Mesh Optimization

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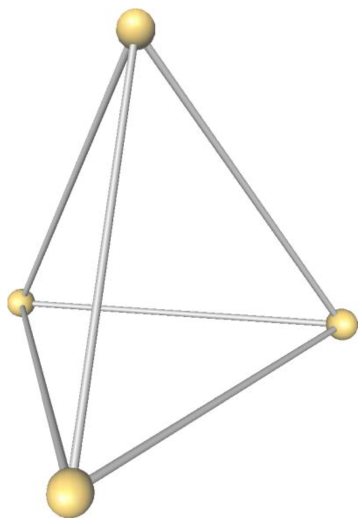
Goals

- 3D simplicial mesh generation
- optimize shape of elements
  - for matrix conditioning
  - isotropic
- control over sizing
  - dictated by simulation
  - constrained by boundary
  - low number of elements desired
    - more elements = slower solution time



Popular Meshing Approaches

- advancing front
- specific subdivision
  - octree
  - lattice (e.g. body centered cubic)
- Delaunay
  - refinement
  - sphere packing

combined with local optimizations

- spring energy
  - Laplacian
  - non-zero rest length
- aspect / radius ratios
- dihedral / solid angles
- max-min/min-max
  - volumes
  - edge lengths
  - containing sphere radii

[Freitag Amenta Bern Eppstein]

- sliver exudation

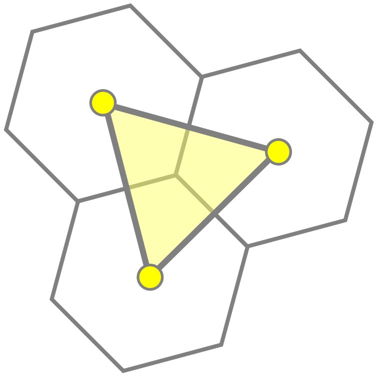
[Edelsbrunner Goy]

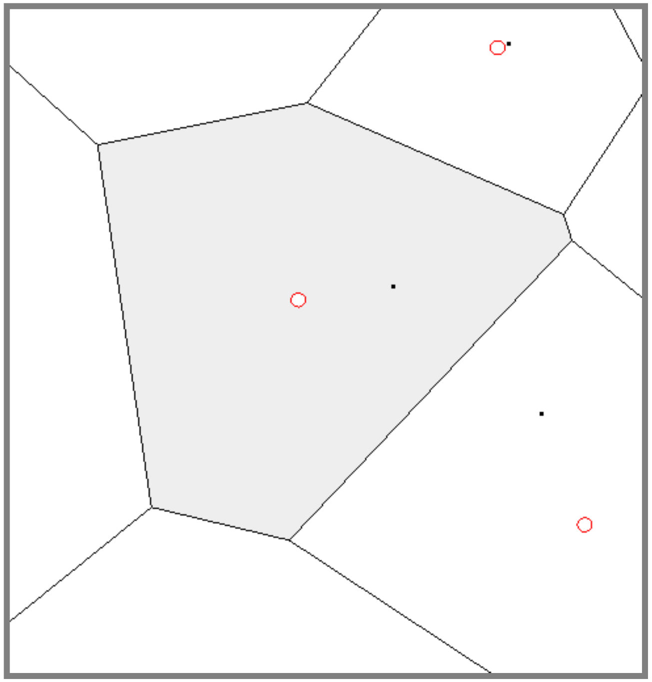
Variational?

- Design one energy function such that good solutions correspond to low energy ones (global minimum in general a mirage).
- Solutions found by optimization techniques.

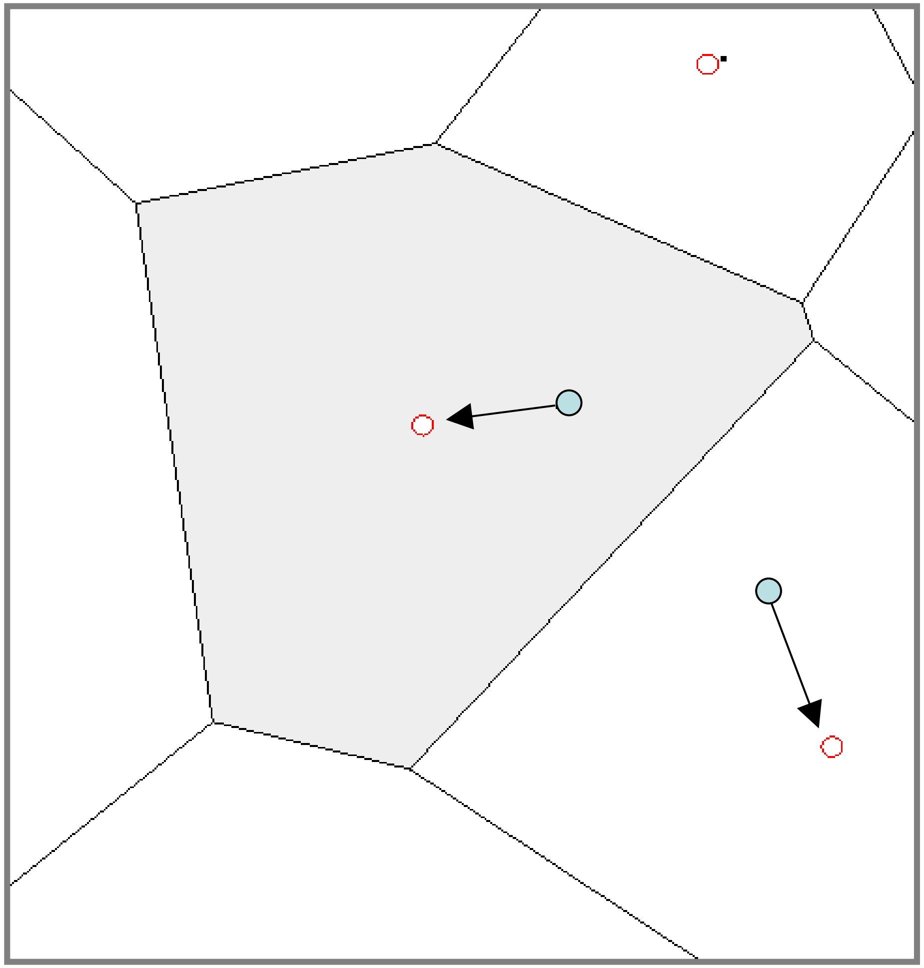
Example Energy in 2D

$$E = \sum_{j=1..k} \int_{x \in R_j} \| x - x_j \|^2 \, dx$$



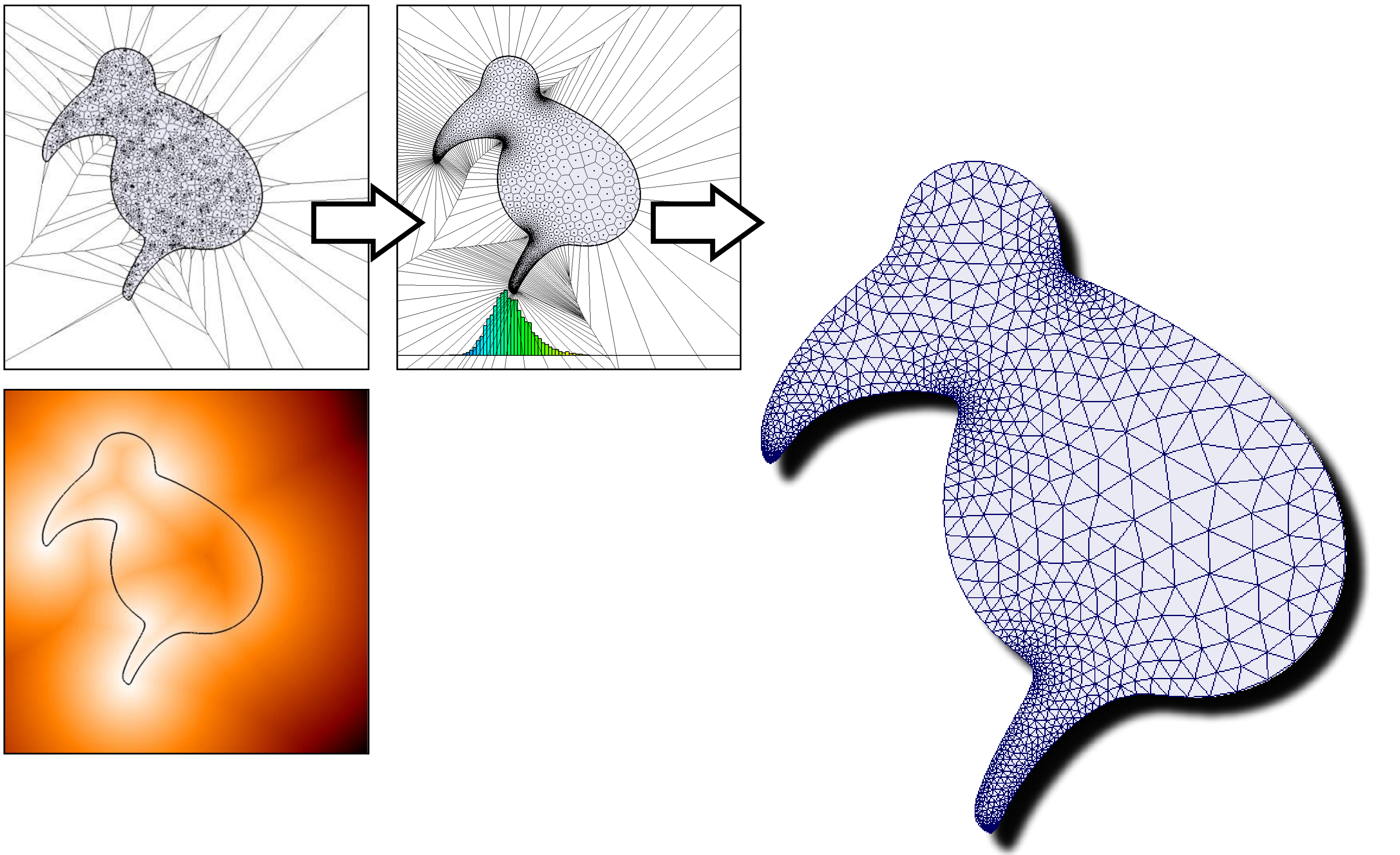


Lloyd Iteration

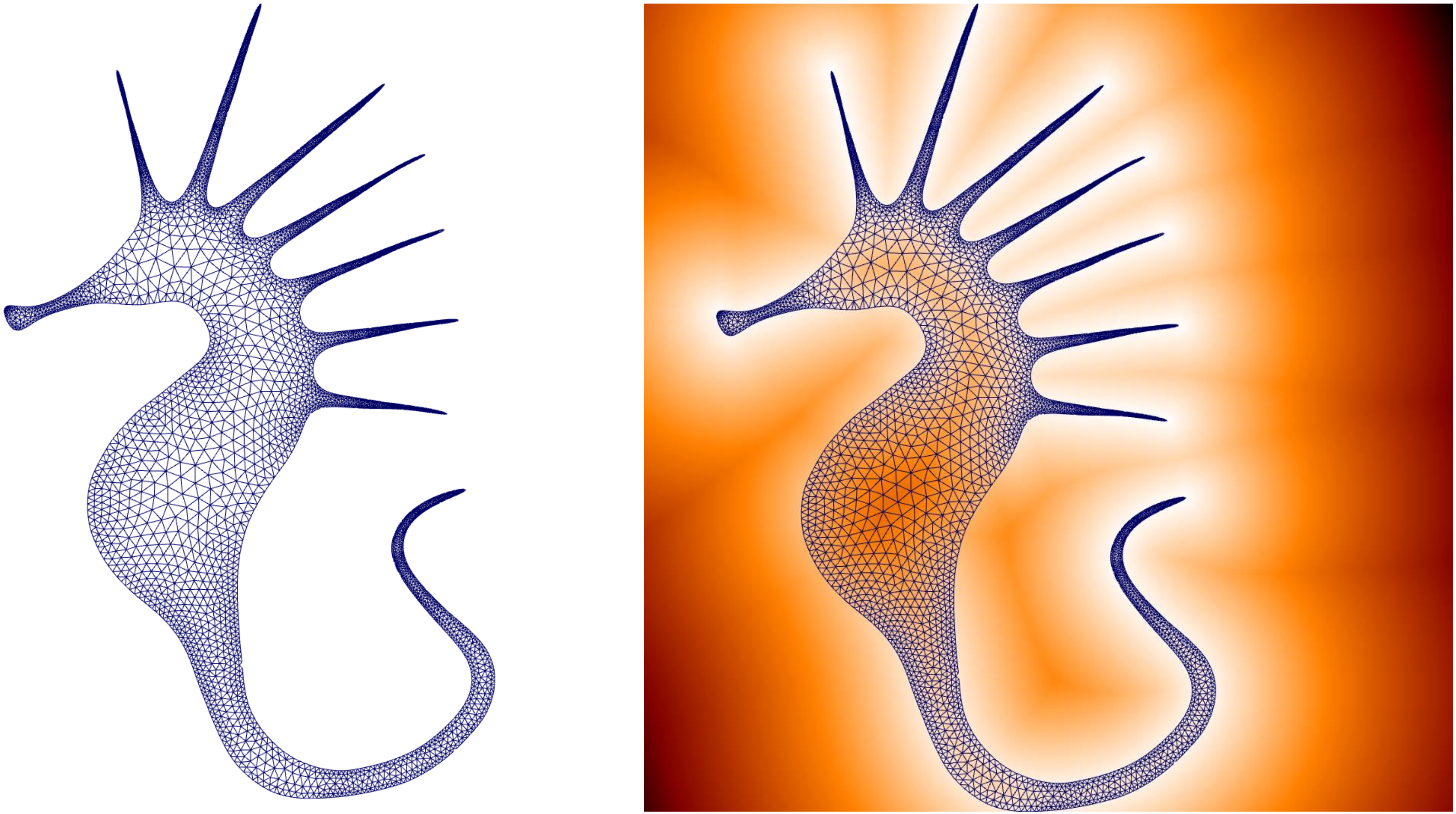


demo

2D Optimized Triangle Meshing



2D Optimized Triangle Meshing

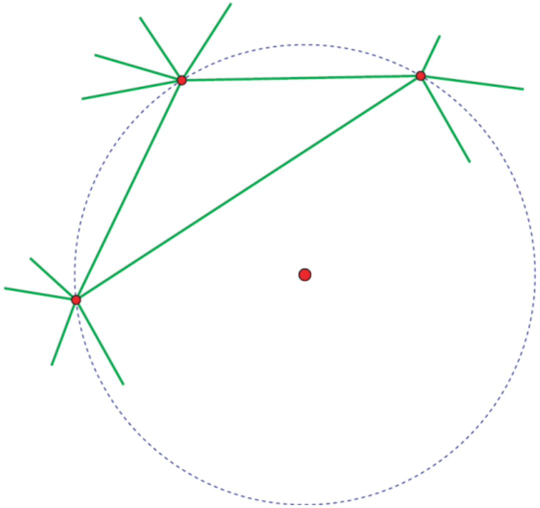


Delaunay refinement

Termination

- shape criterion: radius-edge ratio
- in 2D: max $\sqrt{2}$ (implies min 20.7°)
- in 3D: max 2 (nothing similar on dihedral angles)

[Chew, Ruppert, Shewchuk, ...]



