Shape Reconstruction

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Outline

• Sensors
• Problem statement
• Computational Geometry
  • Convex hull, Voronoi/Delaunay, alpha-shapes
• Variational formulations
  • Poisson / spectral
• Quest for robustness
• Optimal transportation
Laser scanning
Car-based Laser
Airborne Lidar
Multi-View Stereo (MVS)
Depth Sensors
PROBLEM STATEMENT
Reconstruction Problem

**Input**: point set $P$ sampled over a surface $S$:
- Non-uniform sampling
- With holes
- With uncertainty (noise)

**Output**: surface
- Approximation of $S$ in terms of topology and geometry
- Desired:
  - Watertight
  - Intersection free
Ill-posed Problem

Many candidate surfaces for the reconstruction problem!
Ill-posed Problem

Many candidate surfaces for the reconstruction problem! How to pick?
Priors

Smooth

Piecewise Smooth

“Simple”
Surface Smoothness Priors

Local Smoothness
- Local fitting
- No control away from data
- Solution by interpolation

Global Smoothness
- Global: linear, eigen, graph cut, ...
- Robustness to missing data

Piecewise Smoothness
- Sharp near features
- Smooth away from features
Domain-Specific Priors

Surface Reconstruction by Point Set Structuring

[Laforge - A. EUROGRAPHICS 2013]

LOD Reconstruction for Urban Scenes

[Verdie, Laforge - A. ACM Transactions on Graphics 2015]
Previous Work


Warm-up

Smooth

Piecewise Smooth

“Simple”
CONVEX HULL
Convex Hull
VORONOI / DELAUNAY
Voronoi Diagram

Let $\mathcal{E} = \{p_1, \ldots, p_n\}$ be a set of points (so-called sites) in $\mathbb{R}^d$. We associate to each site $p_i$ its Voronoi region $V(p_i)$ such that:

$$V(p_i) = \{x \in \mathbb{R}^d : \|x - p_i\| \leq \|x - p_j\|, \forall j \leq n\}.$$
Delaunay Triangulation

Dual structure of the Voronoi diagram.

The Delaunay triangulation of a set of sites $E$ is a simplicial complex such that $k+1$ points in $E$ form a Delaunay simplex if their Voronoi cells have nonempty intersection.
Delaunay-based

**Key idea:** assuming dense enough sampling, reconstructed triangles are Delaunay triangles.

First define

- Medial axis
- Local feature size
- Epsilon-sampling
Alpha-Shapes [Edelsbrunner, Kirkpatrick, Seidel]

Segments: point pairs that can be touched by an empty disc of radius alpha.
Alpha-Shapes

In 2D: family of piecewise linear simple curves constructed from a point set $P$.
Subcomplex of the Delaunay triangulation of $P$.
Generalization of the concept of the convex hull.
Alpha-Shapes

$\alpha = 0$  Alpha controls the desired level of detail.

$\alpha = \infty$
Computing Alpha-Shapes

Is alpha?
Computing Alpha-Shapes

\[ p_1, p_2, c, Oc, Rc \]
Computing Alpha-Shapes

\[ \text{p1} \]

\[ \text{p2} \]

\[ \text{Rn} \]

\[ \text{On} \]

\[ \text{c} \]

\[ \text{Oc} \]

\[ \text{Rc} \]
is_alpha if
alpha > Rf
AND
alpha <
max(Rc, Rn)

Computing Alpha-Shapes
Computing Alpha-Shapes

is_alpha if
alpha > min(Rc, Rn)
AND
alpha < max(Rc, Rn)
Computing Alpha-Shapes

is_alpha if alpha > Rf
Medial Axis

For a shape (curve/surface) a Medial Ball is a circle/sphere that only meets the shape tangentially, in at least two points.
Medial Axis

For a shape (curve/surface) a *Medial Ball* is a circle/sphere that only meets the shape tangentially, in at least two points.

The centers of all such balls make up the *medial axis/skeleton*. 
Medial Axis
Medial Axis
Medial Axis
Medial Axis

Observation*: For a reasonable point sample, the medial axis is well-sampled by the Voronoi vertices.

*In 3D, this is only true for a subset of the Voronoi vertices - the poles.
Voronoi & Medial Axis
Local Feature Size
Epsilon-Sampling
Crust [Amenta et al. 1998]

If we consider the Delaunay Triangulation of a point set, the shape boundary can be described as a subset of the Delaunay edges.

Q: How do we determine which edges to keep?

A: Two types of edges:

1. Those connecting adjacent points on the boundary
2. Those traversing the shape.

Discard those that traverse.
Observation:

Edges that traverse cross the medial axis. Although we don’t know the axis, we can sample it with the Voronoi vertices. Edges that traverse must be near the Voronoi vertices.
Crust [Amenta et al.]
Delaunay Triangulation
Delaunay Triangulation & Voronoi Diagram
Voronoi Vertices
Refined Delaunay Triangulation
Crust
Crust
Crust (variant)

Algorithm:
1. Compute the Delaunay triangulation.
2. Compute the Voronoi vertices
3. Keep all edges for which there is a circle that contains the edge but no Voronoi vertices.
SPECTRAL « CRUST »
Space Partitioning

Given a set of points, construct the Delaunay triangulation.

