Convex Hull
Convexity and Convex Hull

• A set $S$ is **convex** if any pair of points $p, q \in S$ satisfy $pq \subseteq S$.

• The **convex hull** of a set $S$ is:
  - The minimal convex set that contains $S$, i.e. any convex set $C$ such that $S \subseteq C$ satisfies $\text{CH}(S) \subseteq C$.
  - The intersection of all convex sets that contain $S$.
  - The set of all convex combinations of $p_i \in S$, i.e. all points of the form:
    $$\sum_{i=1}^{n} \alpha_i p_i \ , \quad \alpha_i \geq 0, \quad \sum_{i=1}^{n} \alpha_i = 1$$
Convex Hulls - Some Facts

- The convex hull of a set is unique (up to colinearities).
- The boundary of the convex hull of a point set is a polygon on a subset of the points.
Convex Hull - Naive Algorithm

• **Description:**
  - For each pair of points construct its connecting segment and *supporting line*.
  - Find all the segments whose supporting lines divide the plane into two halves, such that one half plane contains *all* the other points.
  - Construct the convex hull out of these segments.

• **Time complexity:**
  - All pairs:
    \[ O\left(\binom{n}{2}\right) = O\left(\frac{n(n-1)}{2}\right) = O(n^2) \]
  - Check all points for each pair: \( O(n) \)
  - Total: \( O(n^3) \)
Possible Pitfalls

- **Degenerate cases** - e.g. 3 collinear points. Might harm the correctness of the algorithm. Segments AB, BC and AC will all be included in the convex hull.

- **Numerical problems** - We might conclude that *none* of the three segments belongs to the convex hull.
Convex Hull - Graham’s Scan

• **Algorithm:**
  • Sort the points according to their $x$ coordinates.
  • Construct the upper boundary by scanning the points in the sorted order and performing only “right turns”.
  • Construct the lower boundary (with “left turns”).
  • Concatenate the two boundaries.

• **Time Complexity:** $O(n \log n)$

• **May be implemented using a stack**

• **Question:** How do we check for “right turn”? 
The Algorithm

- Sort the points in increasing order of $x$-coord: $p_1, \ldots, p_n$.
- Push($S, p_1$); Push($S, p_2$);
- For $i = 3$ to $n$ do
  - While Size($S$) $\geq 2$ and Orient($p_i, \text{top}(S), \text{second}(S)$) $\leq 0$
    do Pop($S$);
  - Push($S, p_i$);
- Print($S$);
Graham’s Scan - Time Complexity

• Sorting - $O(n \log n)$

• If $D_i$ is number of points popped on processing $p_i$,

\[
\text{time} = \sum_{i=1}^{n} (D_i + 1) = n + \sum_{i=1}^{n} D_i
\]

• Each point is pushed on the stack only once.

• Once a point is popped - it cannot be popped again.

• Hence

\[
\sum_{i=1}^{n} D_i \leq n
\]
Graham’s Scan - a Variant

**Algorithm:**
- Find one point, $p_0$, which must be on the convex hull.
- Sort the other points by the angle of the rays to them from $p_0$.
- **Question:** Is it necessary to compute the actual angles?
- Construct the convex hull using one traversal of the points.

**Time Complexity:** $O(n \log n)$

**Question:** What are the pros and cons of this algorithm relative to the previous?
Convex Hull - Divide and Conquer

• Algorithm:
  • Find a point with a median x coordinate (time: $O(n)$)
  • Compute the convex hull of each half (recursive execution)
  • Combine the two convex hulls by finding common tangents. This can be done in $O(n)$.

• Complexity: $O(n \log n)$
Finding Common Tangents
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To find lower tangent:

- Find \( a \) - the rightmost point of \( H_A \) \( \{ \text{O(n)} \} \)
- Find \( b \) – the leftmost point of \( H_B \)

While \( ab \) is not a lower tangent for \( H_A \) and \( H_B \), do:
  - If \( ab \) is not a lower tangent to \( H_A \) do \( a = a-1 \)
  - If \( ab \) is not a lower tangent to \( H_B \) do \( b = b-1 \)
Output-Sensitive Convex Hull Gift Wrapping

Algorithm:

- Find a point $p_1$ on the convex hull (e.g. the lowest point).
- Rotate counterclockwise a line through $p_1$ until it touches one of the other points (start from a horizontal orientation).

**Question:** How is this done?

- Repeat the last step for the new point.
- Stop when $p_1$ is reached again.

Time Complexity: $O(nh)$, where $n$ is the input size and $h$ is the output (hull) size.
General Position

- When designing a geometric algorithm, we first make some simplifying assumptions, e.g.
  - No 3 collinear points.
  - No two points with the same x coordinate.
  - etc.

- Later, we consider the general case:
  - How should the algorithm react to degenerate cases?
  - Will the correctness be preserved?
  - Will the runtime remain the same?
Lower Bound for Convex Hull

A reduction from sorting to convex hull is:

- Given $n$ real values $x_i$, generate $n$ 2D points on the graph of a convex function, e.g. $(x_i, x_i^2)$.
- Compute the (ordered) convex hull of the points.
- The order of the convex hull points is the numerical order of the $x_i$.

So $\text{CH} = \Omega(n \lg n)$