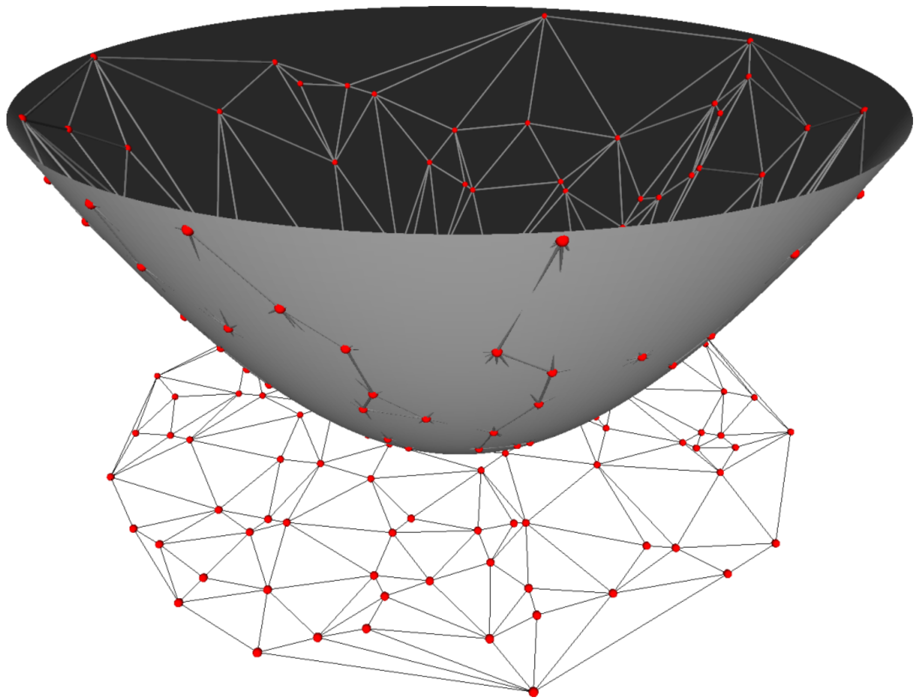
Delaunay refinement

- + greedy (fast)
- + easy incorporation of sizing field
- + allows boundary conforming
  - possibly with Steiner points
  - even for sharp angles on boundary **[Teng]**
- + guaranteed bounds on radius-edge ratio
- - blind to slivers
  - and experimentally...produces slivers

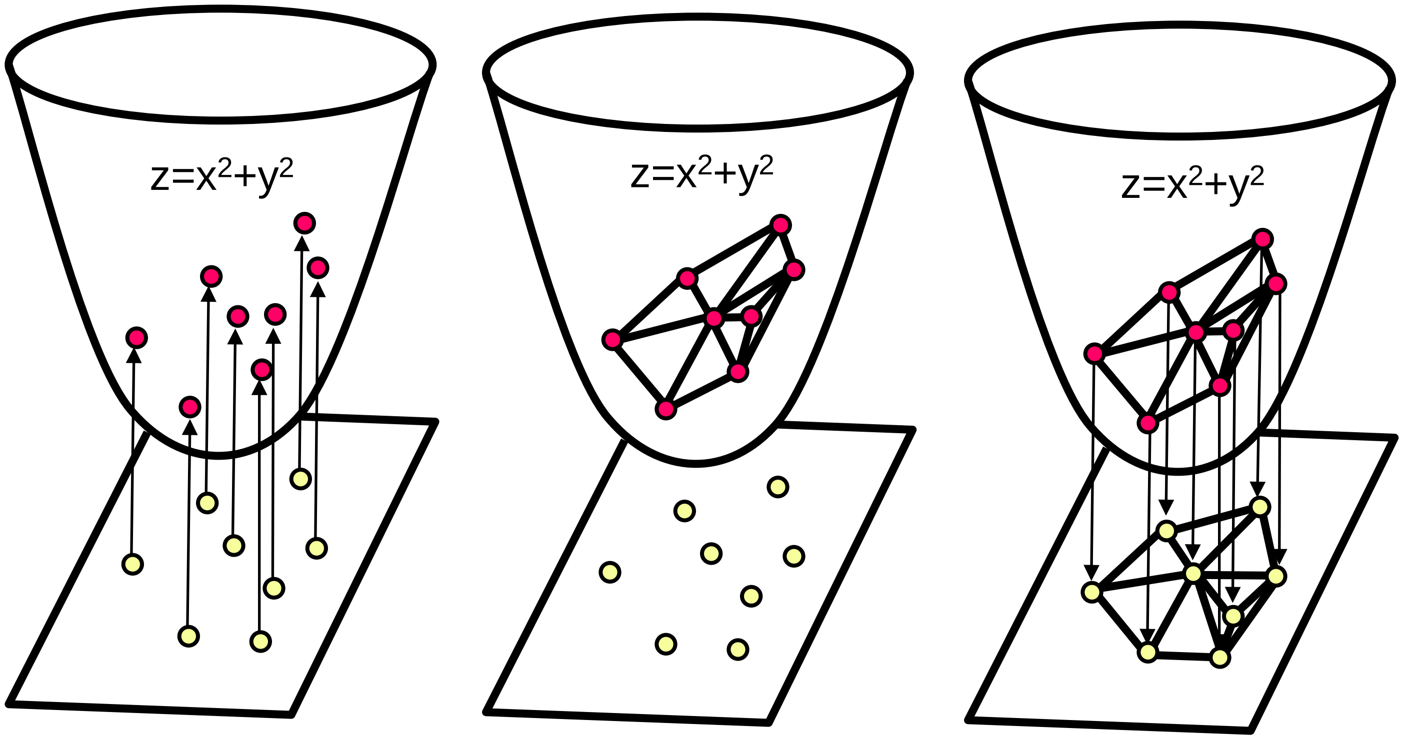
Background

Delaunay Triangulation

- **Duality on the paraboloid**: Delaunay triangulation obtained by projecting the lower part of the convex hull.



Delaunay Triangulation



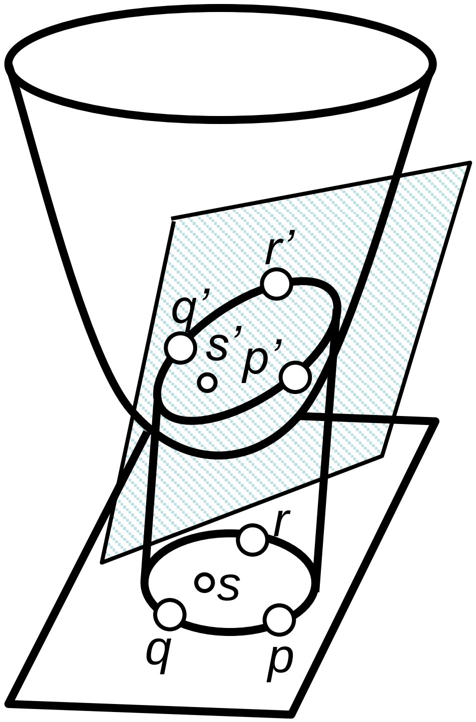
Project the 2D point set onto the 3D paraboloid → Compute the 3D lower convex hull → Project the 3D facets back to the plane.

Proof

- The intersection of a plane with the paraboloid is an ellipse whose projection to the plane is a circle.
- *s* lies within the circumcircle of *p*, *q*, *r* iff *s*’ lies on the lower side of the plane passing through *p*’, *q*’, *r*’.

↓

- *p*, *q*, *r* $\in$ *S* form a Delaunay triangle iff *p*’, *q*’, *r*’ form a face of the convex hull of *S*’.

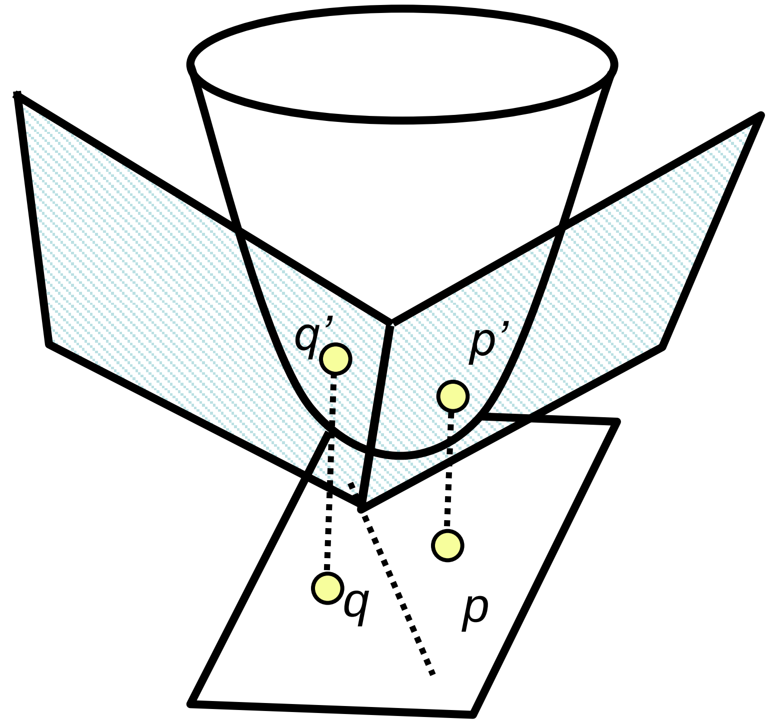


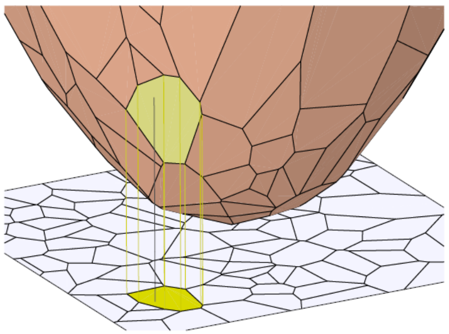
Voronoi Diagram

- Given a set S of points in the plane, associate with each point $p=(a,b)\in S$ the plane tangent to the paraboloid at p:

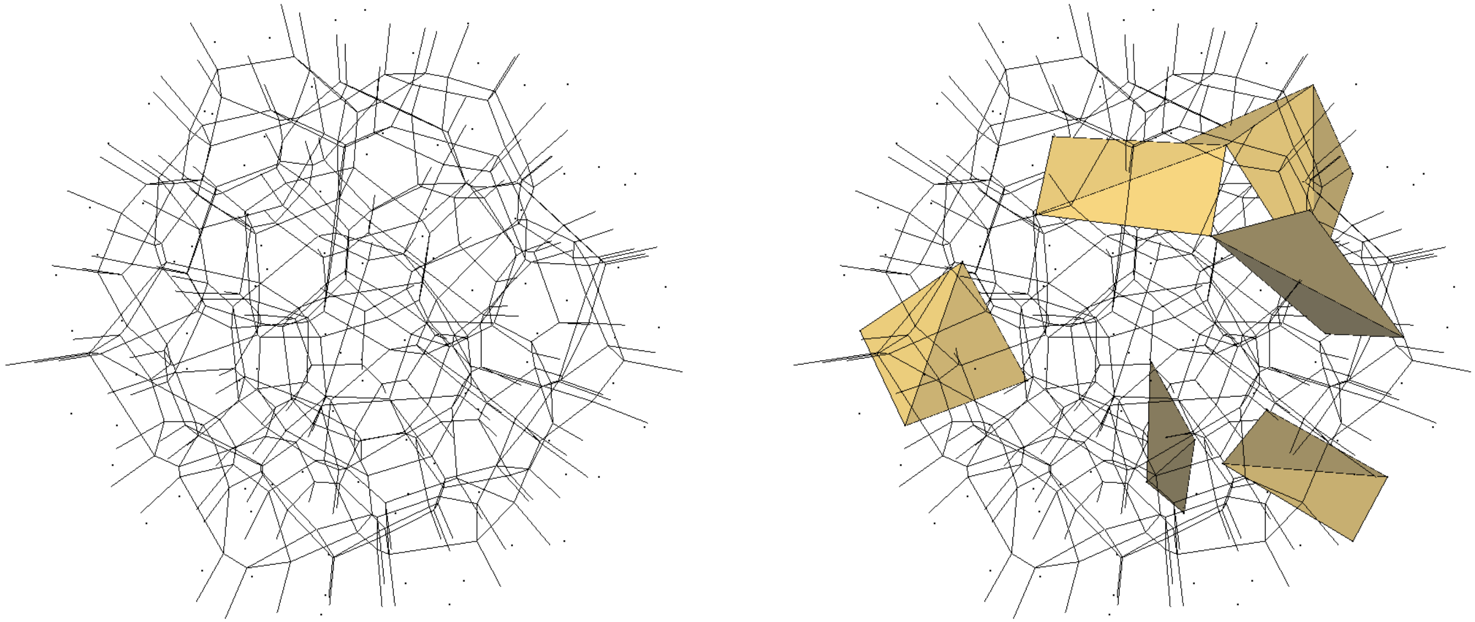
$$z = 2ax+2by-(a2+b2).$$

- VD(S) is the projection to the (x,y) plane of the 1-skeleton of the convex polyhedron formed from the intersection of the halfspaces above these planes.





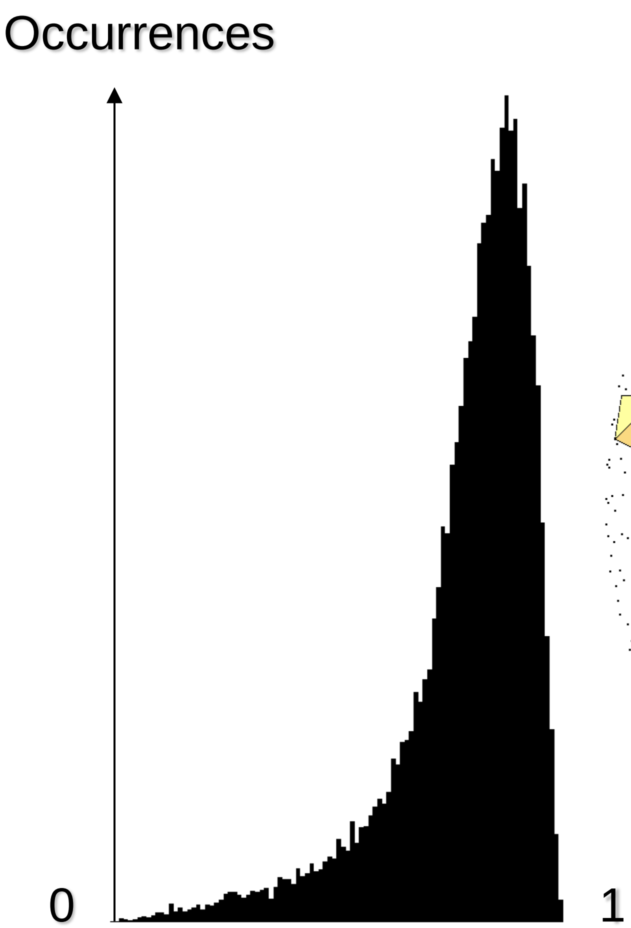
First Idea: Lloyd Algorithm

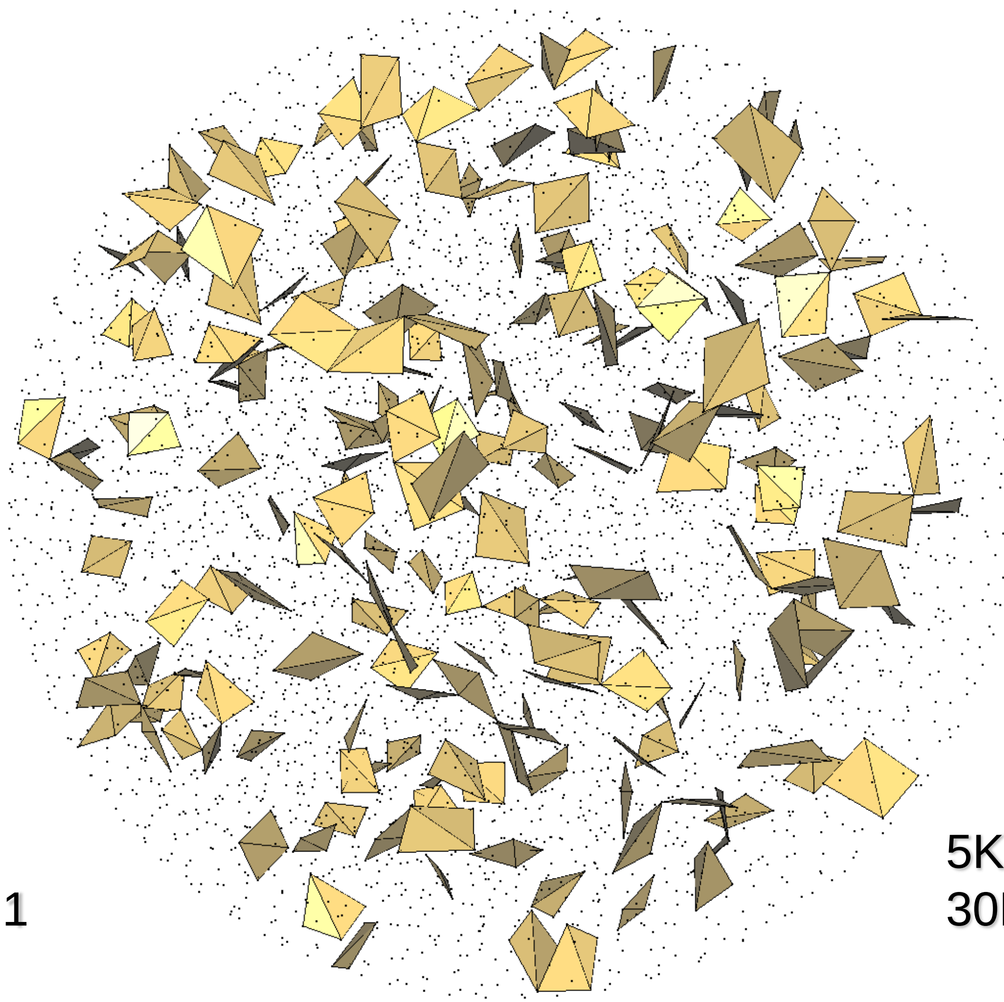


(after Lloyd relaxation)

...back to primal ?

