Mesh Optimization

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Goals

- 3D simplicial mesh generation
- Optimize shape of elements
  - For matrix conditioning
  - Isotropic
- Control over sizing
  - Dictated by simulation
  - Constrained by boundary
  - Low number of elements desired
    - More elements = slower solution time
Popular Meshing Approaches

- advancing front
- specific subdivision
  - octree
  - lattice (e.g. body centered cubic)
- Delaunay
  - refinement
  - sphere packing

- spring energy
  - Laplacian
  - non-zero rest length
- aspect / radius ratios
- dihedral / solid angles
- max-min/min-max
  - volumes
  - edge lengths
  - containing sphere radii

[Freitag Amenta Bern Eppstein]

- sliver exudation

[Edelsbrunner Goy]
Variational?

- Design one energy function such that good solutions correspond to low energy ones (global minimum in general a mirage).
- Solutions found by optimization techniques.
Example Energy in 2D

\[ E = \sum_{j=1}^{k} \int \| x - x_j \|^2 \, dx \]
Lloyd Iteration
2D Optimized Triangle Meshing
2D Optimized Triangle Meshing
Delaunay refinement

Termination

- shape criterion: radius-edge ratio
- in 2D: max $\sqrt{2}$ (implies min 20.7°)
- in 3D: max 2  (nothing similar on dihedral angles)

[Chew, Ruppert, Shewchuk, ...]
Delaunay refinement

+ greedy (fast)
+ easy incorporation of sizing field
+ allows boundary conforming
  - possibly with Steiner points
  - even for sharp angles on boundary [Teng]
+ guaranteed bounds on radius-edge ratio
- blind to slivers
  - and experimentally...produces slivers
Background
Delaunay Triangulation

- **Duality on the paraboloid**: Delaunay triangulation obtained by projecting the lower part of the convex hull.
Delaunay Triangulation

Project the 2D point set onto the 3D paraboloid

Compute the 3D lower convex hull

Project the 3D facets back to the plane.
Proof

- The intersection of a plane with the paraboloid is an ellipse whose projection to the plane is a circle.
- $s$ lies within the circumcircle of $p, q, r$ iff $s'$ lies on the lower side of the plane passing through $p', q', r'$.
- $p, q, r \in S$ form a Delaunay triangle iff $p', q', r'$ form a face of the convex hull of $S'$. 
Voronoi Diagram

- Given a set $S$ of points in the plane, associate with each point $p=(a,b) \in S$ the plane tangent to the paraboloid at $p$:
  \[ z = 2ax + 2by - (a^2 + b^2). \]

- $VD(S)$ is the projection to the $(x,y)$ plane of the 1-skeleton of the convex polyhedron formed from the intersection of the halfspaces above these planes.
First Idea: Lloyd Algorithm

(after Lloyd relaxation)

...back to primal?
Centroidal Voronoi Tessellation

Occurrences

distribution of radius ratios

220 “slivers” (tets with radius ratio < 0.2)

5K sites
30K tets
Tetrahedra Zoo

well-spaced points generate only round or sliver tetrahedra
Key Idea

- adopt the "function approximation" point of view [Chen 04] Optimal Delaunay Triangulation
- 1D: \( f(x) = x^2 \) centered at any vertex
- minimize the \( L^1 \) norm between \( f \) and PWL interpolation
Key Idea

- 3D: $\|x\|^2$ (graph in IR$^4$)
- approximation theory:
  - linear interpolation: optimal shape of the element related to the Hessian of $f$ [Shewchuk]
- Hessian($\|x\|^2$) = Id
  - regular tetrahedron best
- note: FE ~ mesh that best interpolates a function + matrix conditioning
Key Question

- which mesh best approximates the paraboloid?
  - (PWL interpolates)

Answers:

- for fixed point locations
  - Delaunay (lifts to lower facets of convex hull)

- for fixed connectivity
  - quadratic energy
  - closed form for local optimum
Function Approximation

