2D Triangulations in CGAL

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Outline

- Specifications
  - Definition
  - Triangulations in CGAL
  - Features
- Representation
  - As a set of faces
  - Representation based on vertices and cells
- Software design
  - Traits class
  - Triangulation data structure
- Algorithms
  - Point location
- Examples
- Applications

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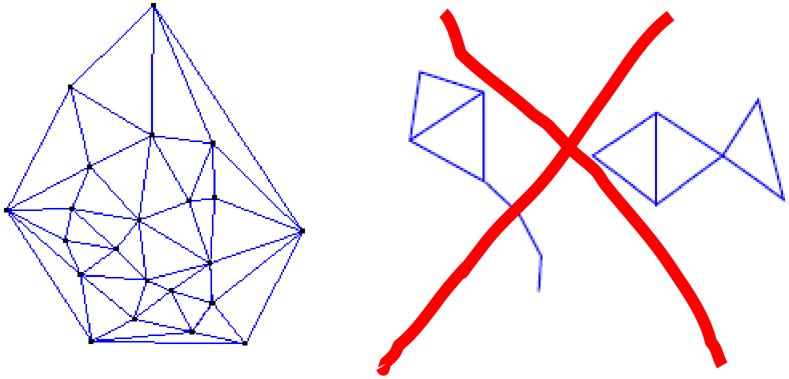
- Exercises

Definitions

- A **2D triangulation** is a set **T** of triangular facets such that:
  - two facets are either disjoint or share a lower dimensional face (edge or vertex).
  - the set of facets in **T** is connected for the adjacency relation.
  - the domain $\mathbf{U}_\mathbf{T}$ which is the union of facets in **T** has no singularity.



Definitions

- A simplicial complex is a set *T* of simplices such that
  - any face of a simplex in *T* is a simplex in *T*
  - two simplices in *T* either are disjoint or share a common subface.
- The dimension *d* of a simplicial complex is the maximal dimension of its simplices.
- A simplicial complex *T* is pure if any simplex of *T* is included in a simplex of *T* with maximal dimension.

Definitions

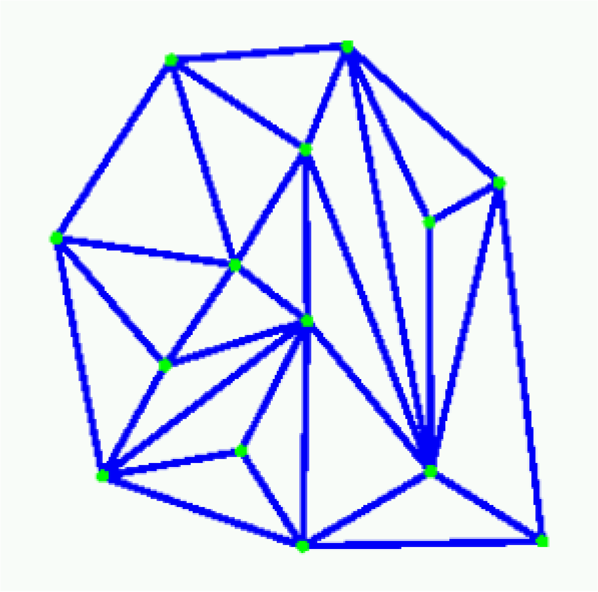
- Two simplexes in T with maximal dimension d are said to be adjacent if they share a $(d-1)$ dimensional subface. A simplicial complex is connected if the adjacency relation defines a connected graph over the set of simplices of T with maximal dimension.
- The union U_T of all simplices in T is called the domain of T.
- A point p in the domain of T is said to singular if its surrounding in U_T is neither a topological ball nor a topological disc.