Centroidal Voronoi Tessellation



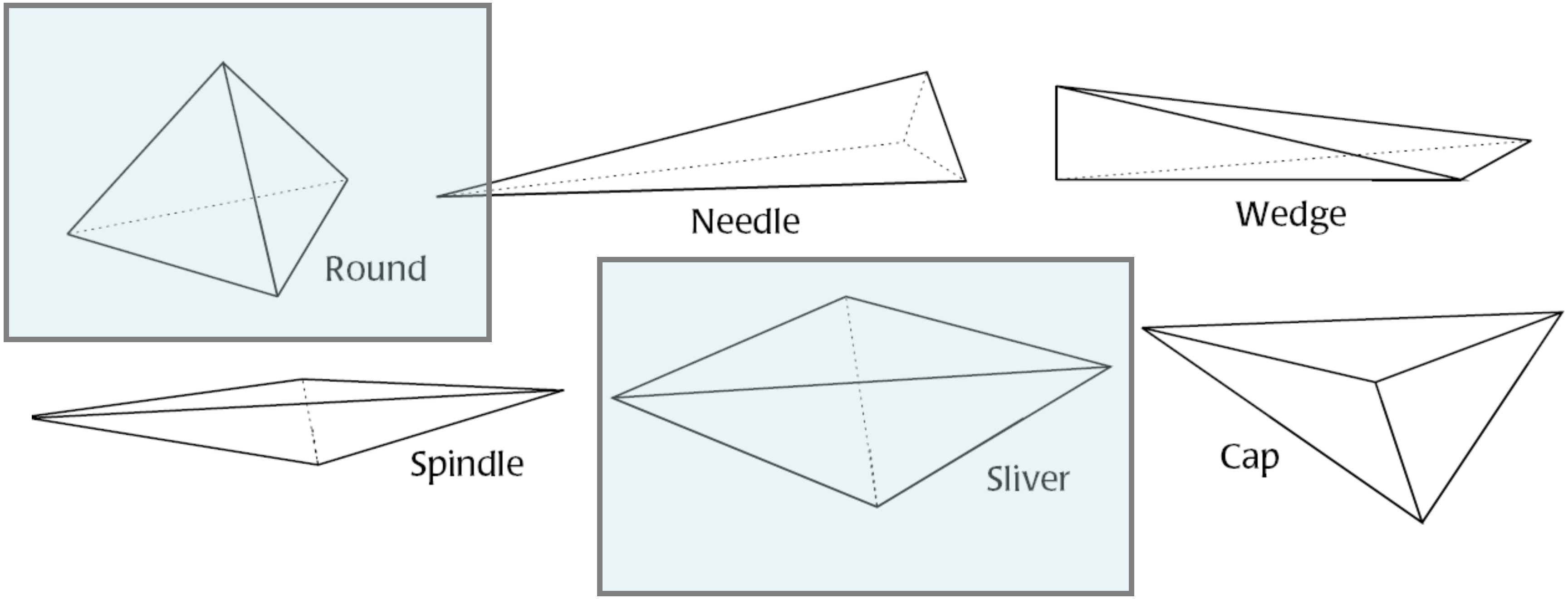


5K sites
30K tets

distribution of radius ratios

220 “slivers” (tets with radius ratio < 0.2)

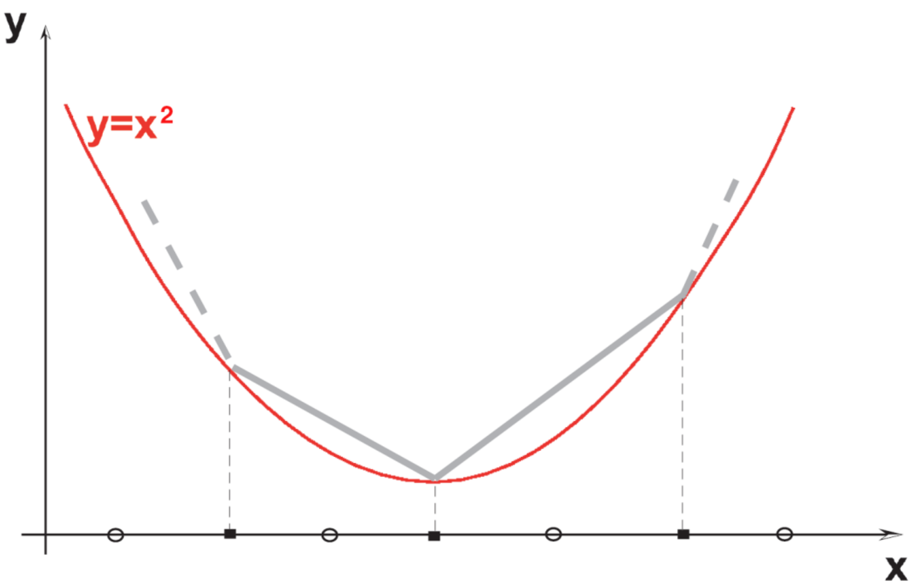
Tetrahedra Zoo



well-spaced points generate only **round** or **sliver** tetrahedra

Key Idea

- adopt the "function approximation" point of view **[Chen 04]** *Optimal Delaunay Triangulation*
- 1D: $f(x)=x^2$ centered at any vertex
- minimize the L^1 norm between f and PWL interpolation



Key Idea

- 3D: $||x||^2$ (graph in $\mathbb{R}^4$)
- approximation theory:
  - linear interpolation: optimal shape of the element related to the Hessian of f **[Shewchuk]**
- Hessian($||x||^2$) = Id
  - regular tetrahedron best
- note: FE ~ mesh that best *interpolates* a function + matrix conditioning

Key Question

- which mesh best approximates the paraboloid?
  - (PWL interpolates)

- Answers:
  - for **fixed point locations**
    - Delaunay (lifts to lower facets of convex hull)
  - for **fixed connectivity**
    - quadratic energy
    - closed form for local optimum

Function Approximation

- Given:
  - triangulation T
  - bounded domain Ω in IR^n
- Consider function approximation error:

$$Q(T, f, p) = \| f - f_{I,T} \|_{L^p, \Omega}$$

linear interpolation

Function Approximation

Theorem [Chen 04]:

$$Q(DT, \|x\|^2, p) = \min_{T \in P_V} Q(T, \|x\|^2, p), 1 \le p \le \infty$$

Isotropic function

set of all triangulations with a **given set** V

□ = convex hull of V

[d'Azevedo-Simpson 89] in IR^2, p = ∞

[Rippa 92] in IR^2, 1 ≤ p ≤ ∞

[Melissaratos 93] in IR^D, 1 ≤ p ≤ ∞

Function Approximation

Let us V vary

Problem:

- find triangulation T^* such that:

$$Q(T^*, f, p) = \inf_{T \in P_N} Q(T, f, p), 1 \leq p \leq \infty$$

↑ set of all triangulations with a **at most N vertices**

Proof:

- existence
- necessary condition for $p=1$

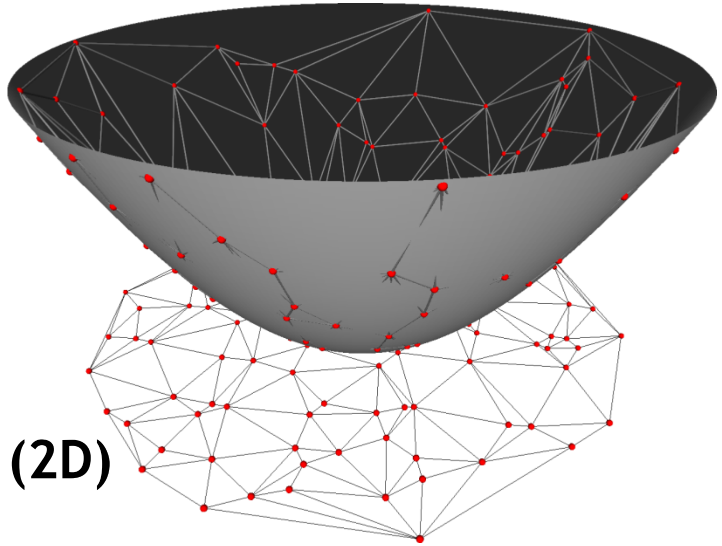
Function Approximation

$$Q(DT, \| x \|^2, p) = \min_{T \in P_V} Q(T, \| x \|^2, p), 1 \leq p \leq \infty$$

set of all triangulations with a **given set** V

▪ = convex hull of V

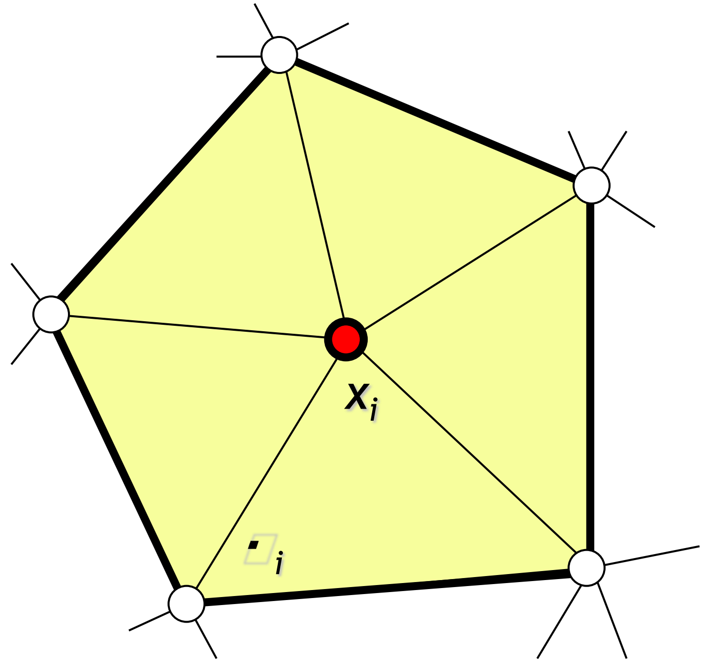
Isotropic function



(2D)

Function Approximation

- x_i: vertex
- Ω_i: *union of simplices incident to* x_i
- $|A|$: *Lebesgue measure of set A in* IR^n



Function Approximation

$$Q(T, f, p) = \| f_{I,T} - f \|_{L^p,\Omega}$$

$$Q(T, f, p) = \left[\int_{\Omega} | f_{I,T}(x) - f(x) |^p dx \right]^{1/p}$$

$$Q(T, f, 1) = \int_{\Omega} (f_{I,T}(x) - f(x)) dx$$

f convex, $f_{I,T}$ PWL interpolant

$$Q(T, f, 1) = \int_{\Omega} f_{I,T}(x) dx - \int_{\Omega} f(x) dx$$

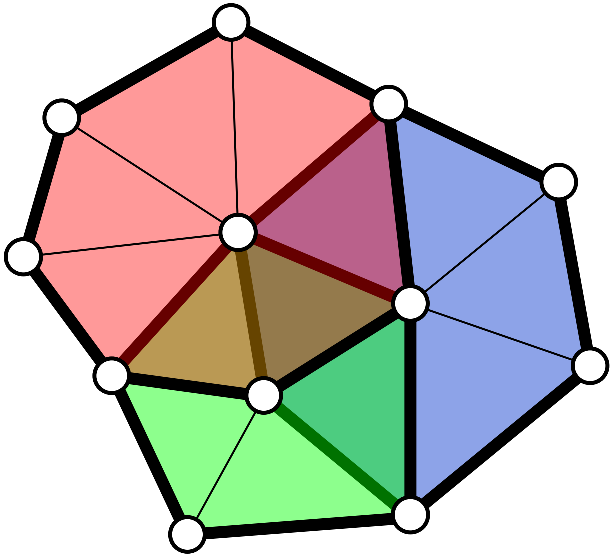
Function Approximation

$$Q(T,f,1)=\int_{\Omega} f_{I,T}(x)dx-\int_{\Omega} f(x)dx$$

$$=\sum_{\tau\in T}\int_{\tau} f_{I,T}(x)dx-\int_{\Omega} f(x)dx$$

$$=\frac{1}{n+1}\sum_{\tau\in T}\left(|\tau|\sum_{k=1}^{n+1} f(x_{\tau},k)\right)-\int_{\Omega} f(x)dx$$

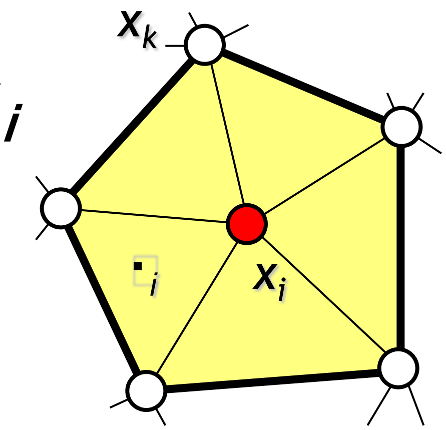
$$=\frac{1}{n+1}\sum_{x_i\in T} f(x_i)|\Omega_i|-\int_{\Omega} f(x)dx$$



n+1 overlaps

Function Approximation

restrict to patch Ω_i incident to vertex x_i



$$Q(\Omega_i, f, 1) = \frac{1}{n+1} \sum_{x_i \in \Omega_i} f(x_i)\,|\Omega_i| - \int_{\Omega_i} f(x)\,dx$$

$$= \frac{1}{n+1} \sum_{\tau_j \in \Omega_i} \left(|\tau_j(x)| \sum_{x_k \in \tau_j, x_k \neq x_i} f(x_k) \right) + \frac{|\Omega_i|}{n+1} f(x_i) - \int_{\Omega_i} f(x)\,dx$$

constant

minimize

$$E_{ODT} \equiv \sum_{\tau_j \in \Omega_i} \left(|\tau_j(x)| \sum_{x_k \in \tau_j, x_k \neq x_i} f(x_k) \right) + |\Omega_i|\, f(x_i)$$

Function Approximation

$$E_{ODT} \equiv \sum_{\tau_j \in \Omega_i} (|\tau_j(x)| \sum_{x_k \in \tau_j, x_k \neq x_i} f(x_k)) + |\Omega_i| \, f(x_i)$$

minimum if $\nabla E_{ODT} = 0$

$$\nabla f(x_i^{*}) = -\frac{1}{|\Omega_i|} \sum_{\tau_j \in \Omega_i} \left(\nabla |\tau_j| (x_i) \sum_{x_k \in \tau_j, x_k \neq x_i} f(x_k) \right)$$

if $f(x) = \| x \|^2$

$$x_i^{*} = -\frac{1}{2|\Omega_i|} \sum_{\tau_j \in \Omega_i} \left(\nabla |\tau_j| (x_i) \sum_{x_k \in \tau_j, x_k \neq x_i} \| x_k \|^2 \right)$$

Geometric Interpretation

$$x_i^* = \frac{1}{|\Omega_i|} \sum_{\tau_j \in \Omega_i} |\tau_j| c_j$$

circumcenter

demo

- **Note**: optimal location depends only on the 1-ring neighbors, not on the current location. If all incident vertices lie on a common sphere, optimal location is at sphere center.