- **Given:**
  - triangulation $T$
  - bounded domain $\Omega$ in $\mathbb{R}^n$

- **Consider function approximation error:**

\[
Q(T, f, p) = \| f - f_{I,T} \|_{L^p,\Omega}
\]

linear interpolation
Function Approximation

**Theorem [Chen 04]:**

\[
Q(DT, \| x \|^2, p) = \min_{T \in P_V} Q(T, \| x \|^2, p), 1 \leq p \leq \infty
\]

- Isotropic function
- Set of all triangulations with a given set \( V \)
- \( \square \) = convex hull of \( V \)

[d’Azevedo-Simpson 89] in \( \mathbb{R}^2 \), \( p = \infty \)

[Rippa 92] in \( \mathbb{R}^2 \), \( 1 \leq p \leq \infty \)

[Melissaratos 93] in \( \mathbb{R}^D \), \( 1 \leq p \leq \infty \)
Function Approximation

Let us V vary

Problem:
- find triangulation $T^*$ such that:

$$Q(T^*, f, p) = \inf_{T \in P_N} Q(T, f, p), 1 \leq p \leq \infty$$

Proof:
- existence
- necessary condition for $p=1$
Function Approximation

\[ Q(DT, \| x \|^2, p) = \min_{T \in P_V} Q(T, \| x \|^2, p), 1 \leq p \leq \infty \]

Isotropic function

set of all triangulations with a given set \( V \)

\( = \) convex hull of \( V \)

(2D)
Function Approximation

- $x_i$: vertex
- $\Omega_i$: union of simplices incident to $x_i$
- $|A|$: Lebesgue measure of set $A$ in $\mathbb{R}^n$
Function Approximation

\[ Q(T, f, p) = \| f_{I,T} - f \|_{L^p, \Omega} \]

\[ Q(T, f, p) = \left[ \int_{\Omega} | f_{I,T}(x) - f(x) |^p \, dx \right]^{1/p} \]

\[ Q(T, f, 1) = \int_{\Omega} (f_{I,T}(x) - f(x)) \, dx \]

\( f \text{ convex, } f_{I,T} \text{ PWL interpolant} \)

\[ Q(T, f, 1) = \int_{\Omega} f_{I,T}(x) \, dx - \int_{\Omega} f(x) \, dx \]
Function Approximation

\[ Q(T, f;1) = \int_{\Omega} f_{I,T}(x) \, dx - \int_{\Omega} f(x) \, dx \]

\[ = \sum_{\tau \in T} \int_{\tau} f_{I,T}(x) \, dx - \int_{\Omega} f(x) \, dx \]

\[ = \frac{1}{n+1} \sum_{\tau \in T} \left( |\tau| \sum_{k=1}^{n+1} f(x_{\tau}, k) \right) - \int_{\Omega} f(x) \, dx \]

\[ = \frac{1}{n+1} \sum_{x_i \in T} f(x_i) |\Omega_i| - \int_{\Omega} f(x) \, dx \]

n+1 overlaps
Function Approximation

restrict to patch $\Omega_i$ incident to vertex $x_i$

\[
Q(\Omega_i, f, 1) = \frac{1}{n+1} \sum_{x_i \in \Omega_i} f(x_i) |\Omega_i| - \int_{\Omega_i} f(x) dx
\]

\[
= \frac{1}{n+1} \sum_{\tau_j \in \Omega_i} \left( |\tau_j(x)| \sum_{x_k \in \tau_j, x_k \neq x_i} f(x_k) \right) + \frac{|\Omega_i|}{n+1} f(x_i) - \int_{\Omega_i} f(x) dx
\]

minimize

\[
E_{ODT} \equiv \sum_{\tau_j \in \Omega_i} \left( |\tau_j(x)| \sum_{x_k \in \tau_j, x_k \neq x_i} f(x_k) \right) + |\Omega_i| f(x_i)
\]
Function Approximation

\[ E_{ODT} \equiv \sum_{\tau_j \in \Omega_i} (|\tau_j(x)| \sum_{x_k \in \tau_j, x_k \neq x_i} f(x_k)) + |\Omega_i| f(x_i) \]

