Optimal Mass Transport

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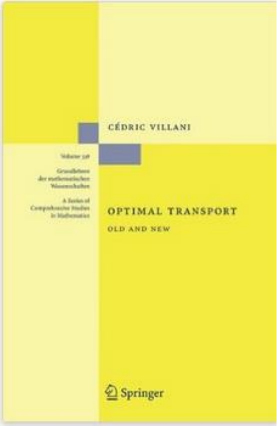
Inria

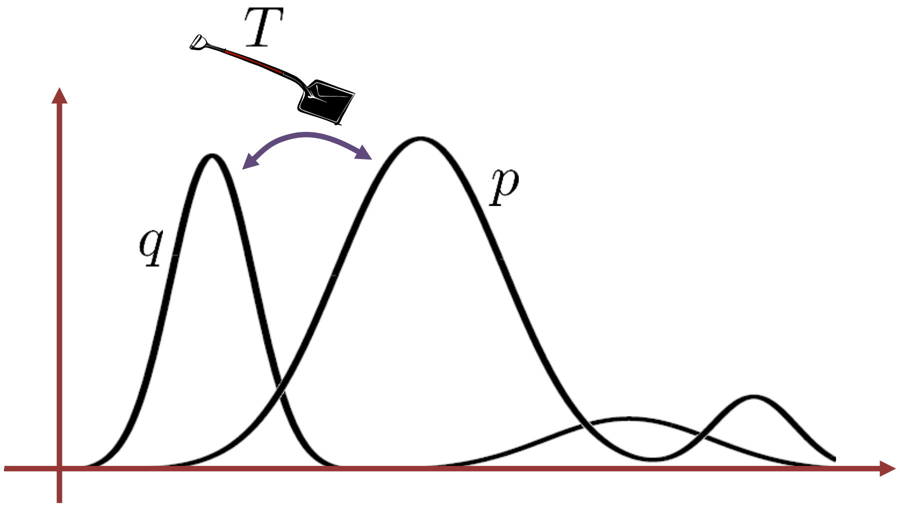
Outline

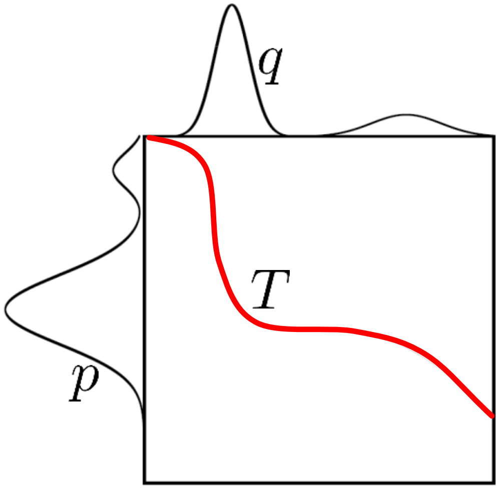
- Optimal mass transport
  - Definition
  - Computational aspects
  - Degrees of freedom
- Applications

Optimal Mass Transport

Optimal Mass Transport



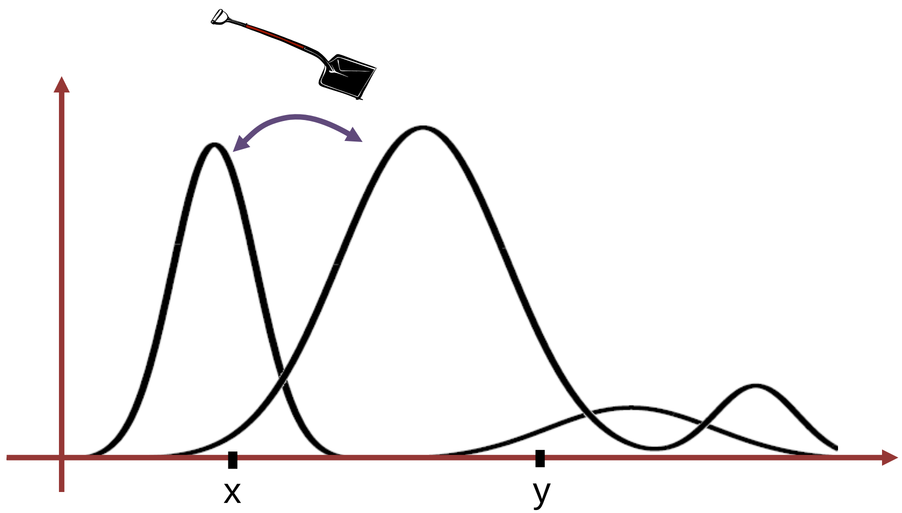




Move mass from one distribution to the other

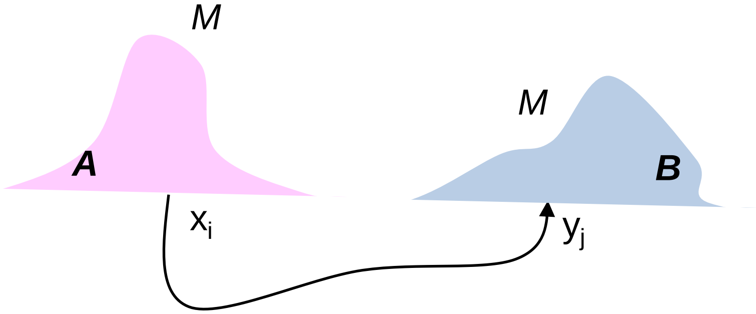
Wasserstein Metric

$$\mathcal{W}_p(\mu,\nu) \equiv \min_{\pi \in \Pi(\mu,\nu)} \left(\iint_{M \times M} d(x,y)^p \, d\pi(x,y) \right)^{1/p}$$