Optimization

- alternate updates of
  - connectivity
  - vertex location

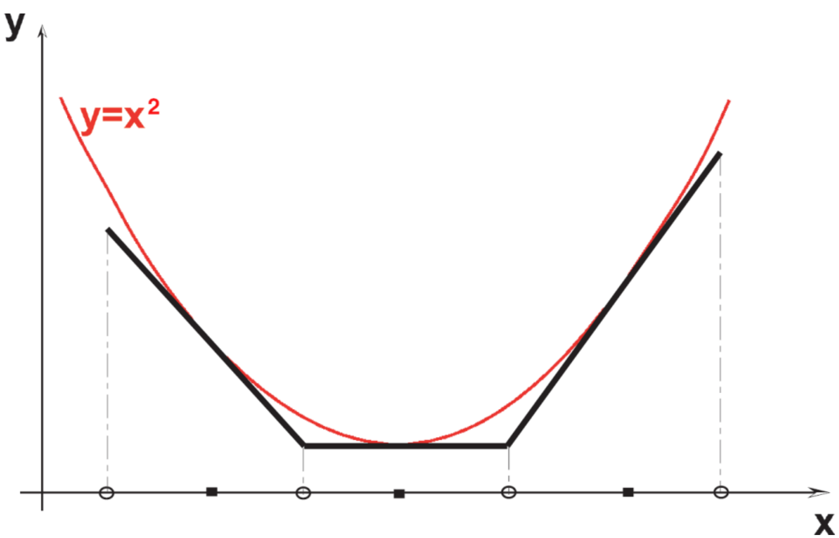
- both steps minimize the same energy
  - as for Lloyd iteration

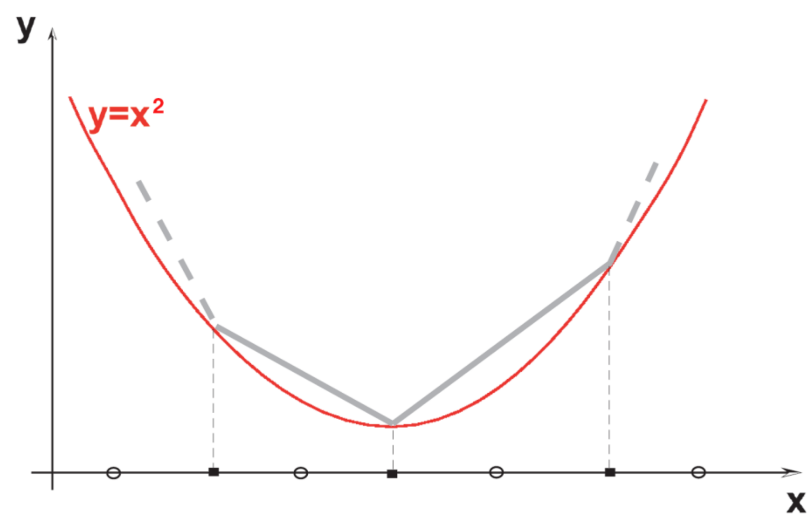
- for convex fixed boundary
  - energy monotonically decreases
  - convergence to a (local) minimum

Underlaid *vs* Overlaid Approximant

- CVT
  - partition
  - approximant
  - compact Voronoi **cells**
  - isotropic **sampling**

- ODT
  - overlapping decomposition
  - PWL interpolant
  - compact **simplices**
  - isotropic **meshing**



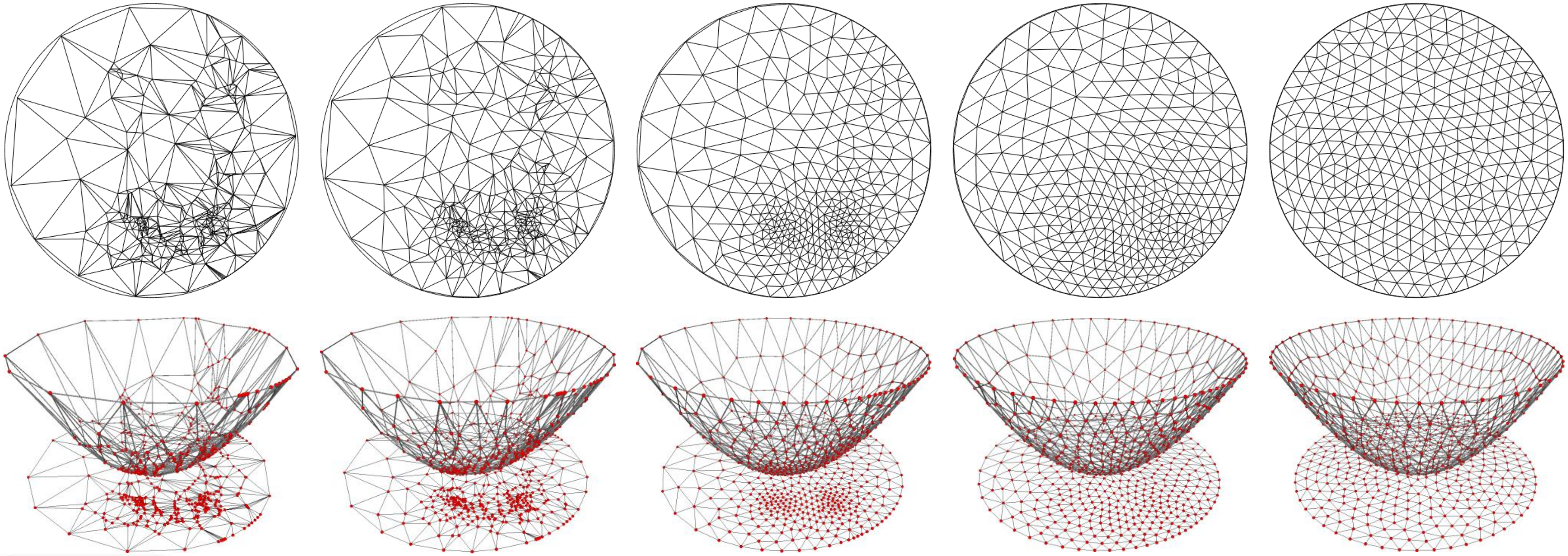


Optimization

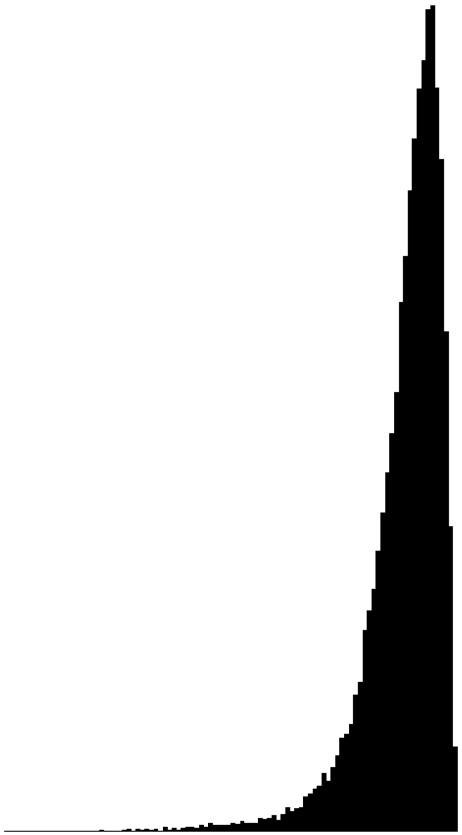
Alternate updates of

- connectivity (Delaunay triangulation)
- vertex locations

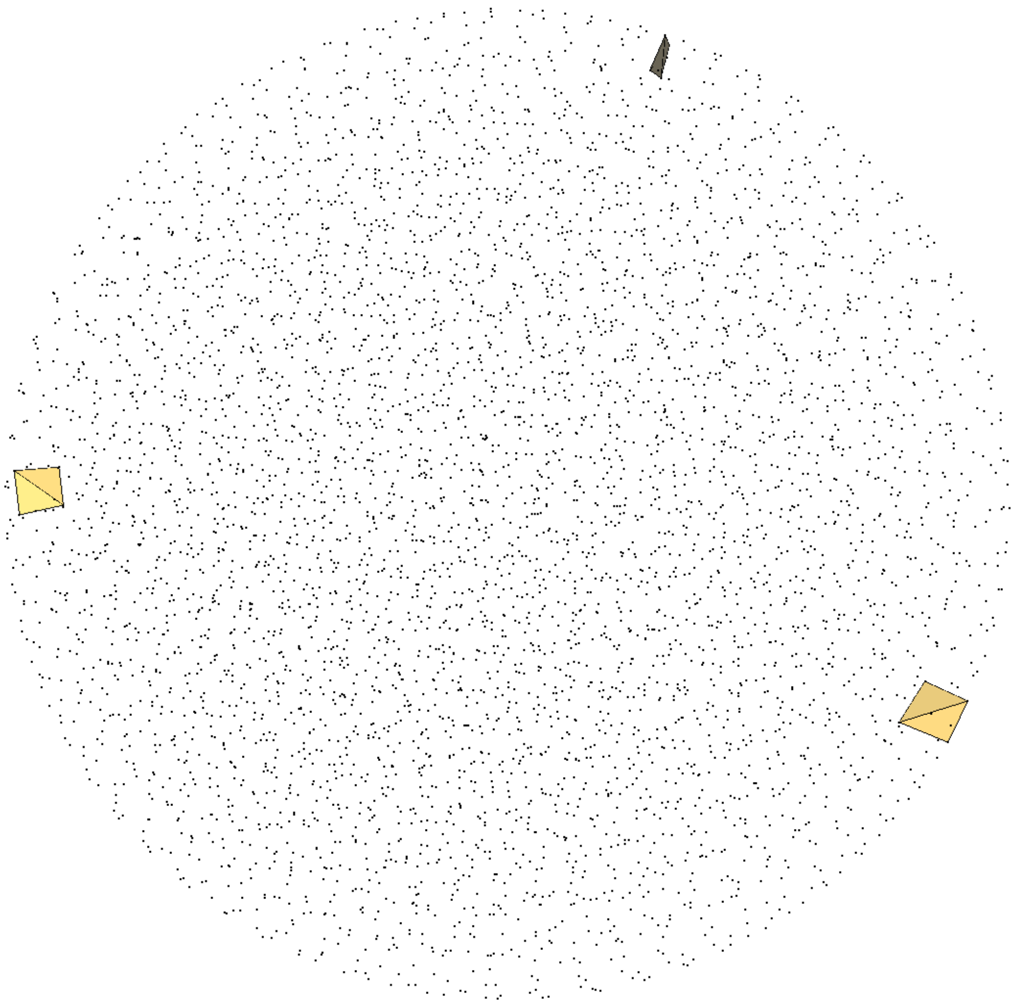
demo



Optimal Delaunay Triangulation



distribution of radius ratios



3 “slivers”, each with two vertices on boundary

Algorithm

- read input boundary $\partial\Omega$
- setup data structure & preprocessing
- compute sizing field
- generate initial sites inside Ω
- do
  - Delaunay triangulation of $\{x_i\}$
  - move sites to optimal locations $\{x_i^*\}$

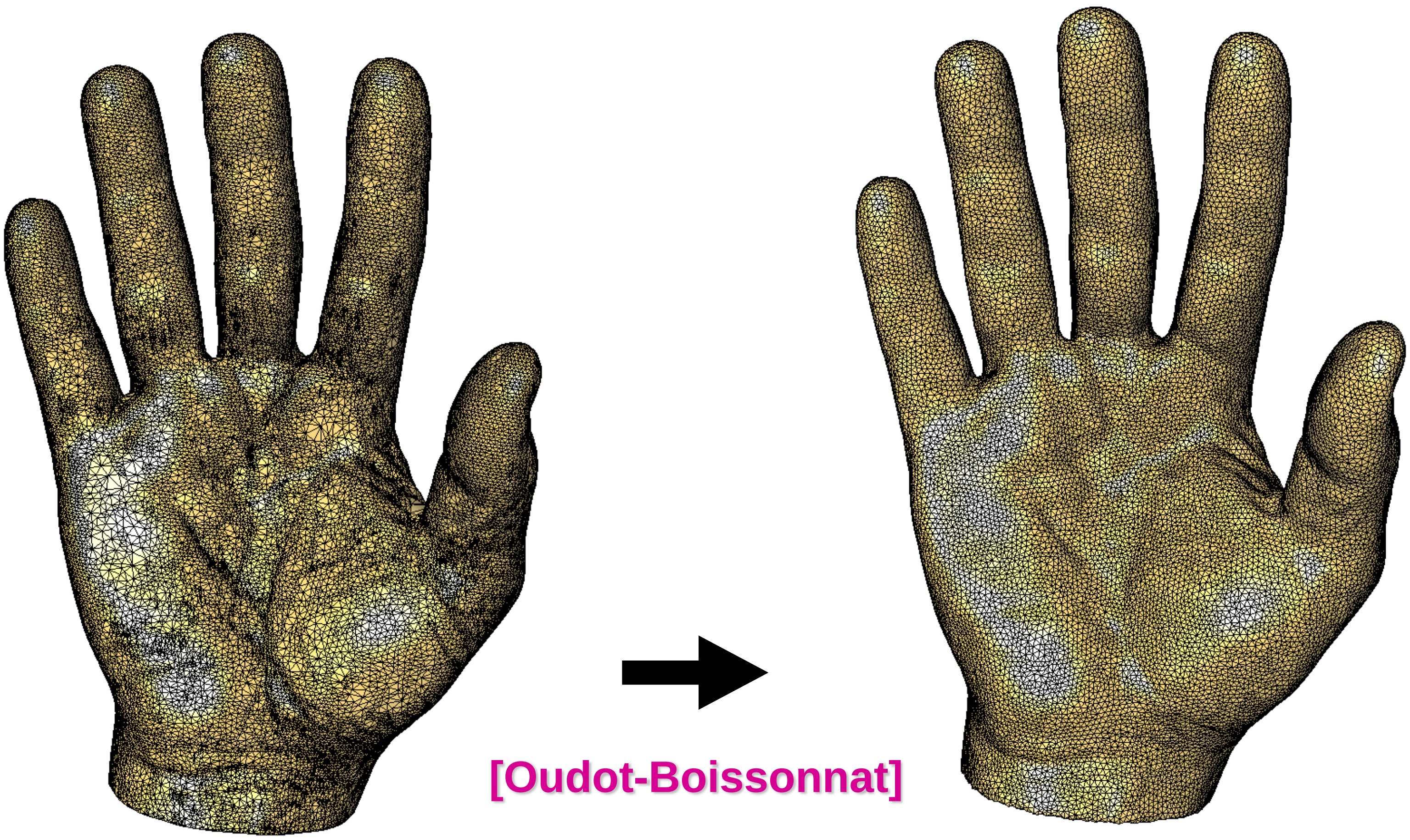
  until convergence or stopping criterion
- extract interior mesh

Input Boundary $\partial\Omega$

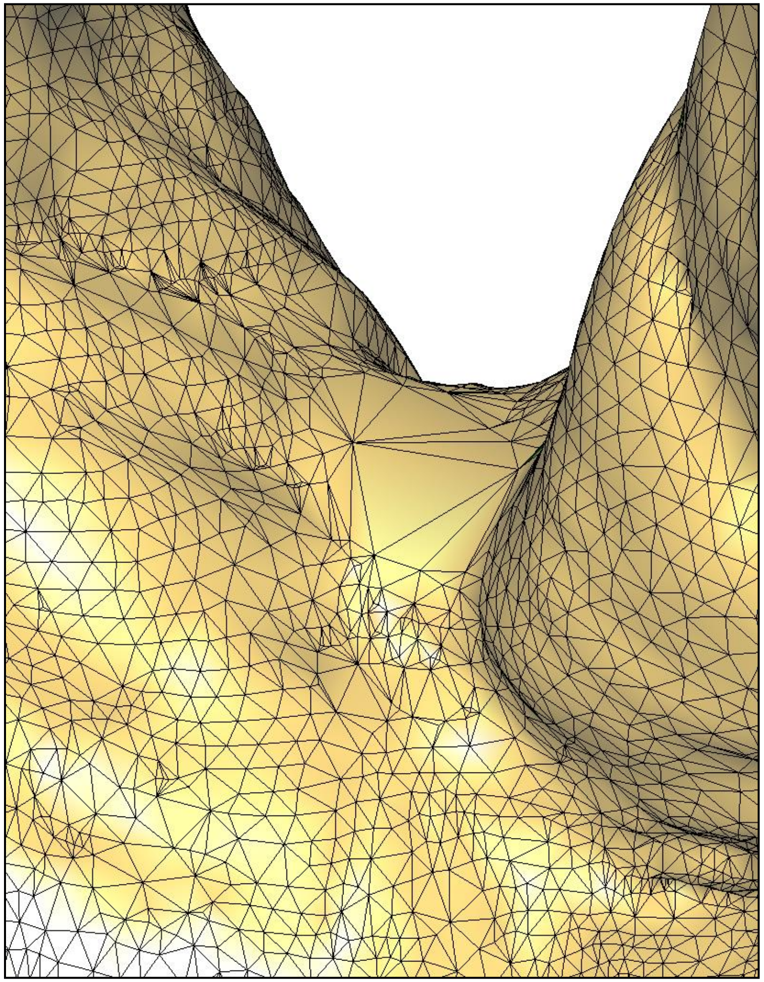
- surface triangle mesh

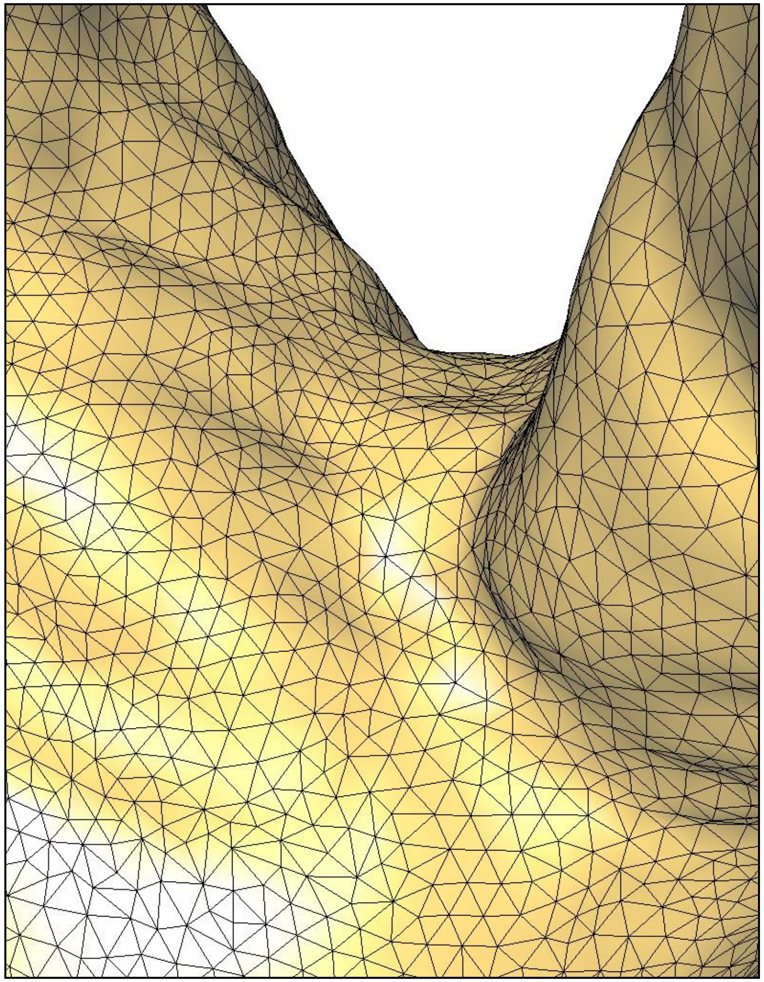
- **Requirements:**
  - intersection free
  - closed
  - restricted Delaunay triangulation of the input vertices [Oudot-Boissonnat, Cohen-Steiner et al.]