**minimum if** \( \nabla E_{ODT} = 0 \)

\[ \nabla f(x^*_i) = -\frac{1}{|\Omega_i|} \sum_{\tau_j \in \Omega_i} \left( \nabla |\tau_j| (x_i) \sum_{x_k \in \tau_j, x_k \neq x_i} f(x_k) \right) \]

**if** \( f(x) = ||x||^2 \)

\[ x^*_i = -\frac{1}{2|\Omega_i|} \sum_{\tau_j \in \Omega_i} \left( \nabla |\tau_j| (x_i) \sum_{x_k \in \tau_j, x_k \neq x_i} ||x_k||^2 \right) \]
Geometric Interpretation

\[ x_i^* = \frac{1}{|\Omega_i|} \sum_{\tau_j \in \Omega_i} |\tau_j| c_j \]

- **Note**: optimal location depends only on the 1-ring neighbors, not on the current location. If all incident vertices lie on a common sphere, optimal location is at sphere center.
Optimization

- alternate updates of
  - connectivity
  - vertex location

- both steps minimize the same energy
  - as for Lloyd iteration

- for convex fixed boundary
  - energy monotonically decreases
  - convergence to a (local) minimum
Underlaid vs Overlaid Approximant

- CVT
  - partition
  - approximant
  - compact Voronoi cells
  - isotropic sampling

- ODT
  - overlapping decomposition
  - PWL interpolant
  - compact simplices
  - isotropic meshing
Optimization

Alternate updates of

- connectivity (Delaunay triangulation)
- vertex locations
Optimal Delaunay Triangulation

- Distribution of radius ratios
- 3 “slivers”, each with two vertices on boundary
Sizing Field

Goal reminder:

- shape of elements
- boundary approximation
- minimize #elements
- note: not independent (well-shape elements force $K$-Lipschitz sizing field [Ruppert, Miller et al.])

Proposal:

- size $\leq lfs$ (local feature size) on boundary
- sizing field = maximal $K$-Lipschitz

$$
\mu(x) = \inf_{y \in \partial \Omega} \left[ K \| x - y \| + lfs(y) \right]
$$
Sizing Field

\[ \mu(x) = \inf_{y \in \partial \Omega} \left[ K \| x - y \| + lfs(y) \right] \]
Sizing Field: Examples

K = 0.1
K = 0.5
K = 1

3
10
100
Algorithm

- read input boundary $\partial \Omega$
- setup data structure & preprocessing
- compute sizing field
- generate initial sites inside $\Omega$

```
  do
  - Delaunay triangulation of $\{x_i\}$
  - move sites to optimal locations $\{x_i^*\}$
  until convergence or stopping criterion
```
- extract interior mesh
Input Boundary $\partial \Omega$

- surface triangle mesh

- Requirements:
  - intersection free
  - closed
  - restricted Delaunay triangulation of the input vertices [Oudot-Boissonnat, Cohen-Steiner et al.]
Input Boundary $\partial \Omega$
Input Boundary $\partial \Omega$
Setup & Preprocessing

- Insertion of input mesh vertices to a 3D Delaunay triangulation
  - control mesh
  - used to answer inside/outside queries
Sizing Field: Poles

control mesh

[Amenta-Bern 98]

poles
Sizing Field: lfs

Approximation: distance to set of poles
CGAL orthogonal search in a kD-tree [Tangelder-Fabri]
Sizing Field: Examples

Approximation: fast marching from boundary
Representation: regular grid or balanced octree
Optimization

- classification boundary/interior vertices by localization of boundary in Voronoi cells
  - CVT on boundary
  - ODT inside
Optimization: init

Distribution of radius ratios
Optimization: step 1
Optimization: step 2
Optimization: step 50
Optimization: step 50
Optimization: step 50
Hand: lfs
Hand: Sizing
Hand: Sizing
Hand: Sizing
Hand: Radius Ratios
Torso
Torso
Torso
Conclusion

- **Meshes**
  - Definition, variety
  - Background
    - Voronoi
    - Delaunay
    - constrained Delaunay
    - restricted Delaunay

- **Optimization**
  - 2D, 3D
  - Lloyd iteration, function approximation approach