Background on Meshes

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- **Mesh**: Cellular complex partitioning an input domain into elementary cells.
- **Elementary cell**: Admits a bounded description.
- **Cellular complex**: Two cells are disjoint or share a lower dimensional face.
- **Domains**: 2D, 3D, nD, surfaces, manifolds, periodic vs non-periodic, etc.
- **Elements**: simplices (triangles, tetrahedra), quadrangles, polygons, hexahedra, arbitrary cells.
- **Structured mesh**: All interior nodes have an equal number of adjacent elements.
- **Unstructured mesh**: Any number of elements can meet at a single vertex.
- Anisotropic vs Isotropic
- Element distribution: Uniform vs adapted
- **Grading**: Smooth vs fast
- control/optimization over:
  - #elements
  - shape
  - orientation
  - size
  - grading
  - Boundary approximation error
  - ... (application dependent)
- **Input:**
  - Domain boundary + internal constraints
  - Constraints
    - Sizing
    - Grading
    - Shape
    - Topology
    - ...
Voronoi Diagram and Delaunay triangulation
Let $\mathcal{E} = \{p_1, \ldots, p_n\}$ be a set of points (so-called sites) in $\mathbb{R}^d$. We associate to each site $p_i$ its Voronoi region $V(p_i)$ such that:

$$V(p_i) = \{x \in \mathbb{R}^d : \|x - p_i\| \leq \|x - p_j\|, \forall j \leq n\}.$$
- The collection of the non-empty Voronoi regions and their faces, together with their incidence relations, constitute a cell complex called the **Voronoi diagram** of E.
- The locus of points which are equidistant to two sites $p_i$ and $p_j$ is called a **bisector**, all bisectors being affine subspaces of $\mathbb{R}^d$ (lines in 2D).
A Voronoi cell of a site $pi$ defined as the intersection of closed half-spaces bounded by bisectors. Implies: All Voronoi cells are **convex**.
Voronoi cells may be **unbounded** with unbounded bisectors. Happens when a site $p_i$ is on the boundary of the convex hull of $E$. 
- **Voronoi cells** have **faces** of different dimensions.
- In 2D, a face of dimension $k$ is the intersection of $3 - k$ Voronoi cells. A **Voronoi vertex** is generically equidistant from three points, and a **Voronoi edge** is equidistant from two points.
- **Dual structure of the Voronoi diagram.**
- The Delaunay triangulation of a set of sites $E$ is a simplicial complex such that $k+1$ points in $E$ form a Delaunay simplex if their Voronoi cells have nonempty intersection.
The Delaunay triangulation of a point set \( E \) covers the convex hull of \( E \).
- canonical triangulation associated to any point set
- **Empty circle**: A triangulation $T$ of a point set $E$ such that any $d$-simplex of $T$ has a circumsphere that does not enclose any point of $E$ is a Delaunay triangulation of $E$. Conversely, any $k$-simplex with vertices in $E$ that can be circumscribed by a hypersphere that does not enclose any point of $E$ is a face of the Delaunay triangulation of $E$. 

![Diagram of Delaunay triangulation with empty circles](image)
- **In 2D: « quality » triangulation**
  - Smallest triangle angle: The Delaunay triangulation of a point set $E$ is the triangulation of $E$ which maximizes the smallest angle.
  - Even stronger: The triangulation of $E$ whose **angular vector** is **maximal** for the lexicographic order is the Delaunay triangulation of $E$.

![Example diagrams](image.png)

**good**

**bad**
- **Thales’ Theorem**: Let $C$ be a circle, and $l$ a line intersecting $C$ at points $a$ and $b$. Let $p, q, r$ and $s$ be points lying on the same side of $l$, where $p$ and $q$ are on $C$, $r$ inside $C$ and $s$ outside $C$. Then:

\[ \angle arb = 2 \angle apb \]

\[ \angle aqb > \angle asb \]
Improving a triangulation:

- In any convex quadrangle, an *edge flip* is possible. If this flip *improves* the triangulation locally, it also improves the global triangulation.

- If an edge flip improves the triangulation, the first edge is called *illegal*.
Illegal edges:

- **Lemma:** An edge $pq$ is illegal iff one of its opposite vertices is inside the circle defined by the other three vertices.
- **Proof:** By Thales’ theorem.

- **Theorem:** A Delaunay triangulation does not contain illegal edges.
- **Corollary:** A triangle is Delaunay iff the circle through its vertices is empty of other sites (the *empty-circle* condition).
- **Corollary:** The Delaunay triangulation is not unique if more than three sites are co-circular.
Duality on the paraboloid: Delaunay triangulation obtained by projecting the lower part of the convex hull.
Delaunay Triangulation

Project the 2D point set onto the 3D paraboloid

Compute the 3D lower convex hull

Project the 3D facets back to the plane.
- The intersection of a plane with the paraboloid is an ellipse whose projection to the plane is a circle.

- \( s \) lies within the circumcircle of \( p, q, r \) iff \( s' \) lies on the lower side of the plane passing through \( p', q', r' \).