Input Boundary ∂Ω

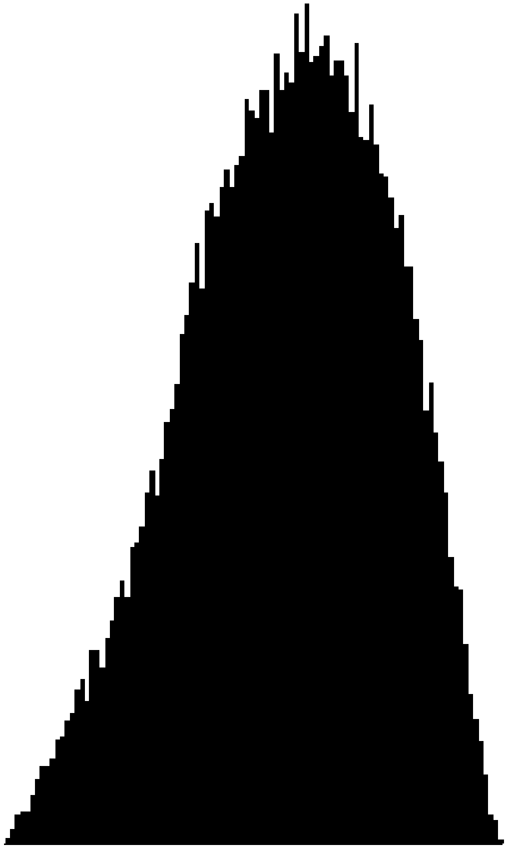


Input Boundary $\partial\Omega$

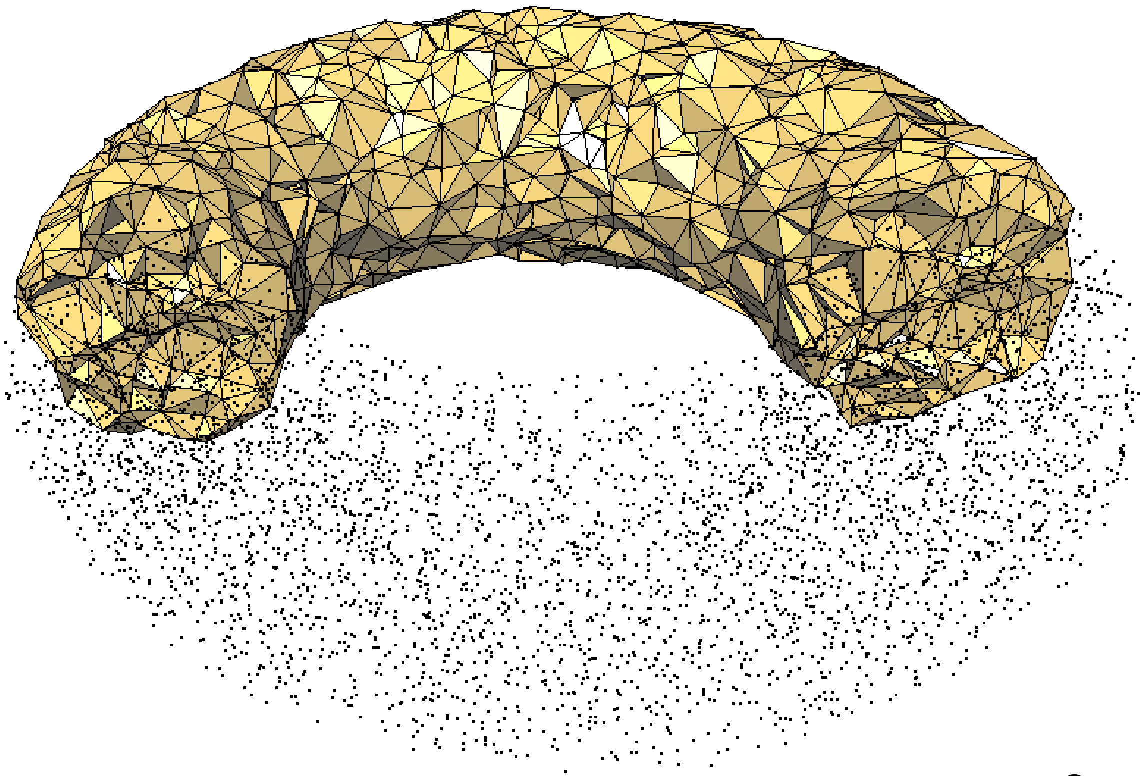




Optimization: init

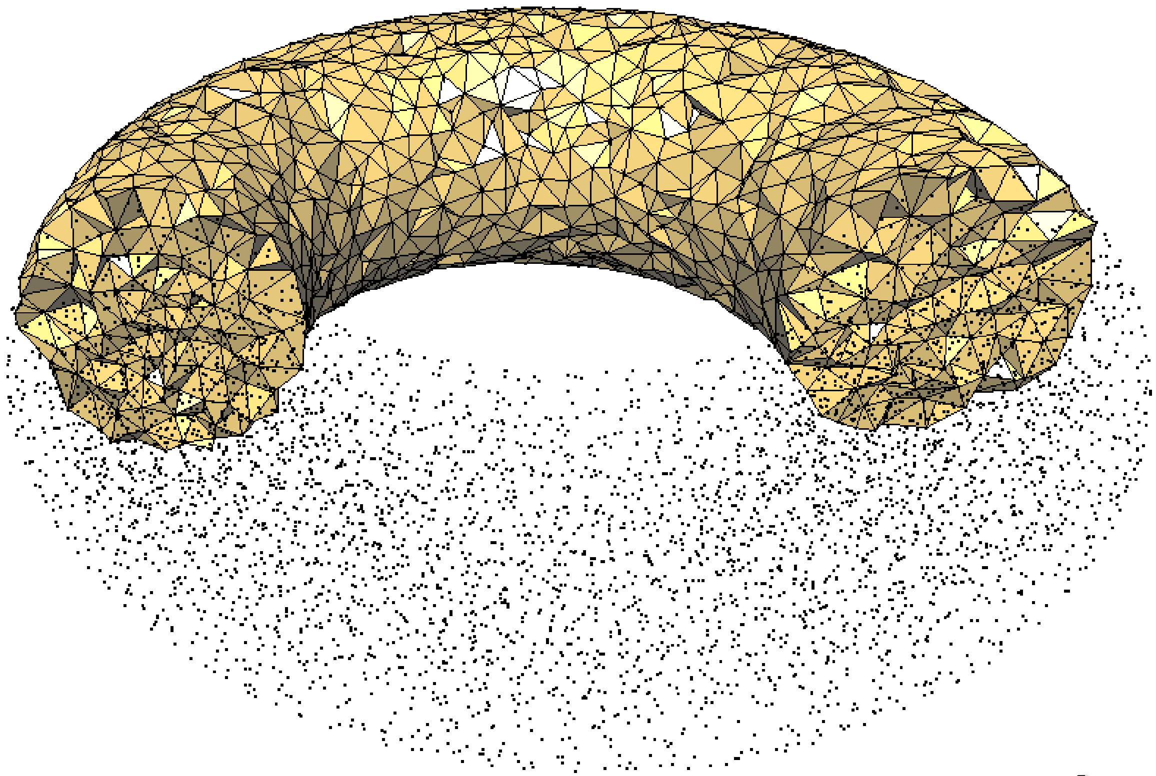


Distribution of radius ratios



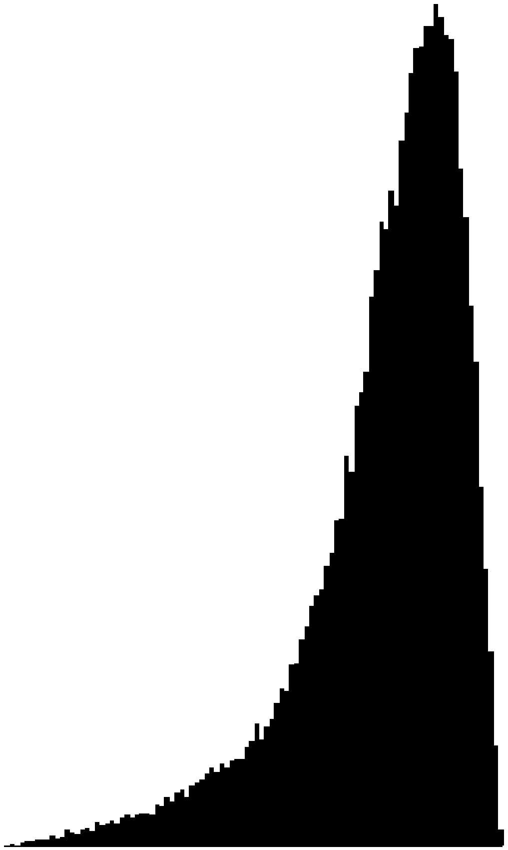
0

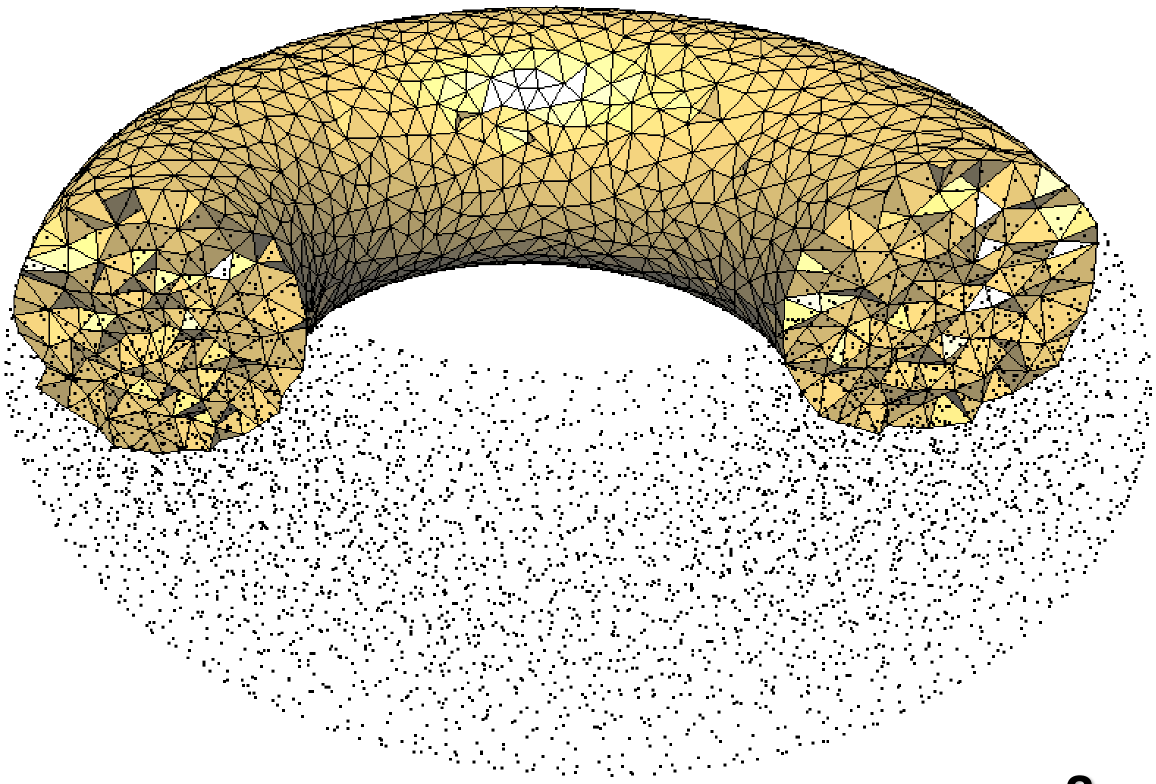
Optimization: step 1



1

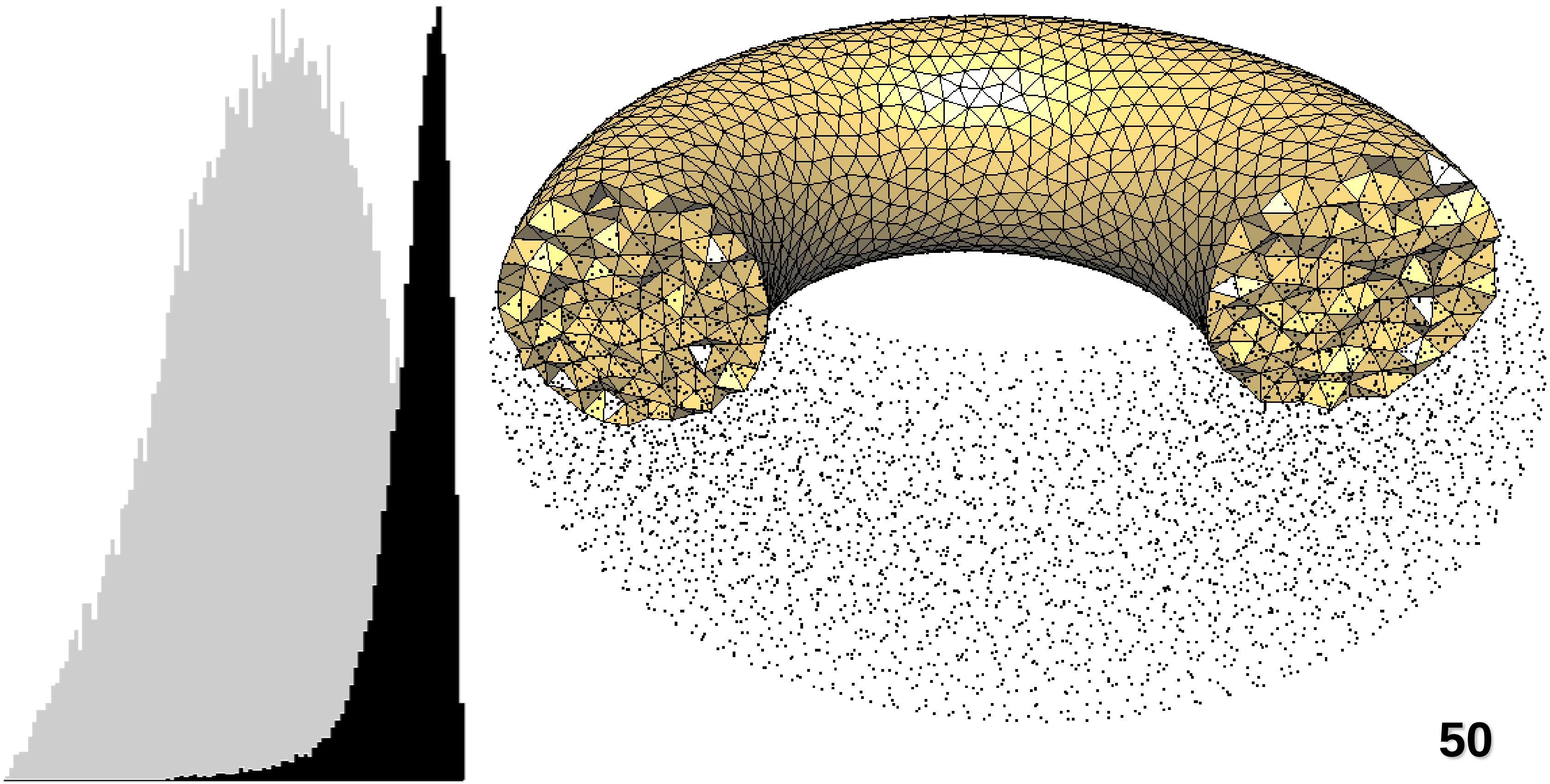