2D Triangulations in CGAL

- Basic
- Delaunay
- Regular
- Constrained
- Constrained Delaunay

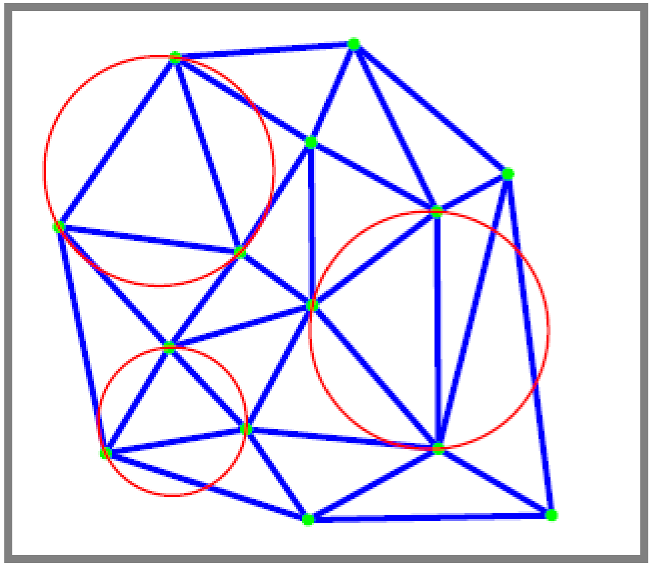
Basic Triangulation

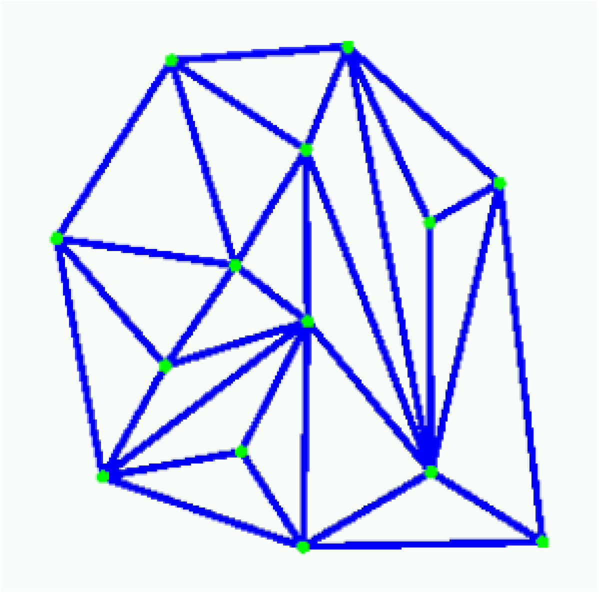
- Lazy incremental construction, no control over the shape of triangles