Distance function on probability distributions

Minimization



$$\text{min. } \mathcal{W}_2^2 = \sum_{ij} \pi_{ij} \|x_i - y_j\|^2$$

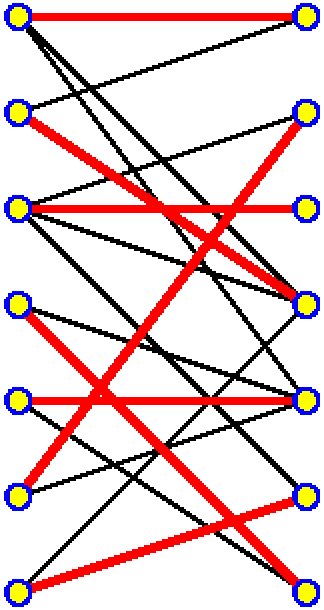
$$\text{s.t.} \quad \begin{cases} \forall \pi_{ij} \geq 0, \\ m_i = \sum_j \pi_{ij}, \\ n_j = \sum_i \pi_{ij}. \end{cases}$$

Mass-preserving constraints

O(n^2) variables

Linear Programming

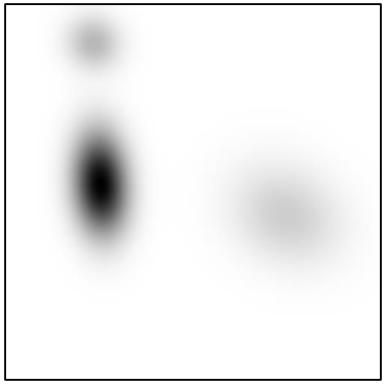
Minimization

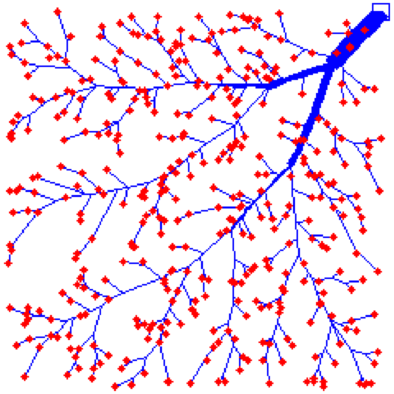


Min weighted complete graph matching
(restricted to binary transport plans)

Degrees of Freedom

- Support of mass
- Transport cost
  - Wasserstein distance
- Transport plan
  - Regularized
  - Relaxed (unbalanced)
  - Ramified
  - ...





Qinglan Xia

Applications

Shape Reconstruction

Approach in 2D

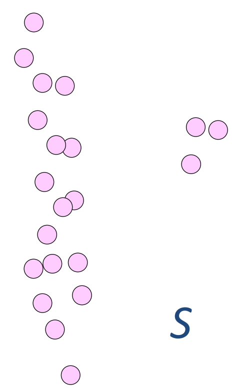
Given a point set S, find a coarse triangulation T such that S is well approximated by uniform measures on the 0- and 1-simplices of T.

How to measure distance D(*S*,*T*)?

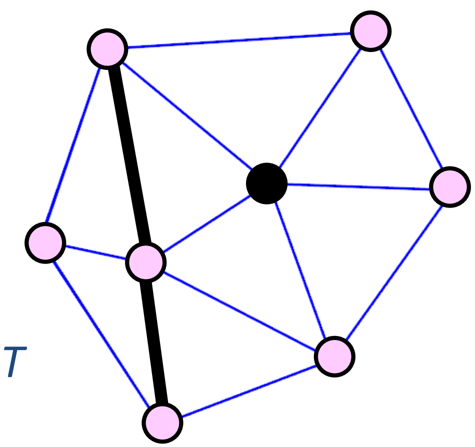
$\Rightarrow$ optimal transport between measures

How to construct *T* that minimizes D(*S*,*T*)?

optimal location problem! $\Rightarrow$ greedy decimation



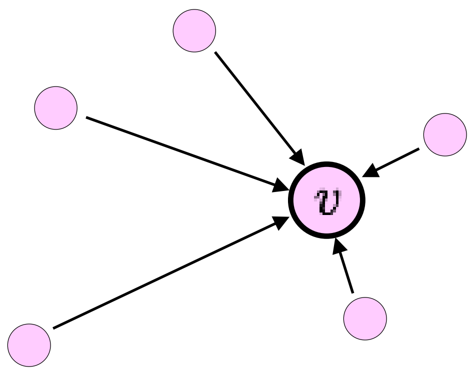
S



T

Mass Transport on a Vertex

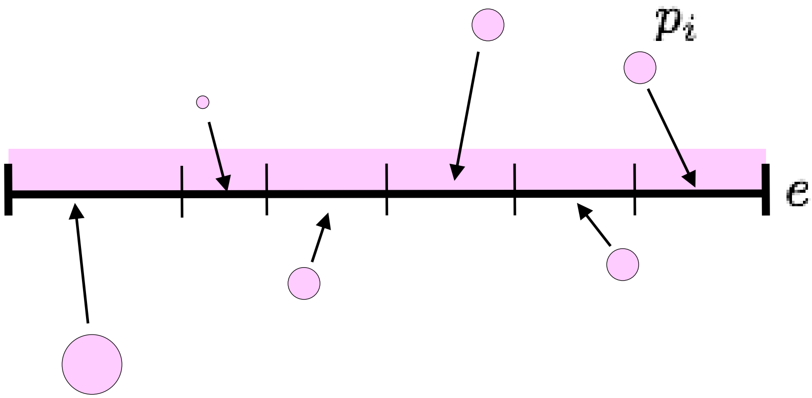
(assume given *binary* transport plan)

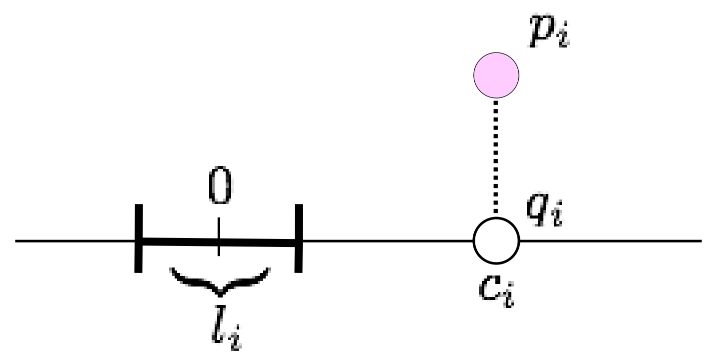


$$W_2(v, S_v) = \sqrt{\sum_{p_i \in S_v} m_i \|p_i - v\|^2}.$$

Mass Transport on an Edge

(assume given *binary* transport plan)