Optimization: step 2



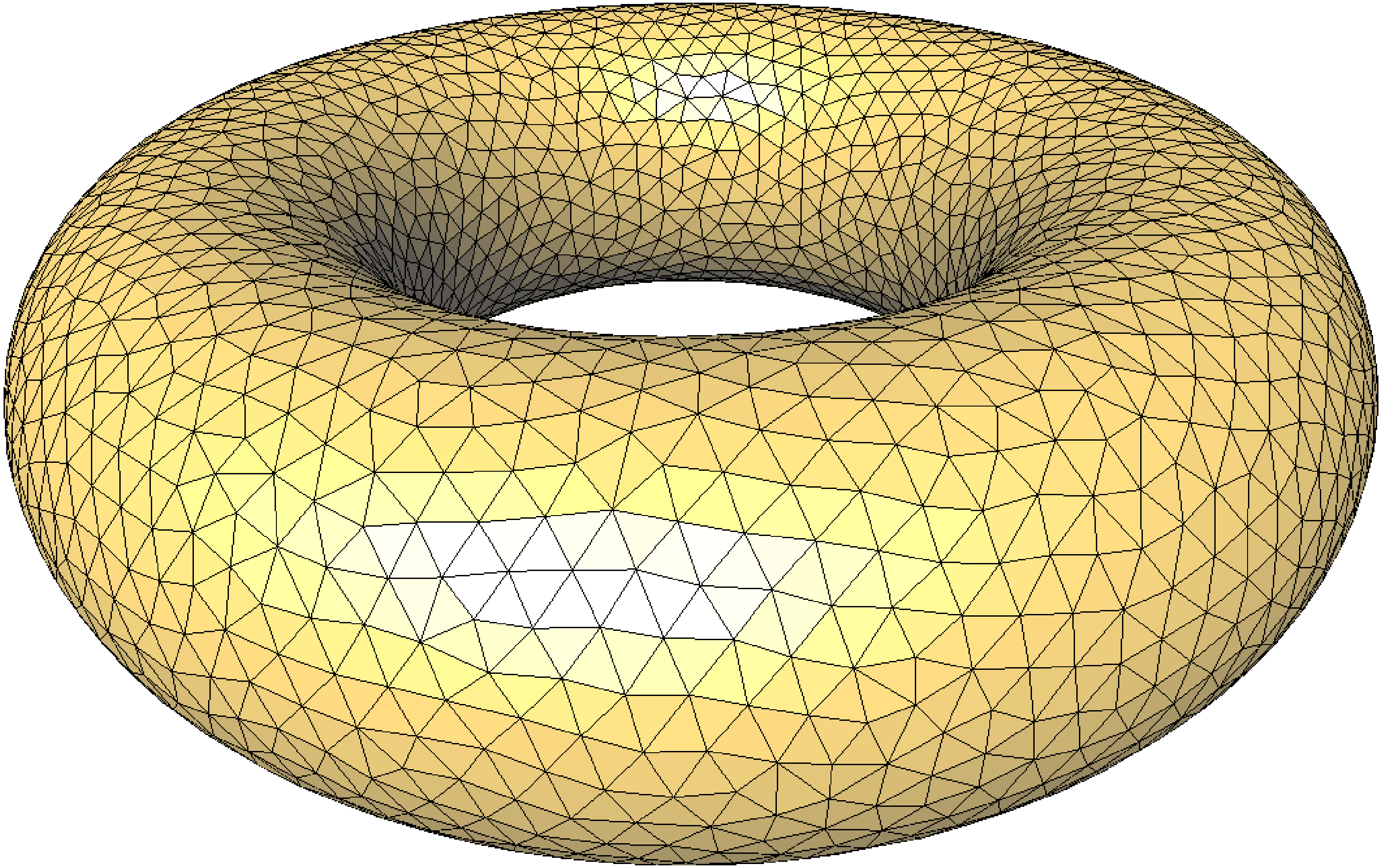


2

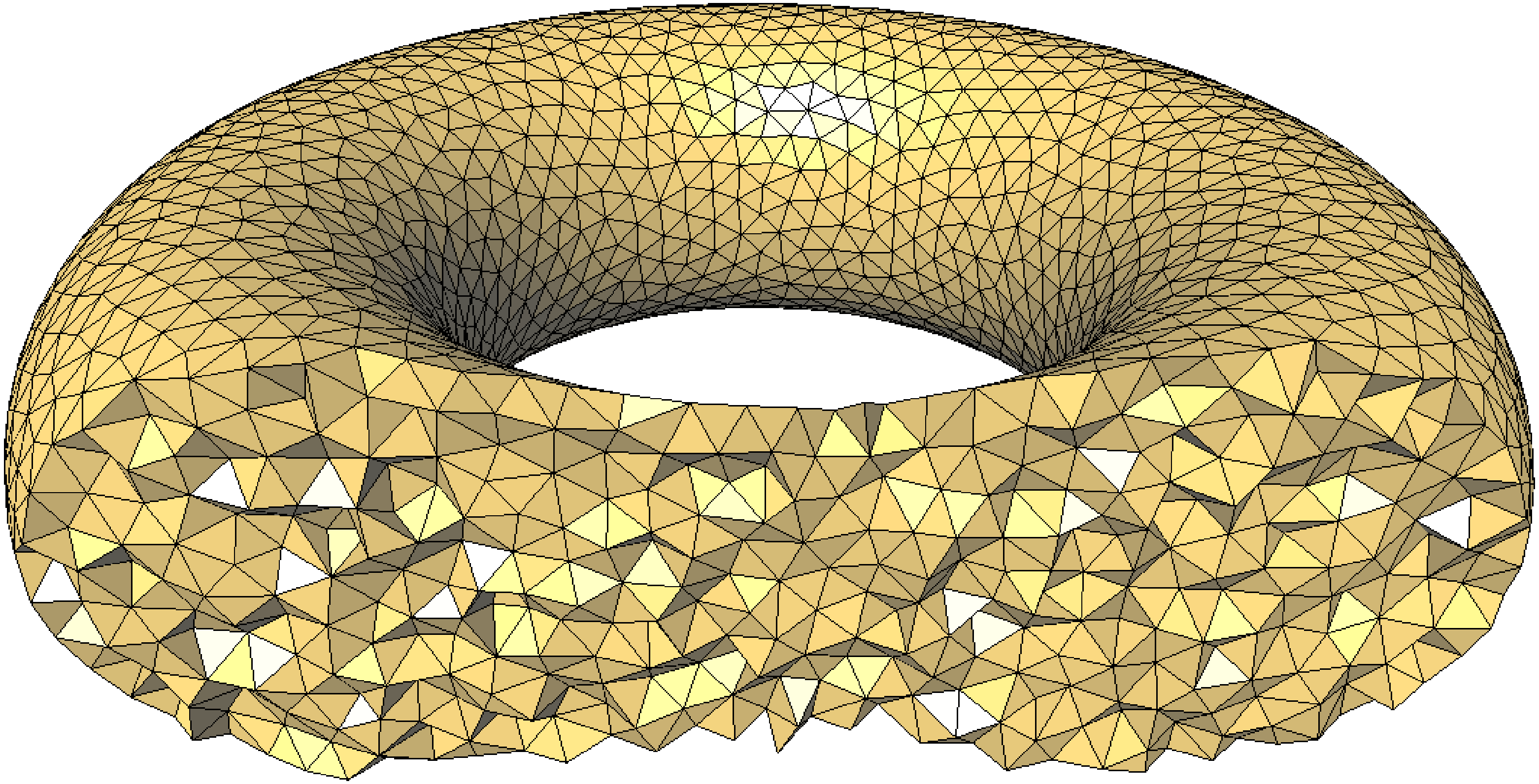
Optimization: step 50



Optimization: step 50

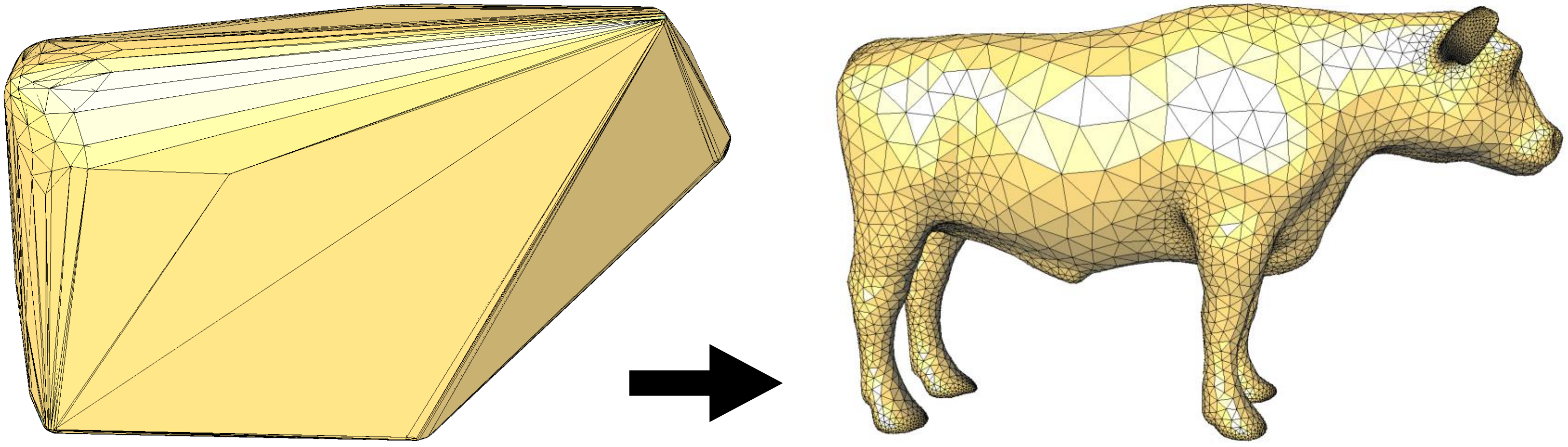


Optimization: step 50

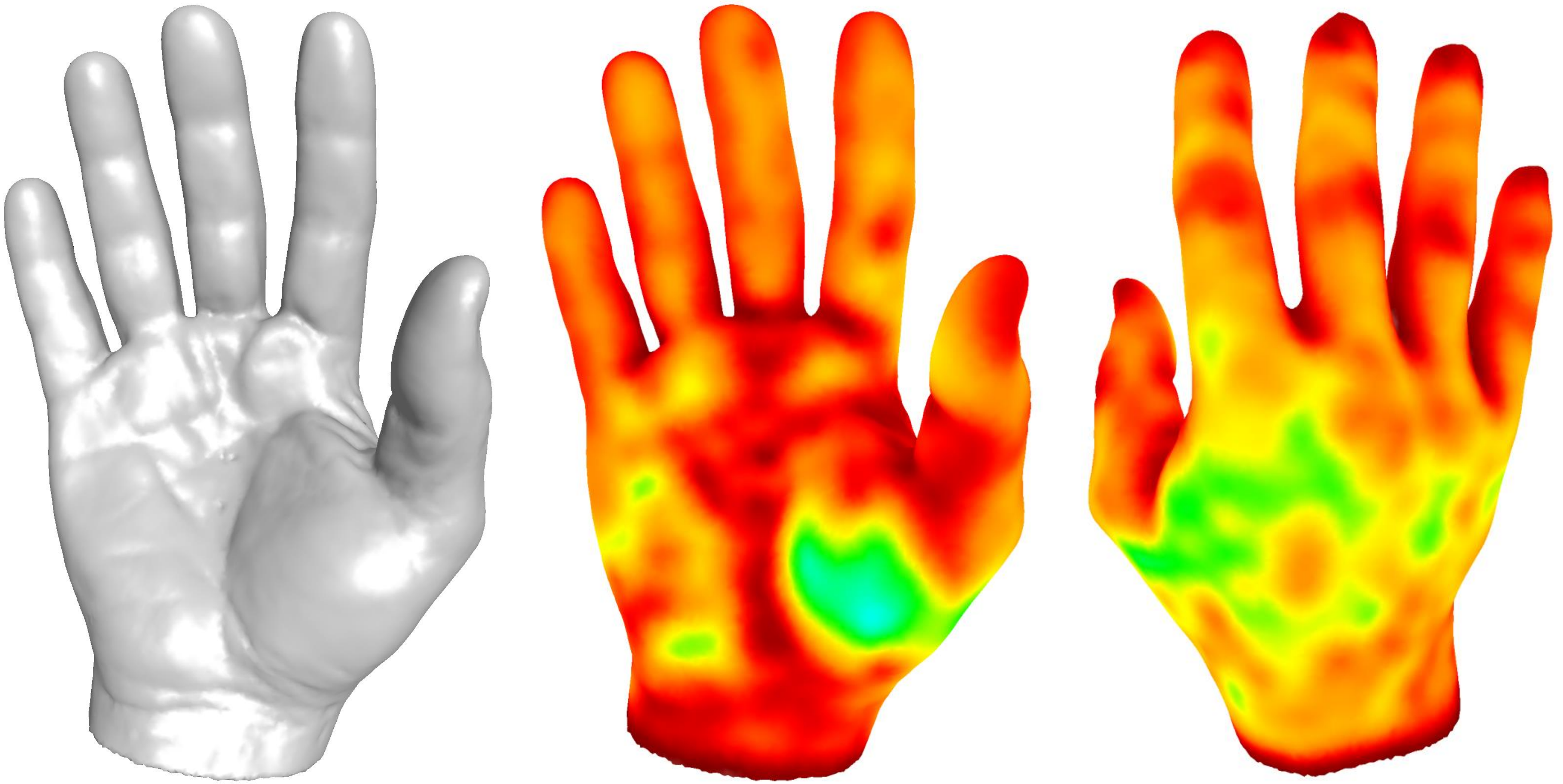


Interior Mesh Extraction

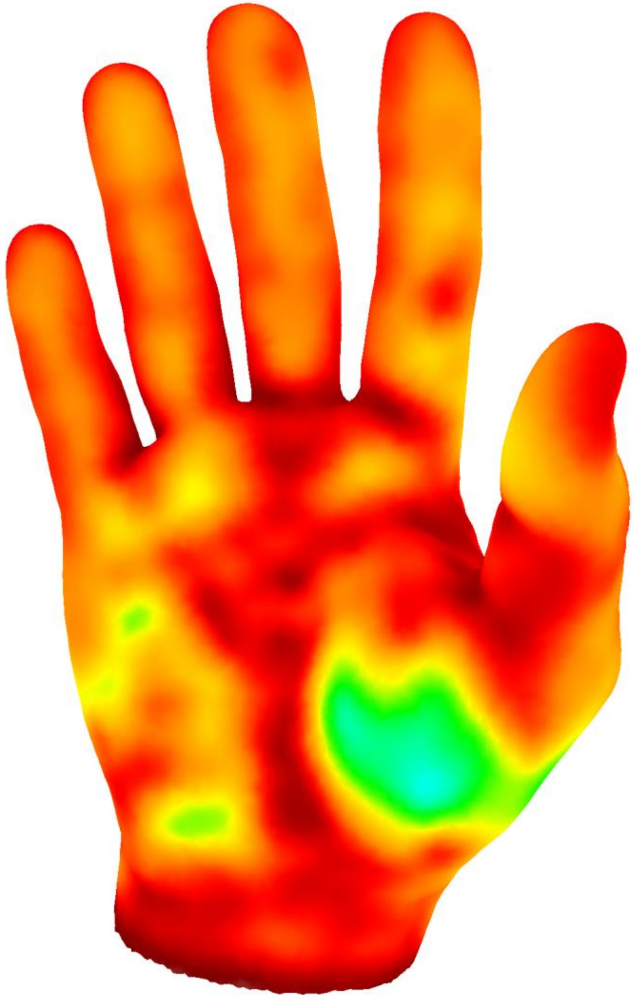
- Delaunay triangulation tessellates the convex hull