Delaunay Triangulation

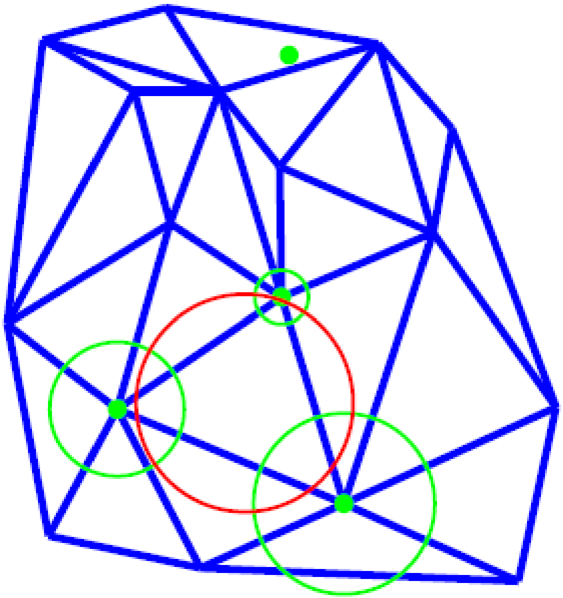
- Empty circle property





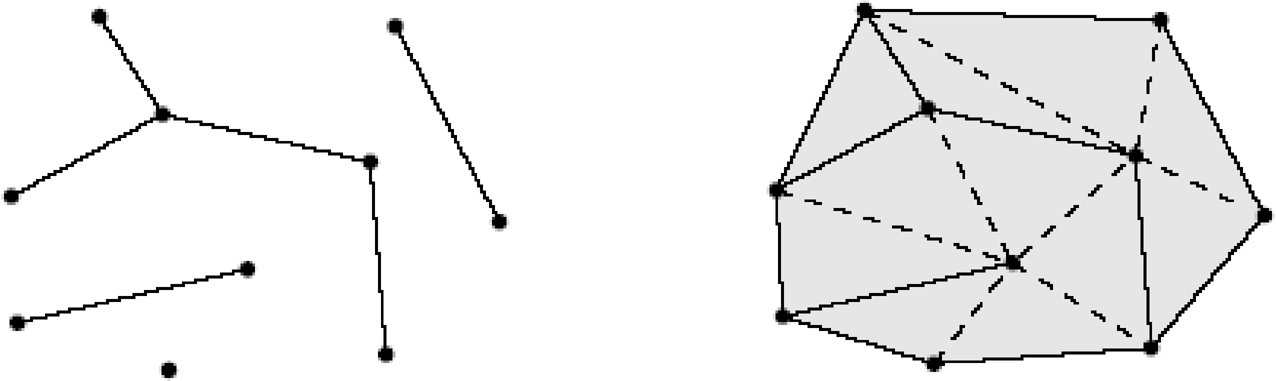
Regular Triangulation

- Generalization of Delaunay triangulation.
- Defined for a set of **weighted points**. Each weighted point can be considered as a sphere whose square radius is equal to the weight. The regular triangulation is the dual of the power diagram.