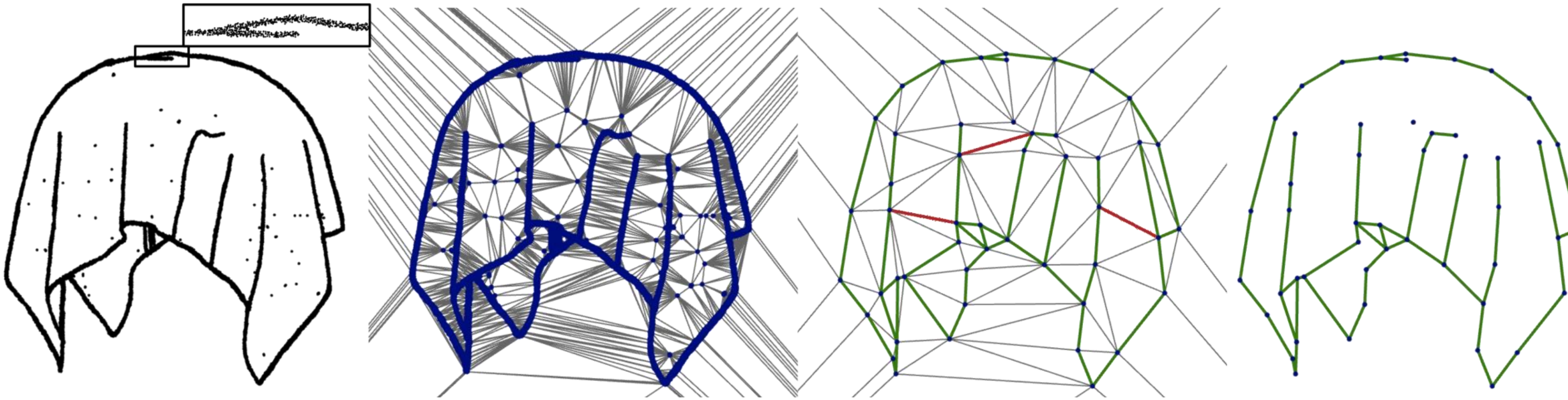


$$N(e,S_e)=\sqrt{\sum_{p_i\in S_e} m_i\|p_i-q_i\|^2}$$

$$T(e,S_e)=\sqrt{\sum_{p_i\in S_e}\frac{M_e}{|e|}\int_{-l_i/2}^{l_i/2}(x-c_i)^2\,dx}=\sqrt{\sum_{p_i\in S_e} m_i\left(\frac{l_i^2}{12}+c_i^2\right)}$$

Overview

An Optimal Transport Approach to Robust Reconstruction and Simplification of 2D Shapes.
De Goes, Cohen-Steiner, P.A., and Desbrun.
Symposium on Geometry Processing, 2011.

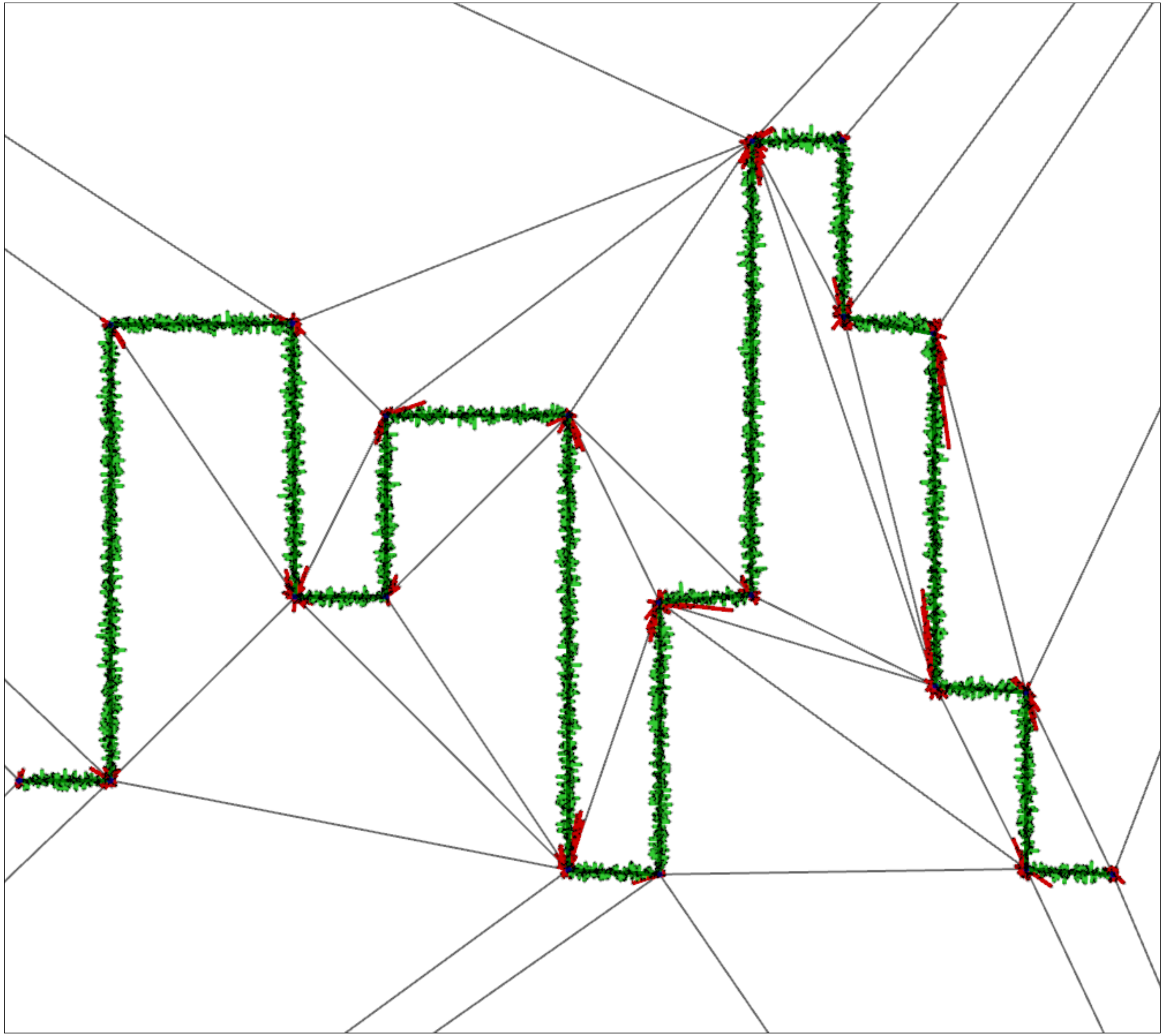


input point set (potentially variable mass)