- \( p, q, r \in S \) form a Delaunay triangle iff \( p', q', r' \) form a face of the convex hull of \( S' \).
- Given a set $S$ of points in the plane, associate with each point $p = (a, b) \in S$ the plane tangent to the paraboloid at $p$:
  \[ z = 2ax + 2by - (a^2 + b^2). \]

- $\text{VD}(S)$ is the projection to the $(x, y)$ plane of the 1-skeleton of the convex polyhedron formed from the intersection of the halfspaces above these planes.
An naïve $O(n^4)$ Construction Algorithm

- Repeat until impossible:
  - Select a triple of sites.
  - If the circle through them is empty of other sites, keep the triangle whose vertices are the triple.
**Theorem:** If $a, b, c, d$ form a CCW convex polygon, then $d$ lies in the circle determined by $a$, $b$ and $c$ iff:

$$
\begin{vmatrix}
  a_x & a_y & a_x^2 + a_y^2 & 1 \\
  b_x & b_y & b_x^2 + b_y^2 & 1 \\
  c_x & c_y & c_x^2 + c_y^2 & 1 \\
  d_x & d_y & d_x^2 + d_y^2 & 1
\end{vmatrix} > 0
$$

**Proof:** We prove that equality holds if the points are co-circular. There exists a center $q$ and radius $r$ such that:

$$
(a_x - q_x)^2 + (a_y - q_y)^2 = r^2
$$

and similarly for $b, c, d$:

$$
\begin{pmatrix}
  a_x^2 + a_y^2 \\
  b_x^2 + b_y^2 \\
  c_x^2 + c_y^2 \\
  d_x^2 + d_y^2
\end{pmatrix} - 2q_x
\begin{pmatrix}
  a_x \\
  b_x \\
  c_x \\
  d_x
\end{pmatrix} - 2q_y
\begin{pmatrix}
  a_y \\
  b_y \\
  c_y \\
  d_y
\end{pmatrix} + (q_x^2 + q_y^2 - r^2)
\begin{pmatrix}
  1 \\
  1 \\
  1 \\
  1
\end{pmatrix} = 0
$$

So these four vectors are linearly dependent, hence their det vanishes.

**Corollary:** $d \in \text{circle}(a,b,c)$ iff $b \in \text{circle}(c,d,a)$ iff $c \notin \text{circle}(d,a,b)$ iff $a \notin \text{circle}(b,c,d)$
Another naive construction:

- Start with an arbitrary triangulation. Flip any illegal edge until no more exist.
- Requires proof that there are no local minima.
- Could take a long time to terminate.
Incremental algorithm:

- Form bounding triangle which encloses all the sites.
- Add the sites one after another in random order and update triangulation.
- If the site is inside an existing triangle:
  - Connect site to triangle vertices.
  - Check if a 'flip' can be performed on one of the triangle edges. If so - check recursively the neighboring edges.
- If the site is on an existing edge:
  - Replace edge with four new edges.
  - Check if a 'flip' can be performed on one of the opposite edges. If so - check recursively the neighboring edges.
A new vertex $p_r$ is added, causing the creation of edges.

The legality of the edge $p_i p_j$ (with opposite vertex) $p_k$ is checked.

If $p_i p_j$ is illegal, perform a flip, and recursively check edges $p_i p_k$ and $p_j p_k$, the new edges opposite $p_r$.

Notice that the recursive call for $p_i p_k$ cannot eliminate the edge $p_r p_k$.

Note: All edge flips replace edges opposite the new vertex by edges incident to it!
Flipping Edges - Example

\[ p_i \quad p_k \quad p_j \]
Theorem: The expected number of edges flips made in the course of the algorithm (some of which also disappear later) is at most $6n$.

Proof: During insertion of vertex $p_i$, $k_i$ new edges are created: 3 new initial edges, and $k_i - 3$ due to flips.

Backward analysis: $E[k_i] = \text{the expected degree of } p_i \text{ after the insertion is complete} = 6$ (Euler).
- Point location for every point: $O(\log n)$ time.
- Flips: $\Theta(n)$ expected time in total (for all steps).
- Total expected time: $O(n \log n)$.
- Space: $\Theta(n)$.
Constrained Delaunay triangulation
Definition 1: Let \((P, S)\) be a PSLG. The constrained triangulation \(T(P, S)\) is constrained Delaunay iff the circumcircle of any triangle \(t\) of \(T\) encloses no vertex visible from a point in the relative interior of \(t\).
Definition 2: Let \((P, S)\) be a PSLG. The constrained triangulation \(T(P, S)\) is constrained Delaunay iff any edge \(e\) of \(T\) is either a segment of \(S\) or is constrained Delaunay.

Simplex \(e\) constrained Delaunay with respect to the PSLG \((P, S)\) iff: \(\text{int}(e) \cap S = 0\)

There exists a circumcircle of \(e\) that encloses no vertex visible from a point in the relative interior of \(e\).
• Any PSLG (P, S) has a constrained Delaunay triangulation. If (P, S) has no degeneracy, this triangulation is unique.
(see [Shewchuk, Si])
Delaunay Filtering
The Voronoi diagram restricted to a curve $S$, $\text{Vor}_{|S}(E)$, is the set of edges of $\text{Vor}(E)$ that intersect $S$. 
The restricted Delaunay triangulation restricted to a curve $S$ is the set of edges of the Delaunay triangulation whose dual edges intersect $S$. 

(2D)
The restricted Delaunay triangulation restricted to a surface $S$ is the set of **triangles** of the Delaunay triangulation whose dual edges **intersect** $S$. 
Delaunay/Gabriel Conforming
An edge is said to be a **Delaunay edge**, if it is inscribed in an empty circle.
An edge is said to be a **Gabriel edge**, if its diametral circle is empty.
A constrained Delaunay triangulation is a **conforming Delaunay triangulation**, if every constrained edge is a Delaunay edge.
A constrained Delaunay triangulation is a conforming Gabriel triangulation, if every constrained edge is a Gabriel edge.
Any constrained Delaunay triangulation can be **refined** into a conforming Delaunay or Gabriel triangulation by adding **Steiner vertices**.