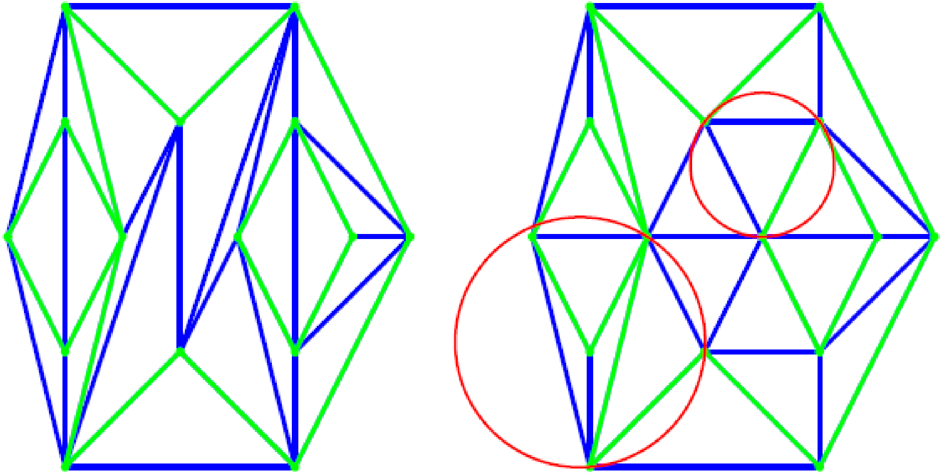
Constrained Triangulation

- Allows to **enforce edges.**

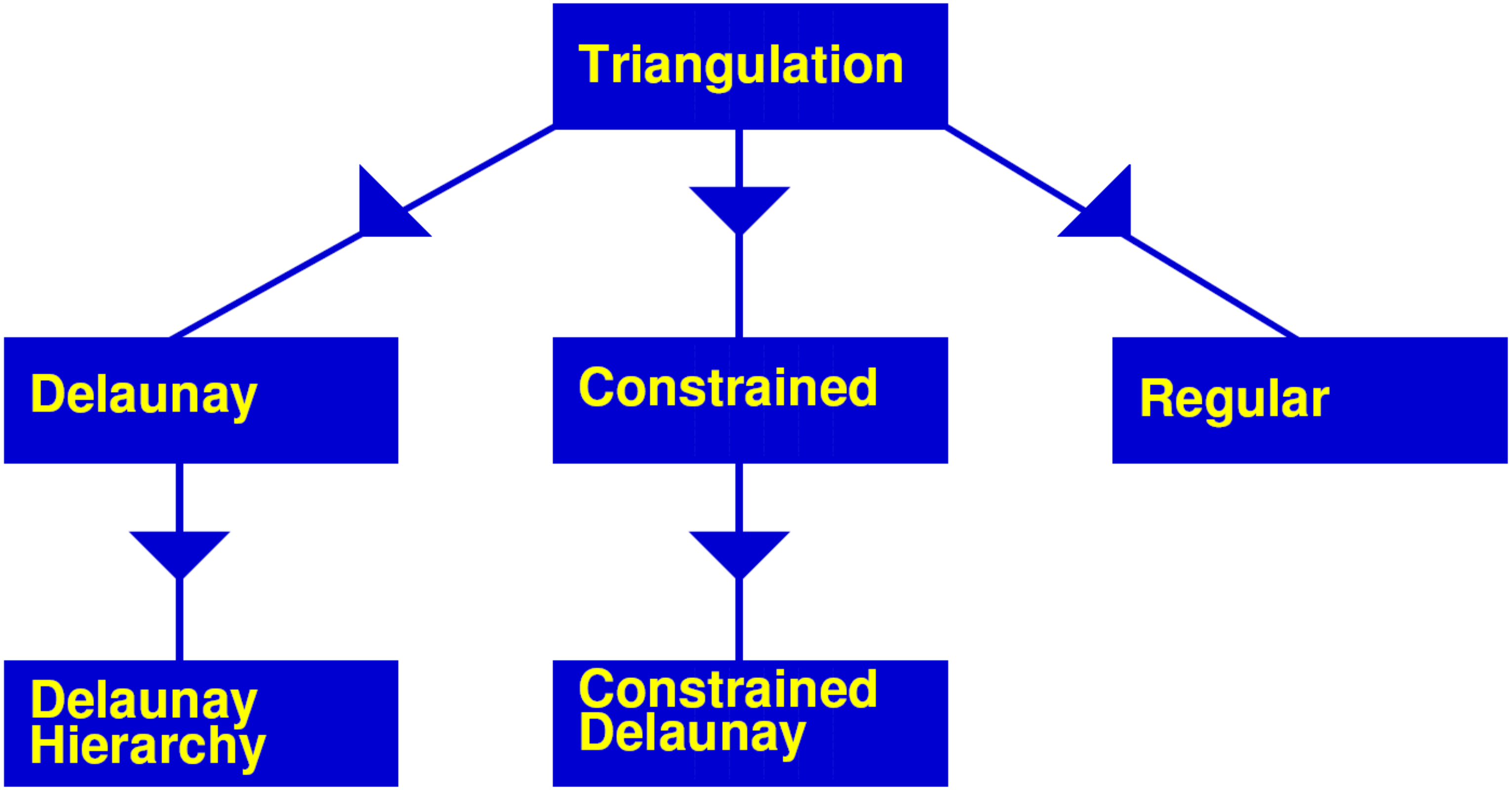


Constrained Delaunay Triangulation

- Constrained triangulation which is **as much Delaunay as possible**. Each triangle satisfies the constrained empty circle property : its circumscribing circle encloses no vertex visible from the interior of the triangle, where enforced edges are considered as visibility obstacles.



Derivation Tree (2D)



Triangulation

Delaunay

Constrained

Regular

Delaunay Hierarchy

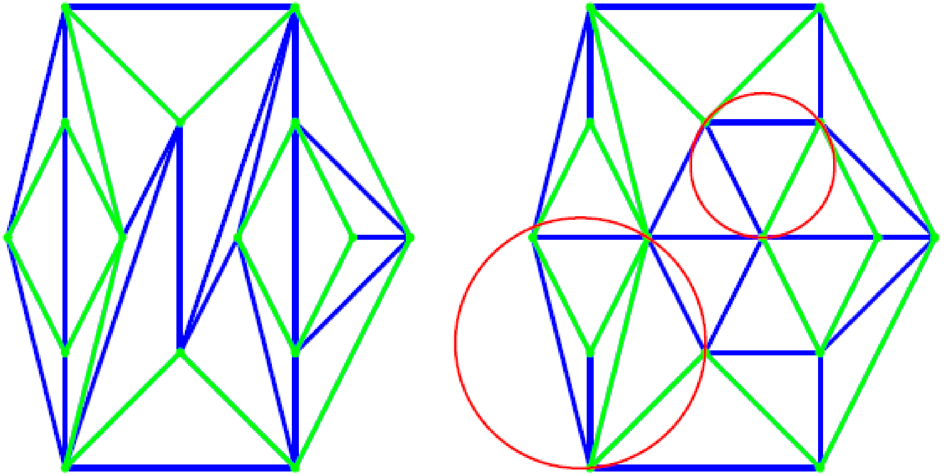
Constrained Delaunay

General Features

- **Traversal:**
  - going from a face to its neighbors
  - iterators to visit all faces of a triangulation
  - circulators to visit all faces around a vertex or all faces intersected by a line.
- **Point location query**
- **Insertion, removal, flips:**
  - Features adapted to each type of triangulations (e.g., the insertions
  - and deletions in a Delaunay triangulation maintain the empty circle property).