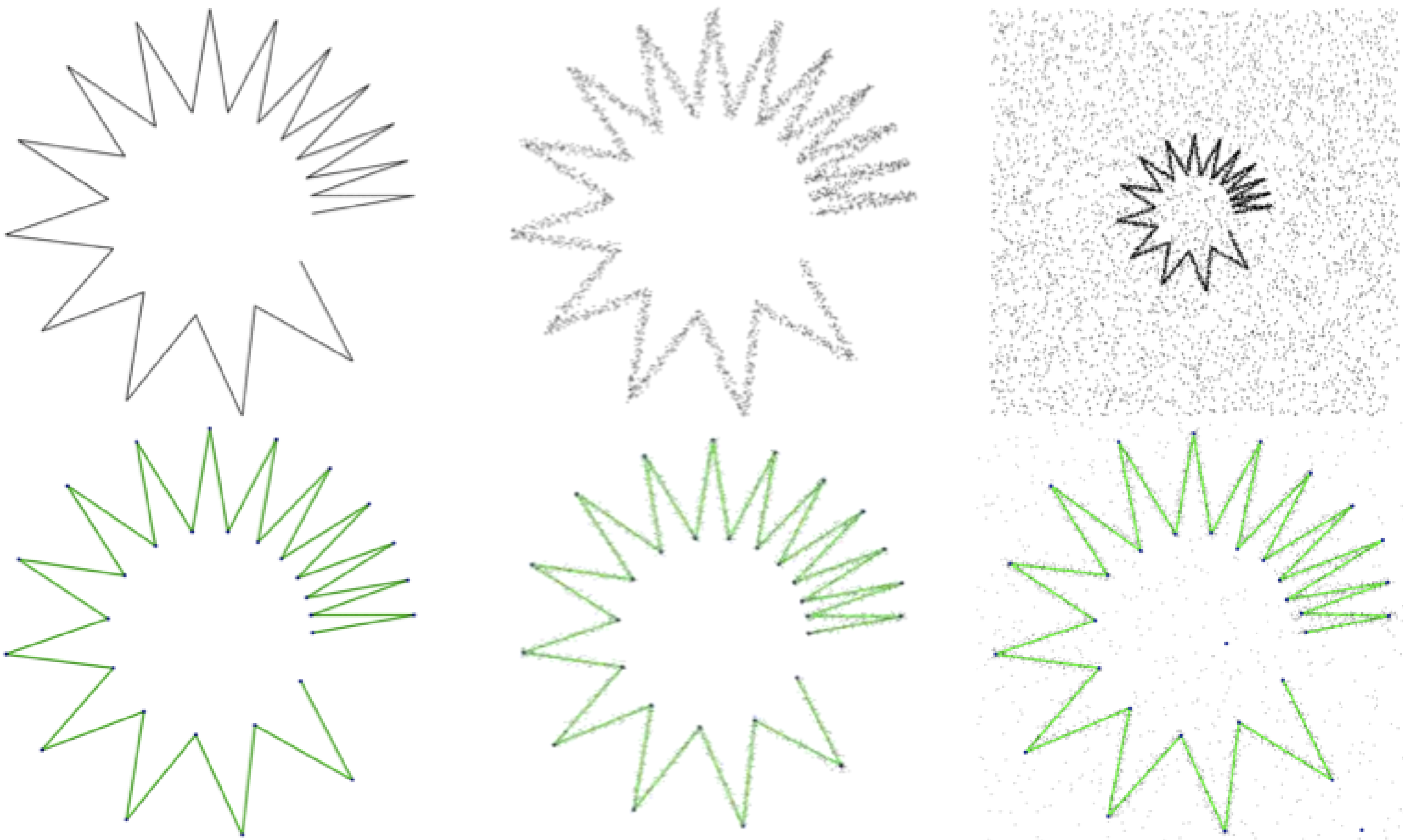
Delaunay triangulation

after decimation

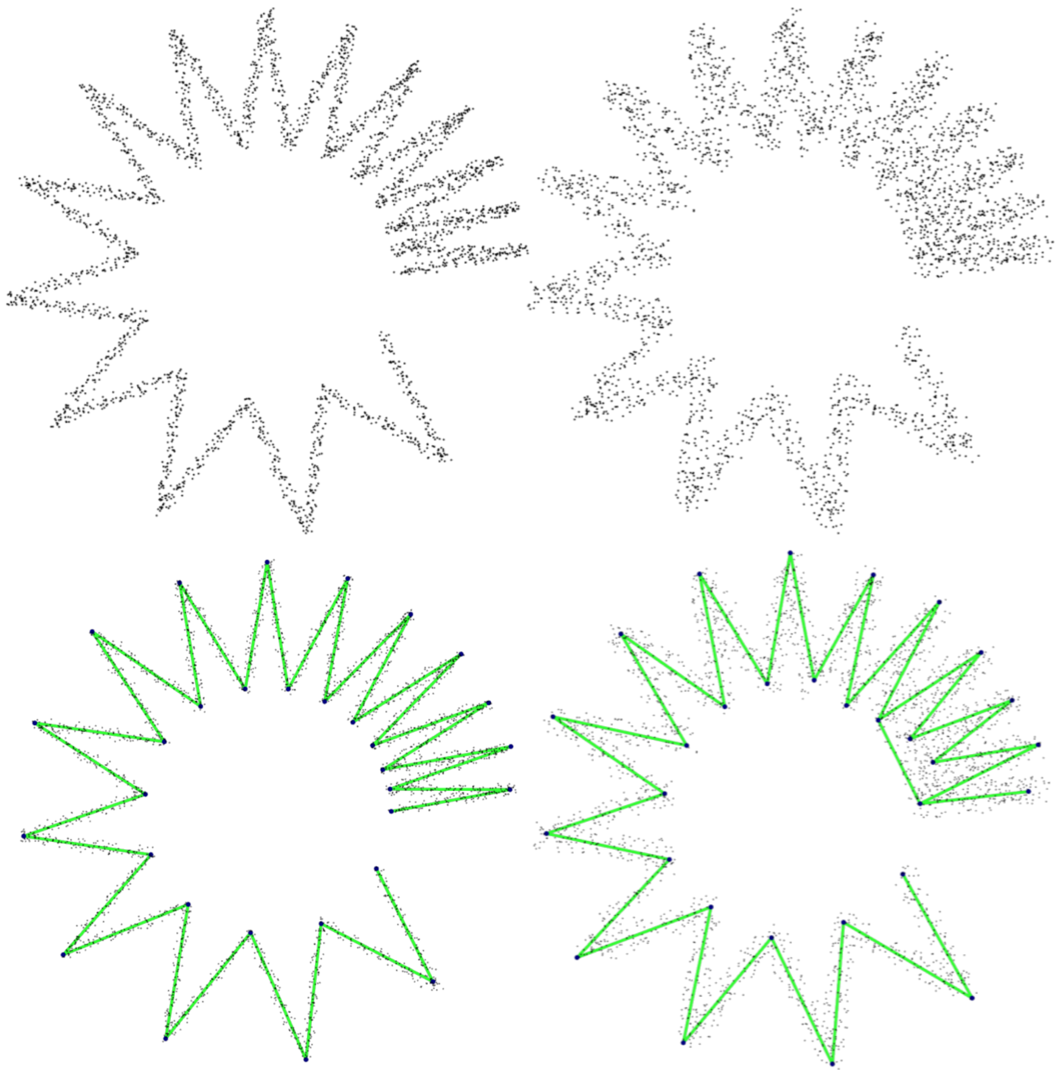
output after edge filtering



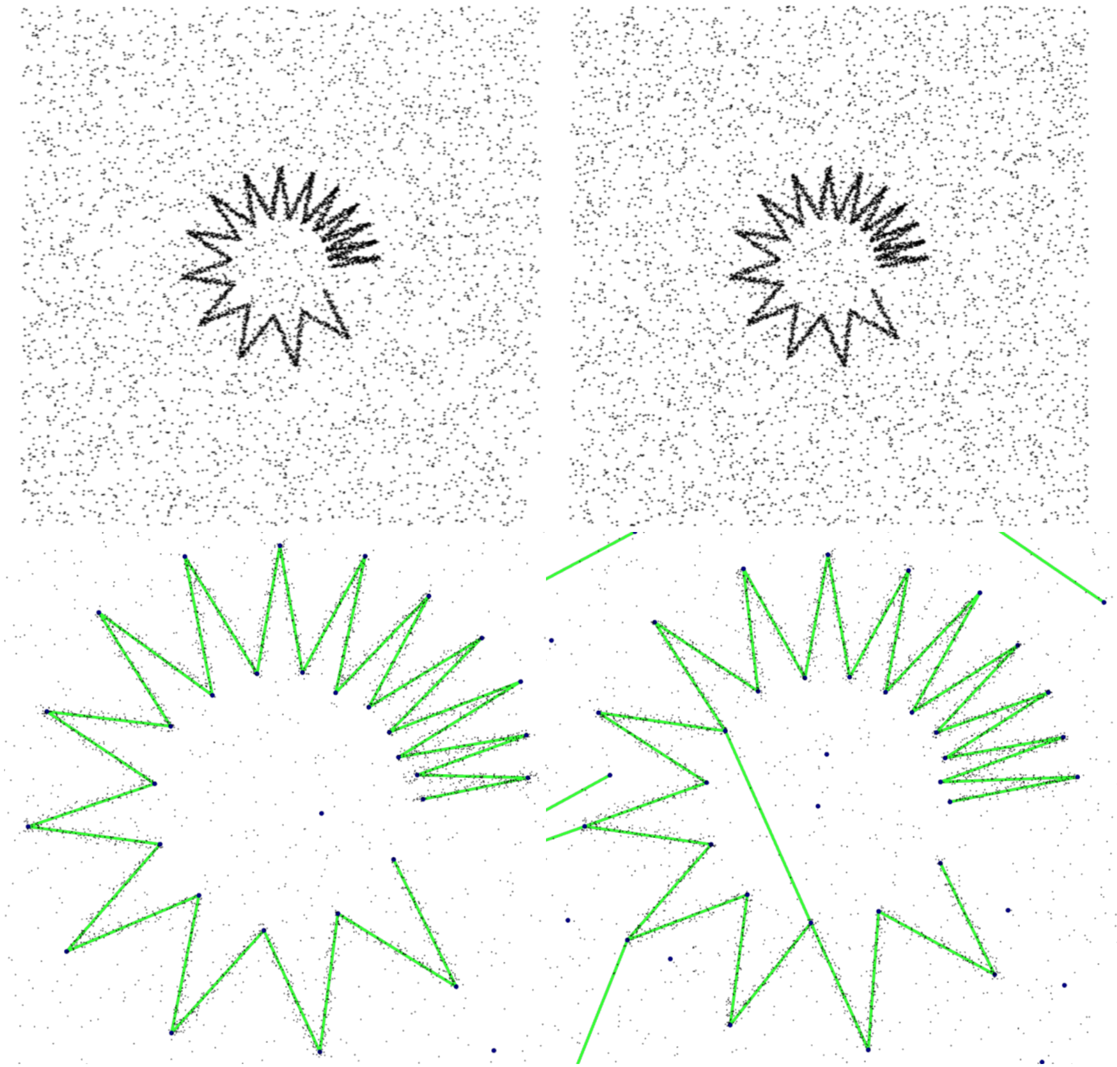