If we label each triangle as inside/outside, then the surface of interest is the set of edges that lie between inside and outside triangles.
Space Partitioning

Q: How to assign labels?
A: Spectral Partitioning

Assign a weight to each edge indicating if the two triangles are likely to have the same label.

[Kolluri et al., 2004]
Space Partitioning

Assigning edge weights

Q: When are triangles on opposite sides of an edge likely to have the same label?

A: If the triangles are on the same side, their circumscribing circles intersect deeply.

Use the angle of intersection to set the weight.
Crust

Several Delaunay algorithms provably correct
Delaunay-based

Several Delaunay algorithms are provably correct... in the absence of noise and undersampling.

perfect data?
Noise & Undersampling
Delaunay-based

Several Delaunay algorithms are provably correct... in the absence of noise and undersampling.

Motivates reconstruction by fitting approximating implicit surfaces
VARIATIONAL FORMULATIONS

Smooth

Piecewise Smooth

“Simple”
Poisson Surface Reconstruction

[Kazhdan et al. SGP’06]
Construct indicator function from point samples
Construct indicator function from point samples

$$\min_{\chi} \int \| \nabla \chi(x) - N(x) \|^2 dx$$

$$\Delta \chi = \nabla \cdot N$$

Variational calculus

Splatted normals

Sparse linear system
2D Poisson Reconstruction
Poisson Reconstruction

Requires oriented normals, as many other implicit approaches.
Poisson Reconstruction

Requires **oriented normals**, as many other implicit approaches.

Normal estimation
Normal orientation

\[\text{ill-posed problems}\]
Poisson Reconstruction

Can we deal with unoriented normals?
QUEST FOR ROBUSTNESS
Quest for Robustness

Inferred shape  Perfect point set  Noise  Outliers  Non-uniform sampling density  Missing data  Variable noise

?
Noise-adaptive Reconstruction

Smooth

[Giraudot, Cohen-Steiner, A. SGP’13]
Robust Distance Function

[Chazal, Cohen-Steiner, Mérigot 11] Noise and outlier robust. Based on optimal transport distance between geometric measures (Wasserstein-2 distance)
Robust Distance Function

Unsigned distance function to a measure [Chazal et al., 2011]

\[ d_{\mu,m}^2 : \mathbb{R}^n \to \mathbb{R}, \quad q \mapsto \frac{1}{m} \int_{B(q,r_{\mu,m}(q))} \|q - y\|^2 d\mu(y) \]
Robust Distance Function

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Note: scale parameter \( m \)
- User-specified
- Depends on point set properties
- Global: not noise-adaptive
Non-adaptive Distance Function

$K = 6$

$K = 70$

Variable noise
Case of Ambient Noise

Uniform measure in $d$-dimensional space

$$d_{\mu,m}(q) = c \cdot m^{\frac{2}{d}}$$

$$d_{\mu,m}(q) \propto m^{\frac{1}{d}}$$ for $q$ fixed

$$\frac{d_{\mu,m}(q)}{m^{\alpha}}$$ decreasing for $\alpha > \frac{1}{d}$
Case of Submanifold

Uniform measure on $k$-submanifold

\[ d_{\mu,m}(q) = c \cdot m^{\frac{2}{k}} + h^2 \]

\[ d_{\mu,m}(q) \propto m^{\frac{1}{k}} \text{ for } q \text{ fixed} \]

\[ \frac{d_{\mu,m}(q)}{m^\alpha} \text{ increasing for } \alpha < \frac{1}{k} \]
Noisy Case

Scale $m = 10$ nearest neighbors

- Apparent dimension = 2
- Ambient noise in 2D
- $\frac{d_{\mu,m}(q)}{m^\alpha}$ decreasing for $\alpha > \frac{1}{2}$
Noisy Case

Scale $m = 10$ nearest neighbors

- Apparent dimension $= 2$
- Ambiant noise in 2D
- $\frac{d_{\mu,m}(q)}{m^\alpha}$ decreasing for $\alpha > \frac{1}{2}$

Scale $m = 30$ nearest neighbors

- Apparent dimension $= 1$
- 1-submanifold in 2D
- $\frac{d_{\mu,m}(q)}{m^\alpha}$ increasing for $\alpha < 1$
Noise-adaptive Distance Function

Assumption
Inferred shape is a submanifold of known dimension

For a $k$-submanifold in $d$-dimensional space:

$$\delta_{\mu} = \inf_{m > 0} \frac{d_{\mu,m}}{m^\alpha},$$

with $\alpha \in \left[\frac{1}{d}; \frac{1}{k}\right]$
Noise-adaptive Distance Function

\[ \delta_\mu = \inf_{m > 0} \frac{d_{\mu,m}}{m^\alpha} \]