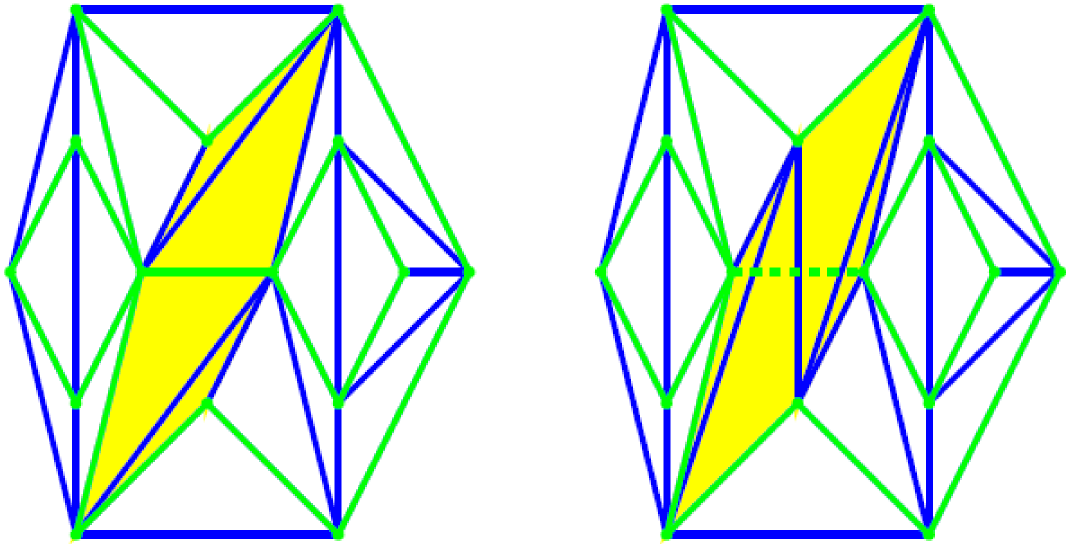
Additional Features

- For some triangulations



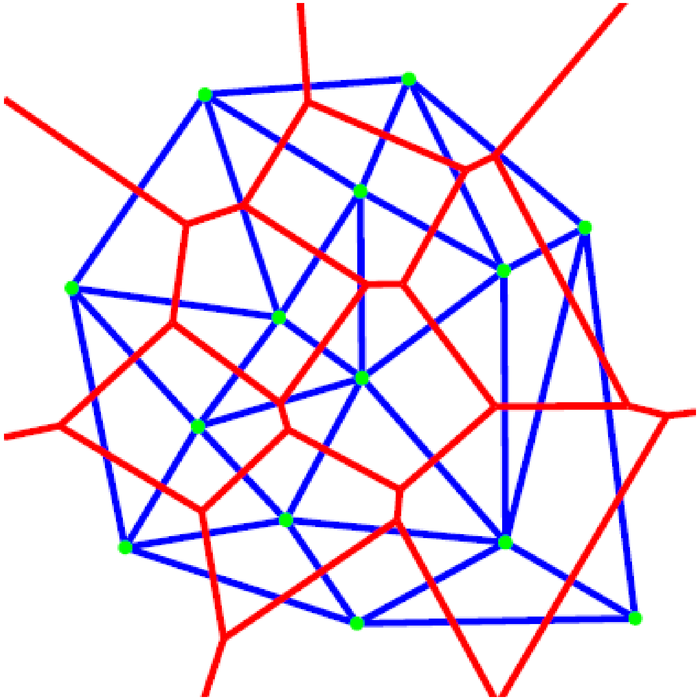
Additional Features

- Example for **constrained** and **Delaunay constrained** triangulations:
  - Insertion and removal of constraints