Robustness



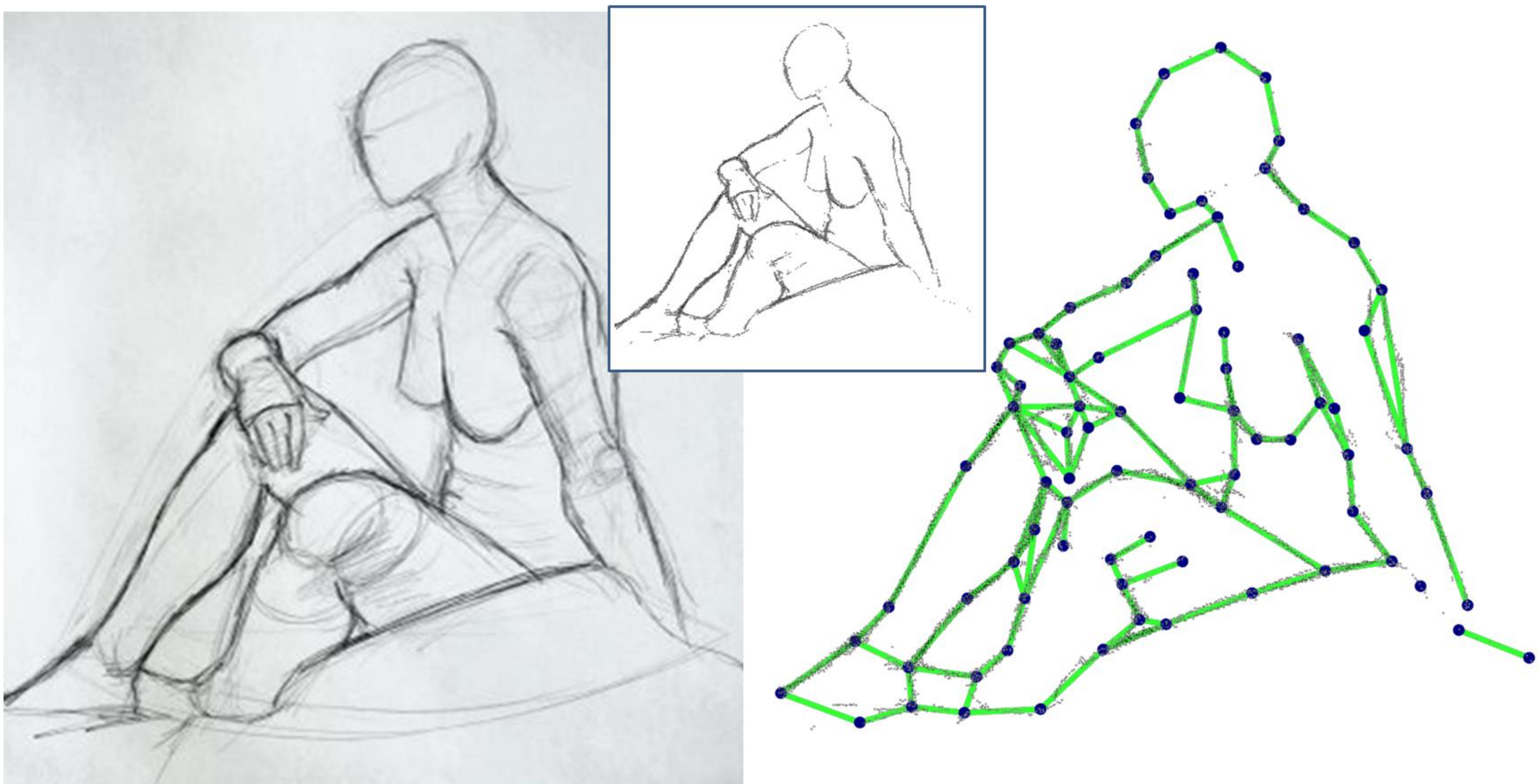
More Noise



More Outliers



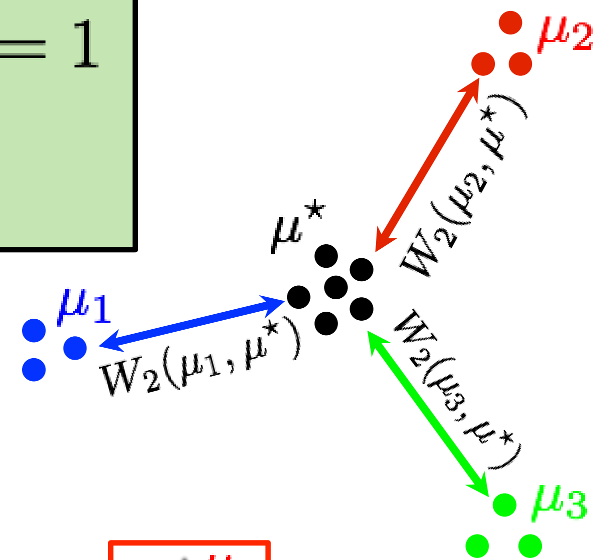
Variable Mass



Barycenters
(Examples from G. Peyré)

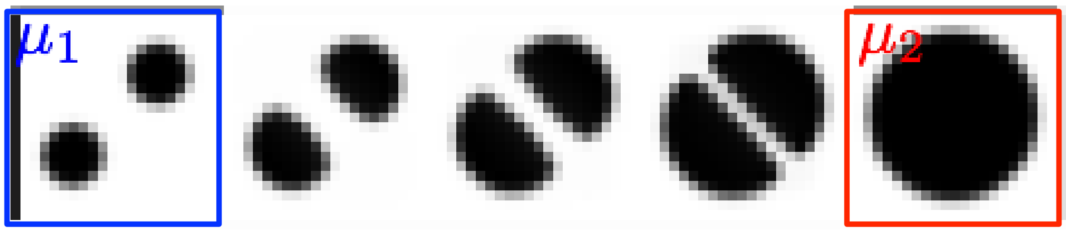
Barycenters of measures $(\mu_k)_k$: $\sum_k \lambda_k = 1$

$$\mu^\star \in \underset{\mu}{\operatorname{argmin}} \sum_k \lambda_k W_2^2(\mu_k, \mu)$$