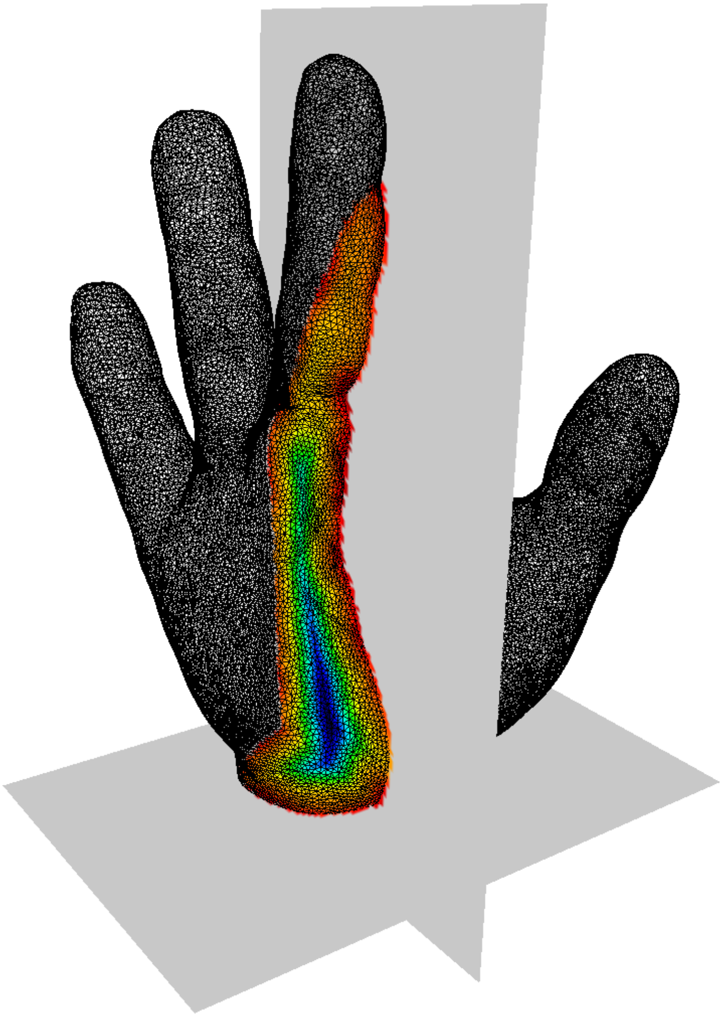


Hand: lfs

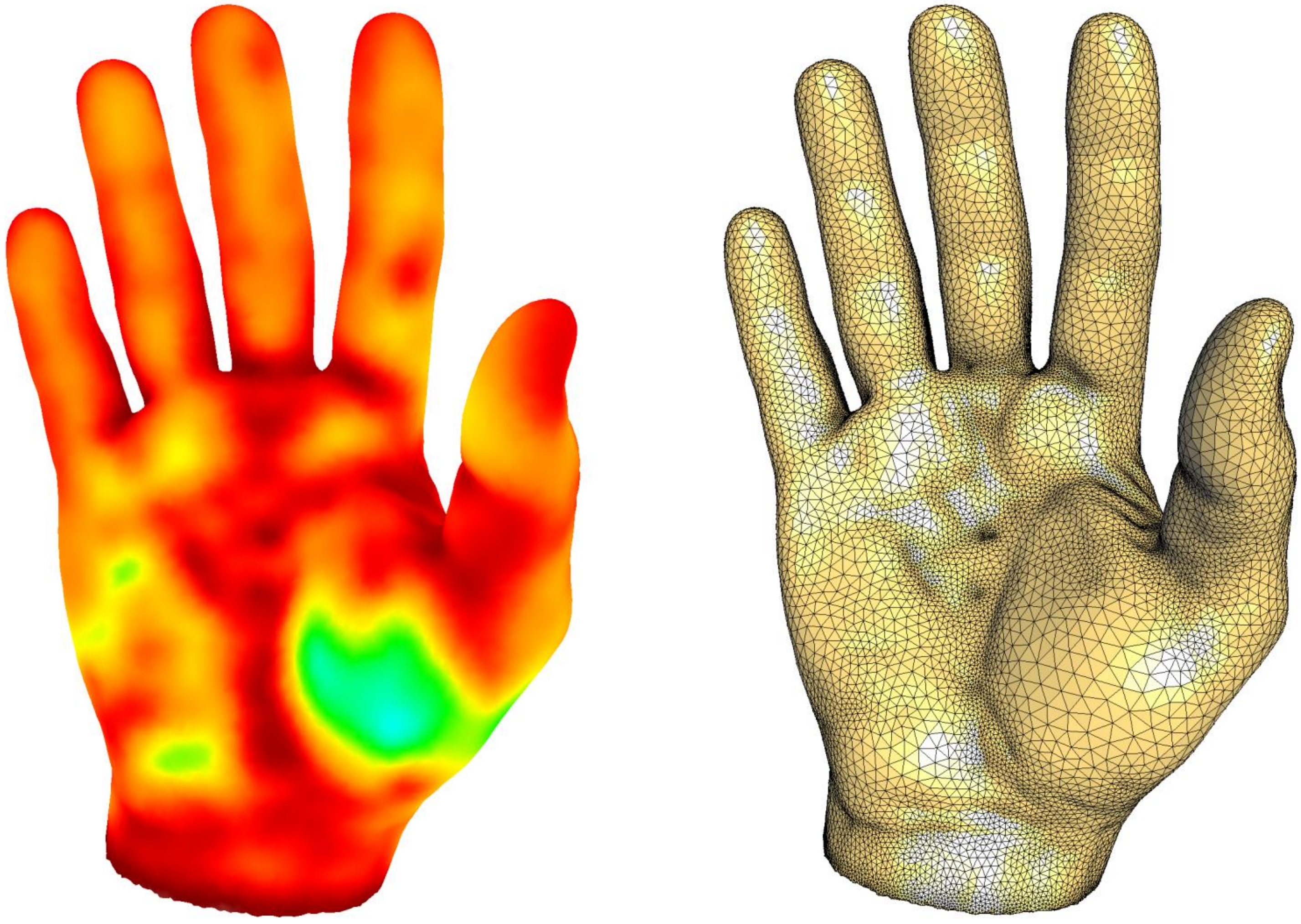


Hand: Sizing

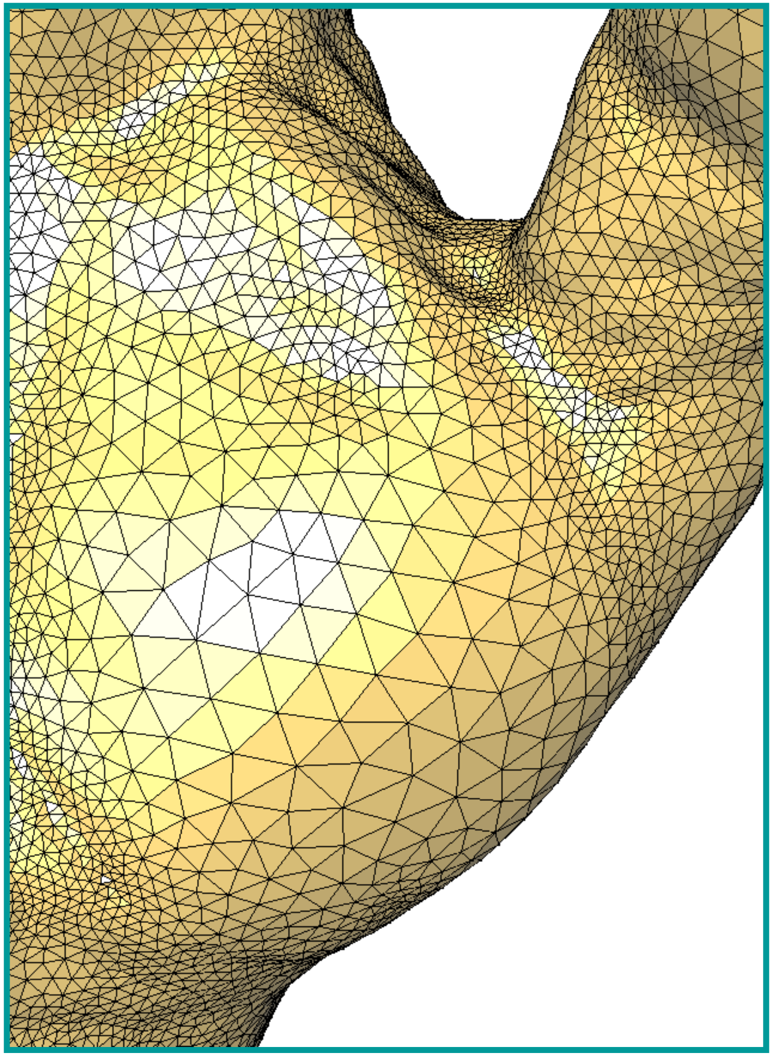


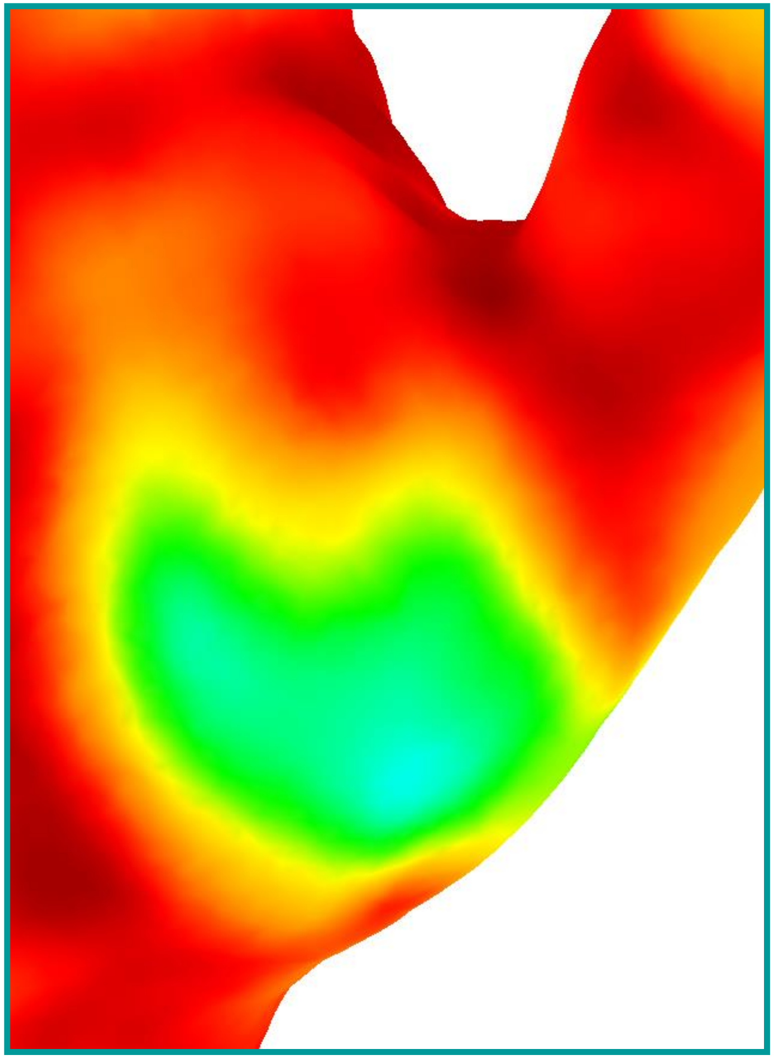


Hand: Sizing

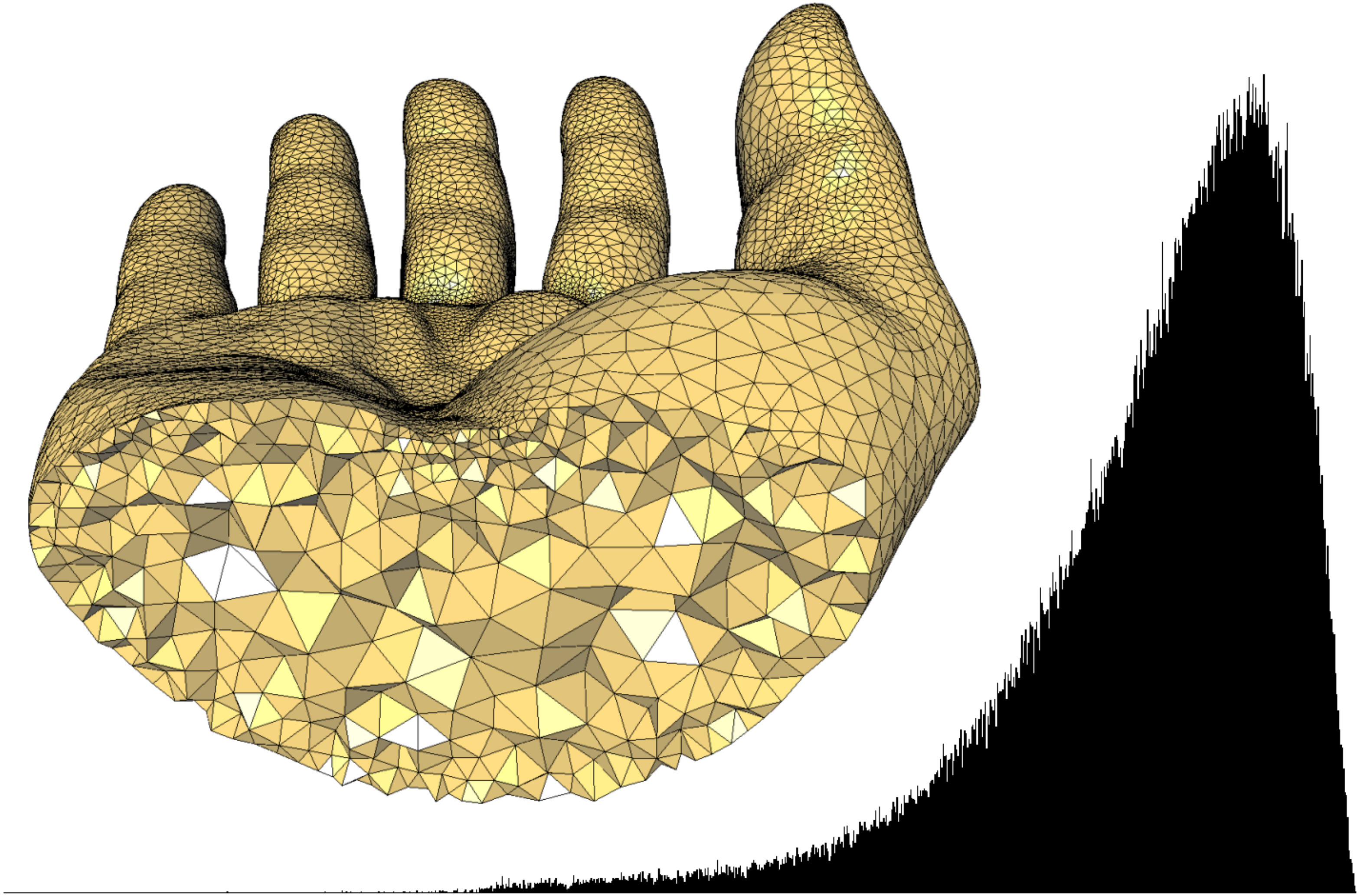


Hand: Sizing

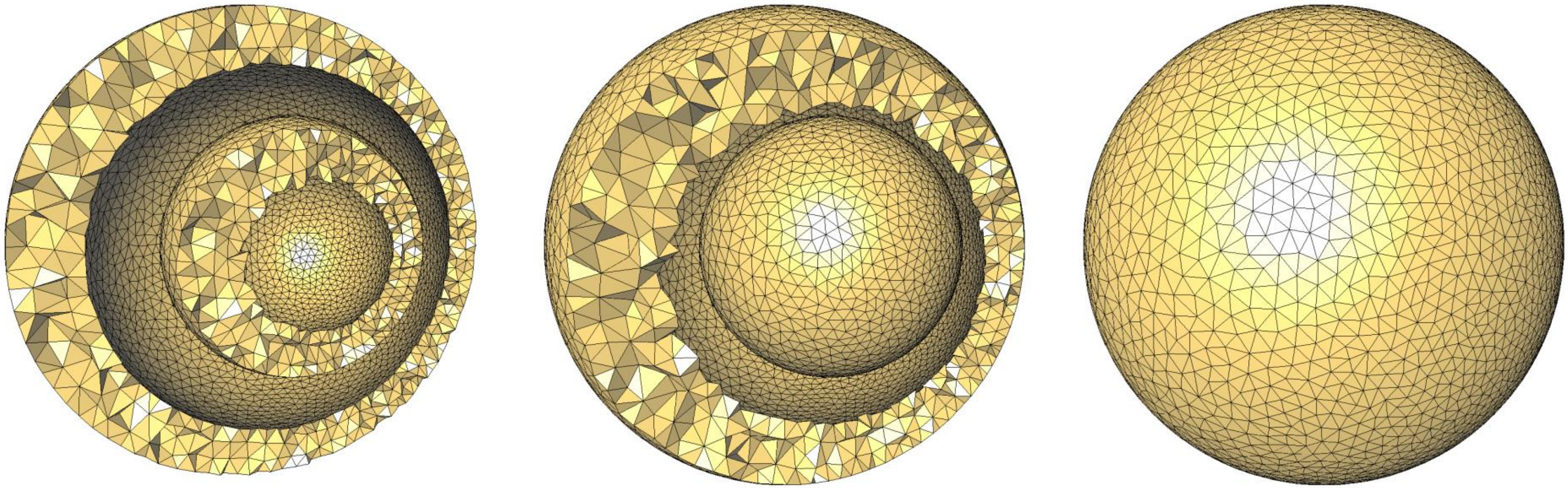




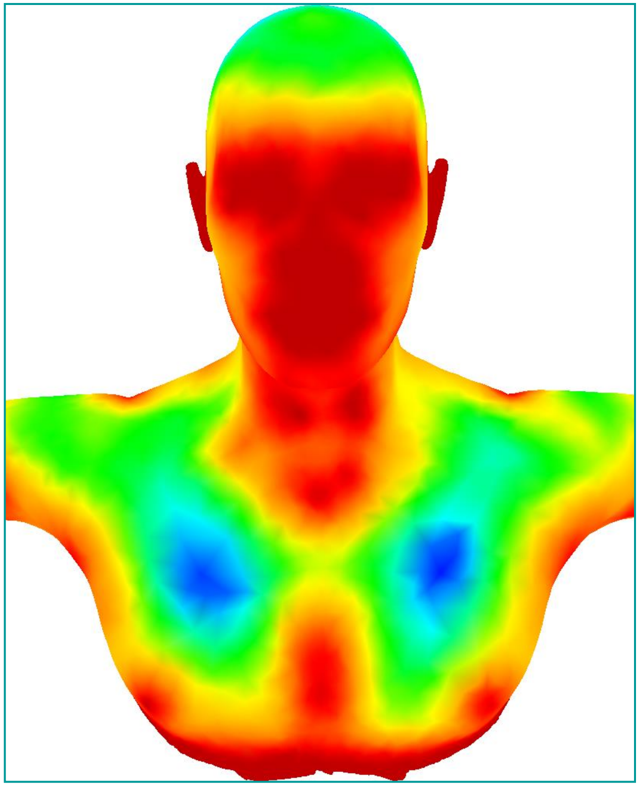
Hand: Radius Ratios

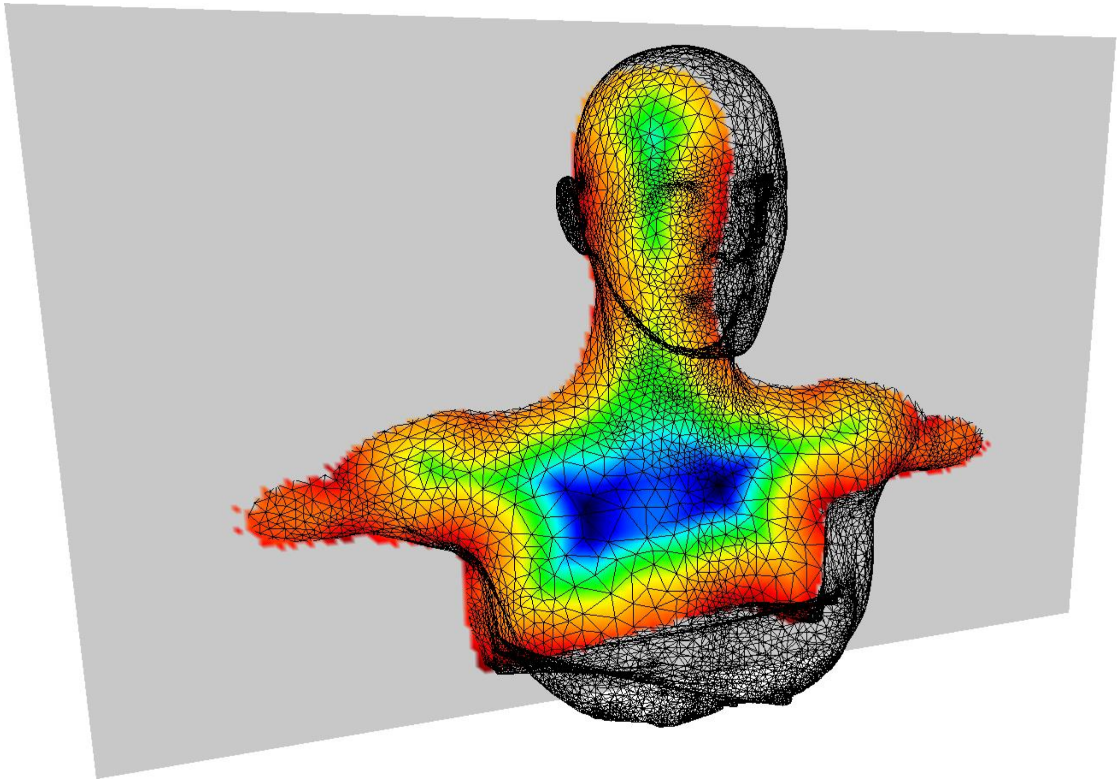


Nested Spheres

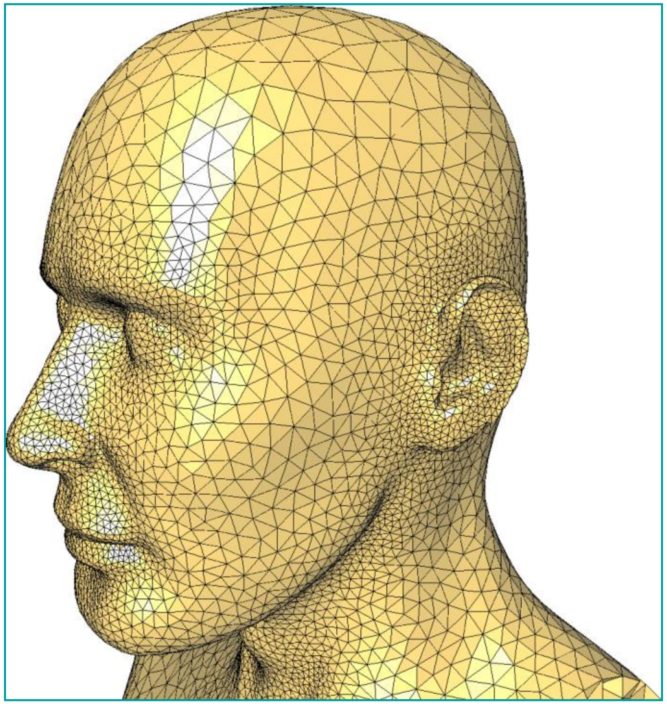


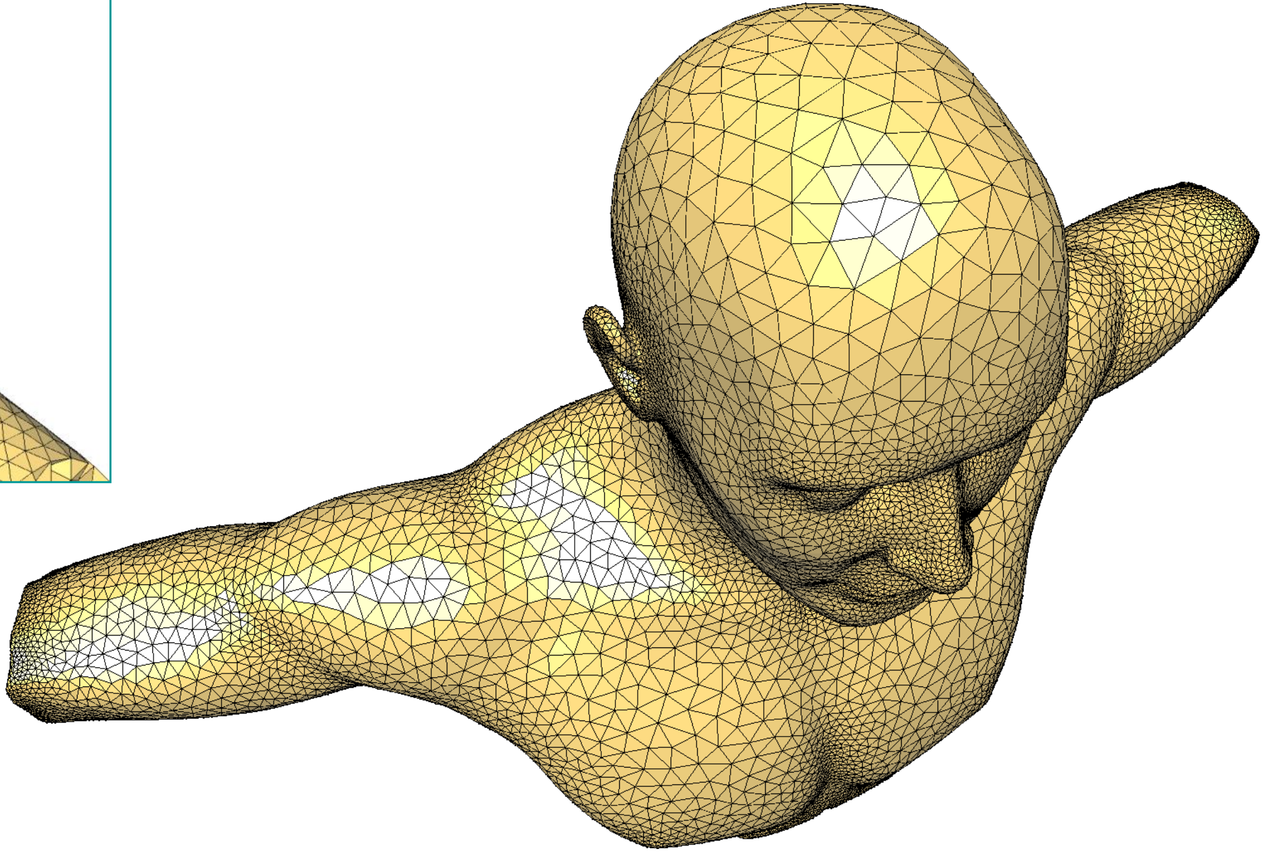
Torso





Torso