Additional Features

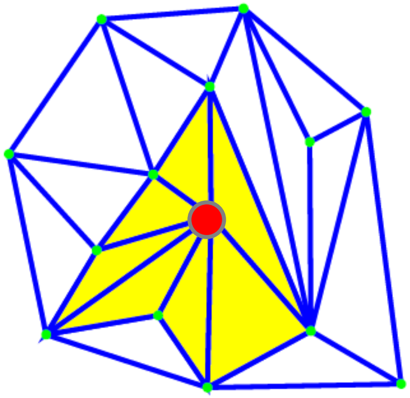
- For **Delaunay triangulation**
  - Nearest neighbor queries
  - Voronoi diagram



Traversal (1)

- **Iterators**
  - All faces iterator
  - All vertices iterator
  - All edges iterator

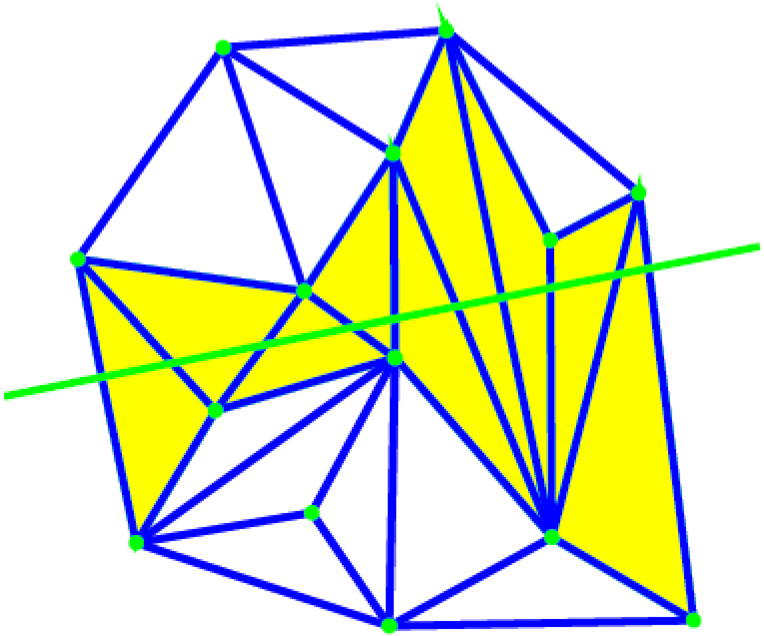
Traversal (2)



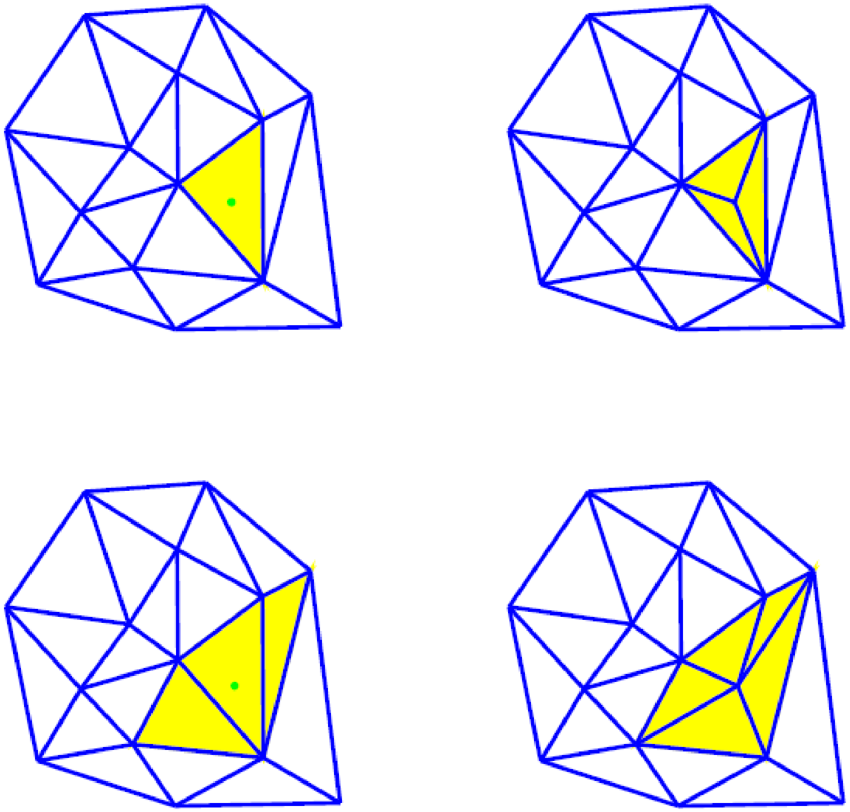
- Circulators
  - Face circulator
    - faces incident to a vertex
  - Edge circulator
    - edges incident to a vertex
  - Vertex circulator
    - vertices incident to a vertex