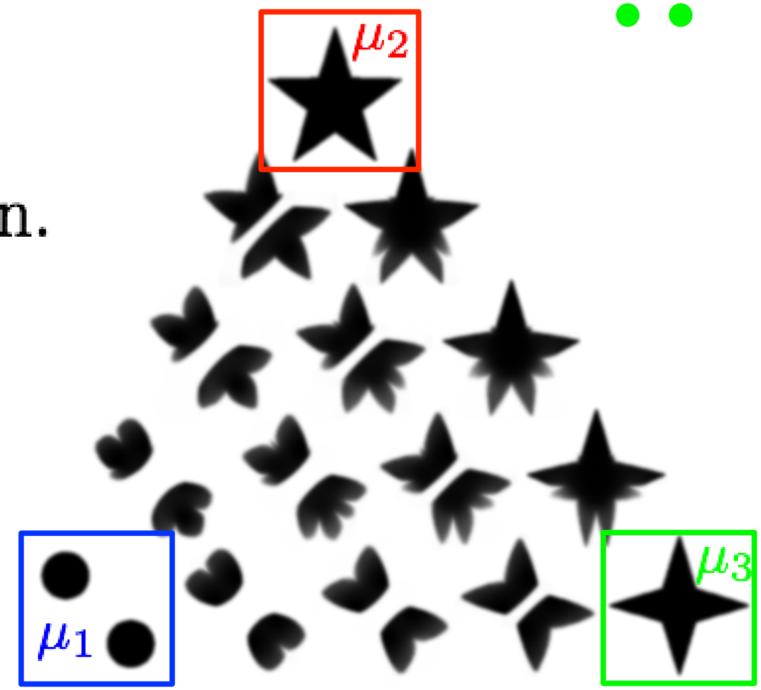


Generalizes Euclidean barycenter:

If $\mu_k = \delta_{x_k}$ then $\mu^\star = \delta_{\sum_k \lambda_k x_k}$



Mc Cann's displacement interpolation.



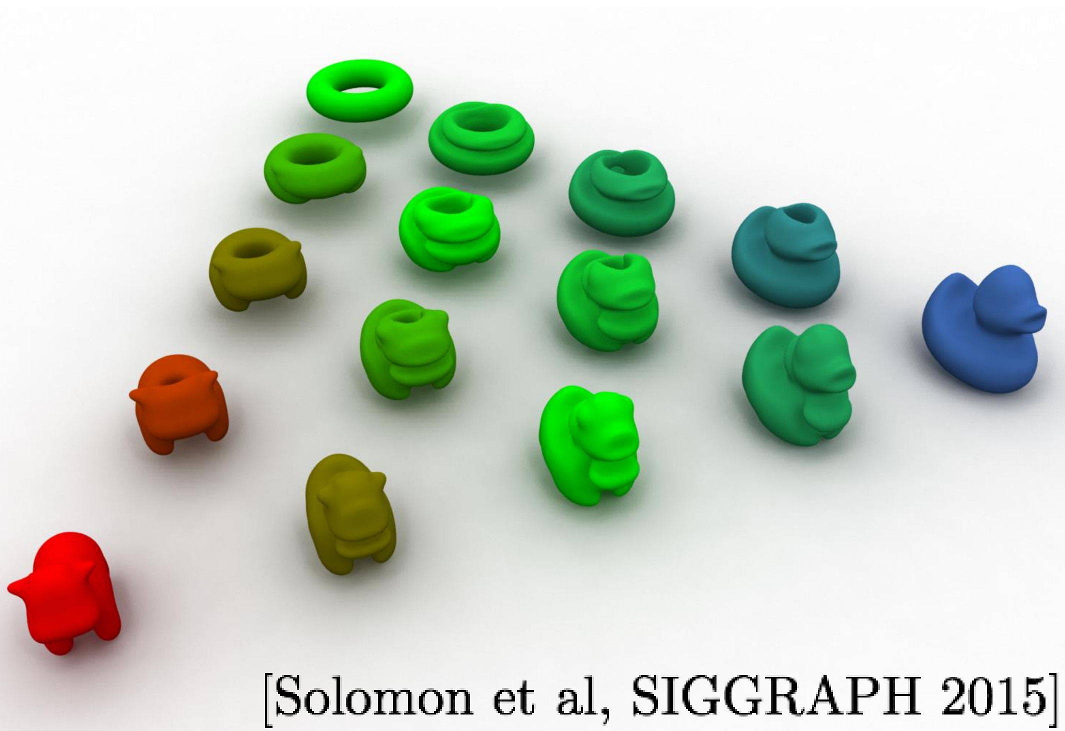
Theorem: [Agueh, Carlier, 2010]

(for $c(x,y) = \|x-y\|^2$)