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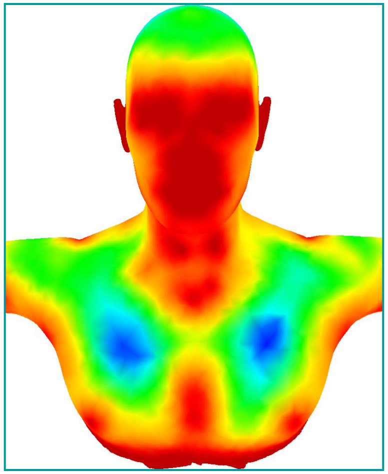

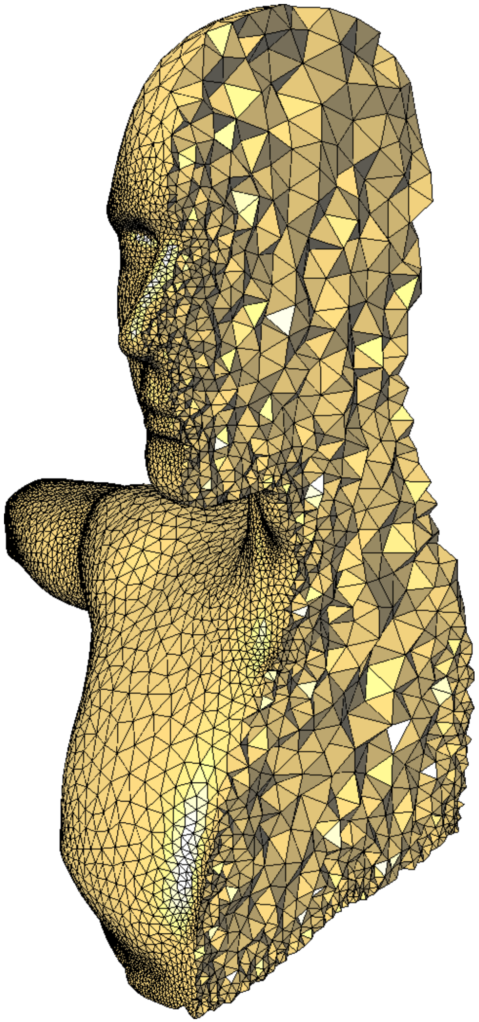

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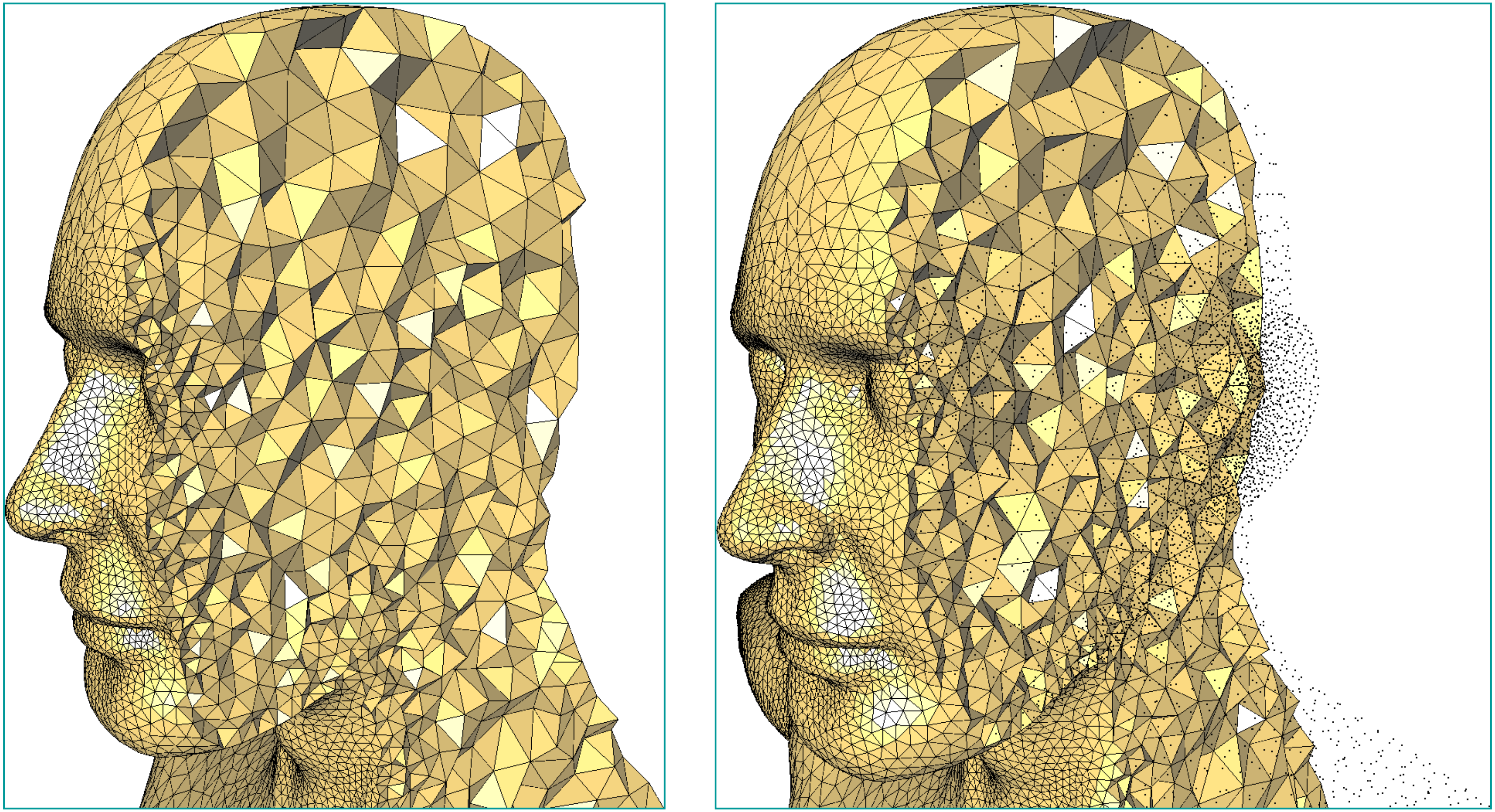

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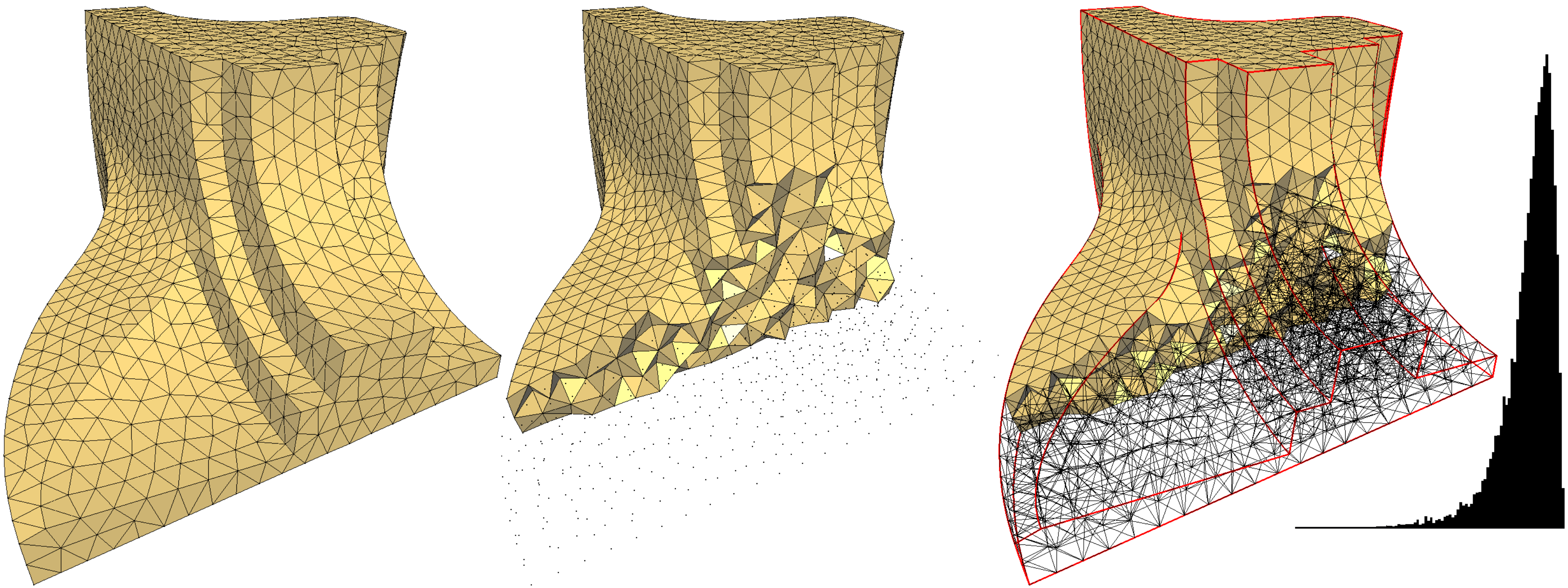

Conclusion

- Meshes
  - Definition, variety
  - Background
    - Voronoi
    - Delaunay
    - constrained Delaunay
    - restricted Delaunay
  - Optimization
    - 2D, 3D
    - Lloyd iteration, function approximation approach