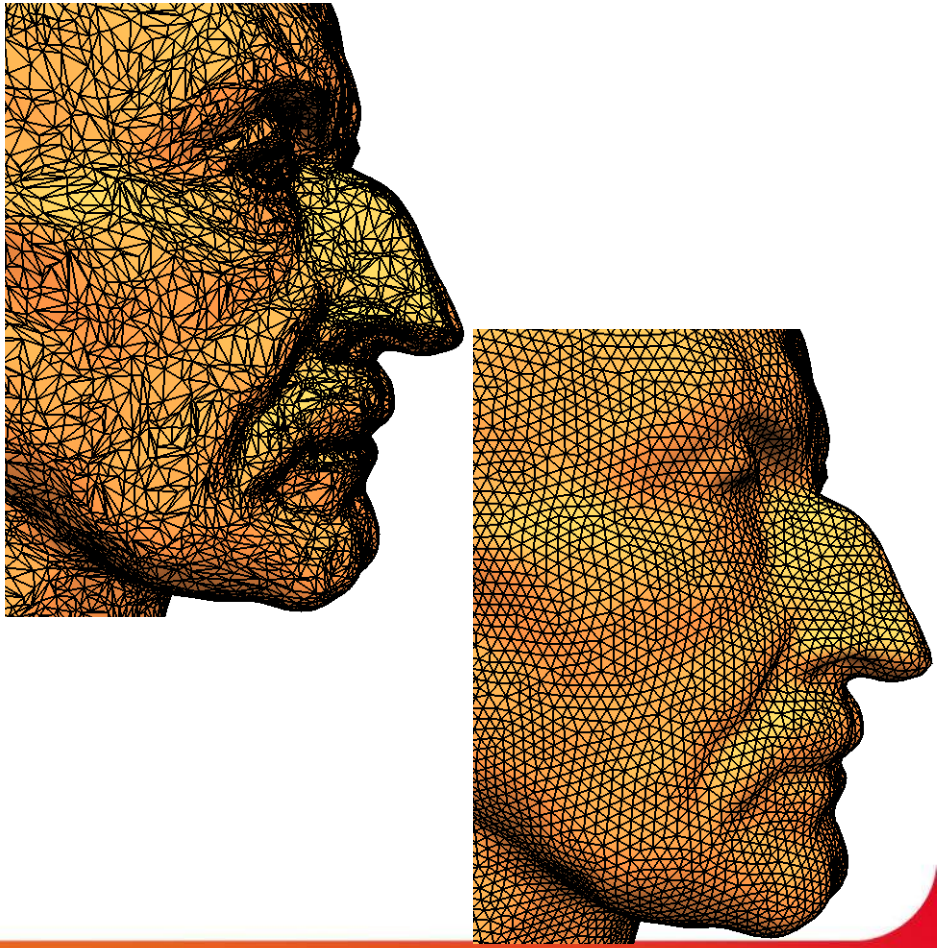
Mesh quality criteria

- Smoothness
  - Low geometric noise
- Fairness
  - Simplest shape
- Adaptive tessellation
  - Low complexity

Mesh quality criteria

- Smoothness
  - Low geometric noise
- Fairness
  - Simplest shape
- Adaptive tessellation
  - Low complexity
- Triangle shape
  - Numerical robustness



Mesh quality criteria

- Smoothness
  - ➞ Smoothing
- Fairness
  - ➞ Fairing
- Adaptive tessellation
  - ➞ Decimation
- Triangle shape
  - ➞ Remeshing