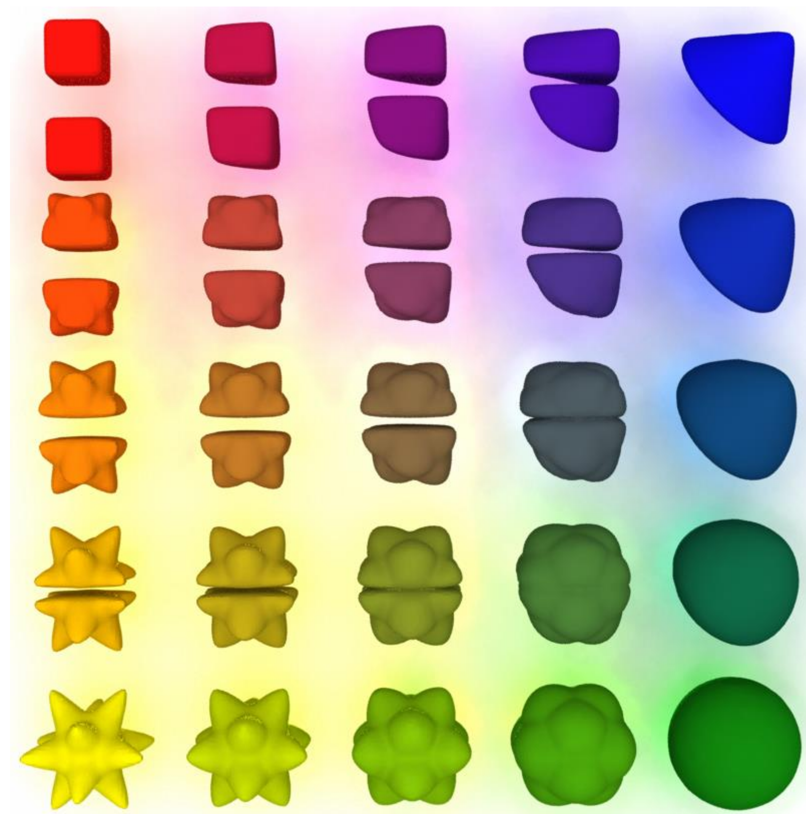
if μ_1 does not vanish on small sets,

$\mu^\star$ exists and is unique.

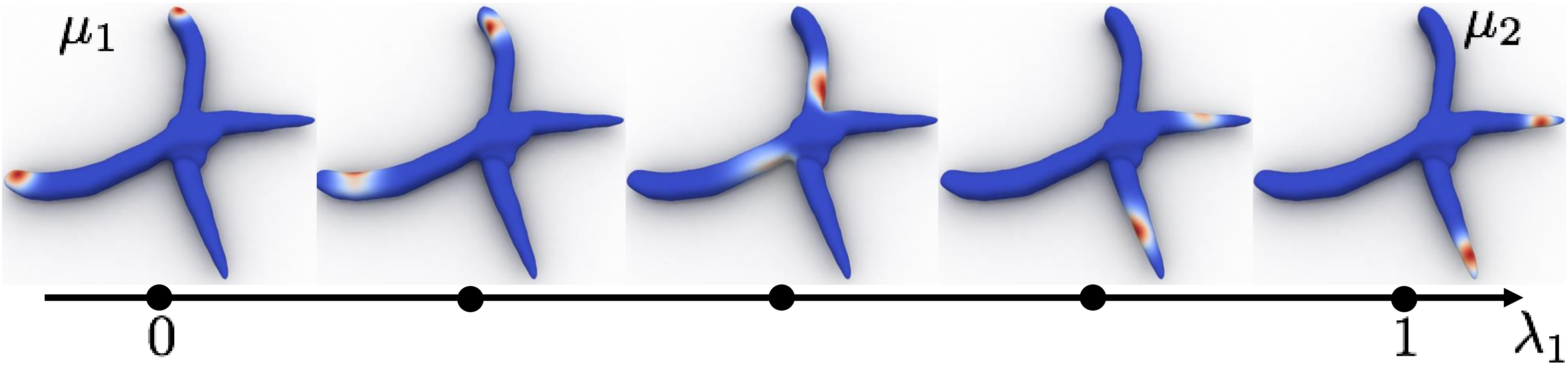
Regularized Barycenters

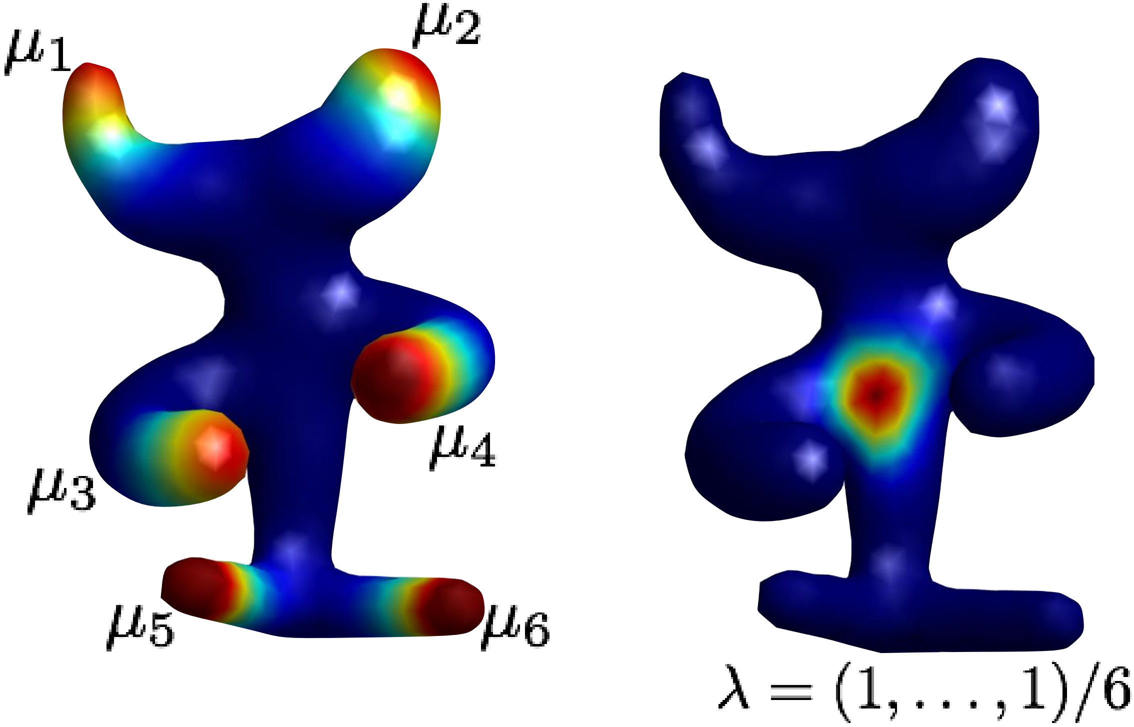


[Solomon et al, SIGGRAPH 2015]



Barycenter on a Surface





Color Transfer

Input images: (f, g) (chrominance components)

Input measures: $\mu(A) = \mathcal{U}(f^{-1}(A)), \nu(A) = \mathcal{U}(g^{-1}(A))$