**Infimum:**

1. \( m \) as small as possible \( \rightarrow \) no oversmoothing
2. \( m \) large enough \( \rightarrow \) point subset appears as \( k \)-submanifold

**Setting** \( \alpha \) \( (\alpha \in \left[ \frac{1}{d}; \frac{1}{k} \right]) \)

- Curve in 2D: \( \alpha = \frac{3}{4} \) to satisfy \( \alpha \in \left[ \frac{1}{2}; 1 \right] \)
- Surface in 3D: \( \alpha = \frac{5}{12} \) to satisfy \( \alpha \in \left[ \frac{1}{3}; \frac{1}{2} \right] \)
Noise-adaptive Distance Function

Function $\delta_{\mu}$

Local scale $m$
Noise-adaptive Distance Function

Wasserstein robustness & semiconcavity

- $\forall m, \ d_{\mu,m}$ is $1/\sqrt{m}$-robust and $1$-semiconcave
- Limiting infimum over values of $m$ above $m_0$: properties preserved for $\delta_{\mu}$

→ correct topological inference (Geometric Inference for Probability Measures [F. Chazal, D. Cohen-Steiner, Q. Mérigot, 2011])
Experiment

Our reconstruction

Poisson reconstruction
Outlier Robustness
OPTIMAL TRANSPORTATION

Smooth

Piecewise Smooth

“Simple”
Motivations

Geometric data:
Defect-laden sampling

- Sparse
- Non-uniform
- Widely variable
- Missing data
Motivations

Geometric data:

Uncertain

• Measurement noise
• Registration noise
• Outliers
• Spurious topology

Heterogeneous

Calls for Robustness
Motivations

Complex shapes:

• Sharp features
• Non-manifold features
• Boundaries

Calls for feature preservation
## Previous Work

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Approach in 2D

Given a point set $S$, find a coarse triangulation $T$ such that $S$ is well approximated by uniform measures on the 0- and 1-simplices of $T$.

How to measure distance $D(S, T)$?

$\Rightarrow$ optimal transport between measures

How to construct $T$ that minimizes $D(S, T)$?

optimal location problem! $\Rightarrow$ greedy decimation
Distance between Measures (1D)

Transport plan: \( \pi \) on \( \mathbb{R} \times \mathbb{R} \) whose marginals are \( A \) and \( B \)

Transport cost: \( W_2(A, B, \pi) = \left( \int_{\mathbb{R} \times \mathbb{R}} \|x - y\|^2 d\pi(x, y) \right)^{1/2} \)

Optimal transport: \( W_2(A, B) = \inf_{\pi} W_2(A, B, \pi) \)
Distance between Measures

Transport plan: $\pi$ on $\mathbb{R} \times \mathbb{R}$ whose marginals are $A$ and $B$

Transport cost: $W_2(A, B, \pi) = \left( \int_{\mathbb{R} \times \mathbb{R}} \|x - y\|^2 d\pi(x, y) \right)^{1/2}$

Optimal transport: $W_2(A, B) = \inf_{\pi} W_2(A, B, \pi)$

(discrete measure)
Optimal Transport on a Vertex

(assume given *binary* transport plan)

\[ W_2(v, S_v) = \sqrt{\sum_{p_i \in S_v} m_i \| p_i - v \|^2}. \]
Optimal Transport on an Edge

(assume given binary transport plan)

\[ N(e, S_e) = \sqrt{\sum_{p_i \in S_e} m_i \| p_i - q_i \|^2} \]

\[ T(e, S_e) = \sqrt{\sum_{p_i \in S_e} \frac{M_e}{|e|} \int_{-l_i/2}^{l_i/2} (x - c_i)^2 \, dx} = \sqrt{\sum_{p_i \in S_e} m_i \left( \frac{l_i^2}{12} + c_i^2 \right)} \]
Boundary Preserving?

density
Feature Preserving?
Current (Binary) Transport Plan

Total cost $W_2(\pi) = \sqrt{\sum_{e \in T} [N(e,S_e)^2 + T(e,S_e)^2] + \sum_{v \in T} W_2(v,S_v)^2}$
Solid vs Ghost Edge
Algorithm Overview

An Optimal Transport Approach to Robust Reconstruction and Simplification of 2D Shapes.

- input point set (potentially variable mass)
- Delaunay triangulation
- after decimation
- output after edge filtering
Robustness
More Noise
More Outliers
Features and Robustness
Variable Mass

An Optimal Transport Approach to Robust Reconstruction and Simplification of 2D Shapes.
Symposium on Geometry Processing, 2011.
Limitation: Binary Transport Plan
WHAT NEXT
What Next

Robustness

• Structured outliers
• Heterogeneous data

New acquisition paradigms

• Super-resolution
• Community data
« La Lune »
« La Lune »