Traversal (3)

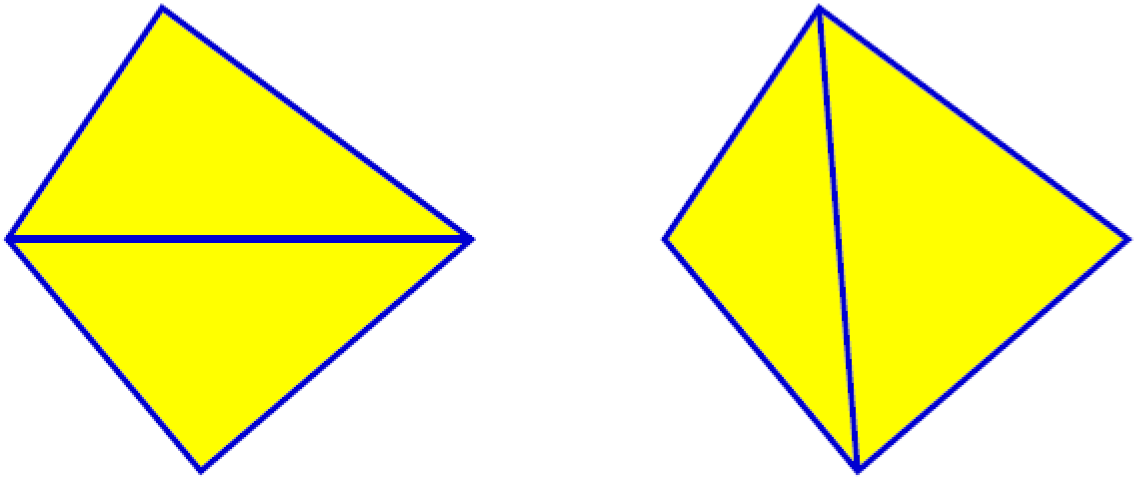
- **Line face circulator**



Point Location & Insertion



Edge Flip



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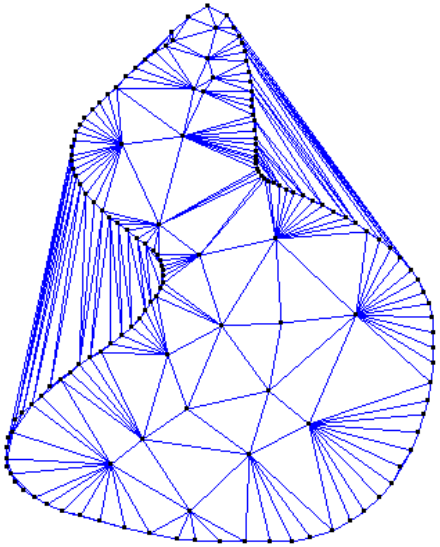
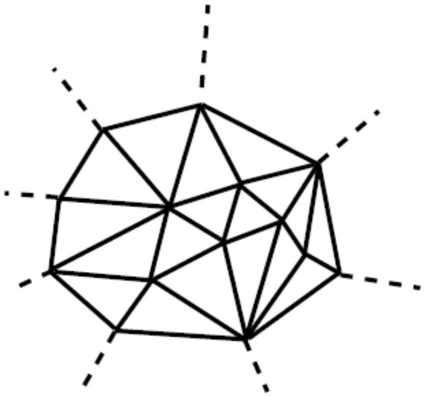
Triangulations as a Set of Faces

- All triangulations in CGAL **tile the convex hull** of their vertices. Triangulated polygonal regions can be obtained through constrained triangulations.



Triangulations as a Set of Faces

- All triangulations in CGAL **tile the convex hull** of their vertices. Triangulated polygonal regions can be obtained through constrained triangulations.
- An imaginary vertex (so-called **infinite vertex** is added).

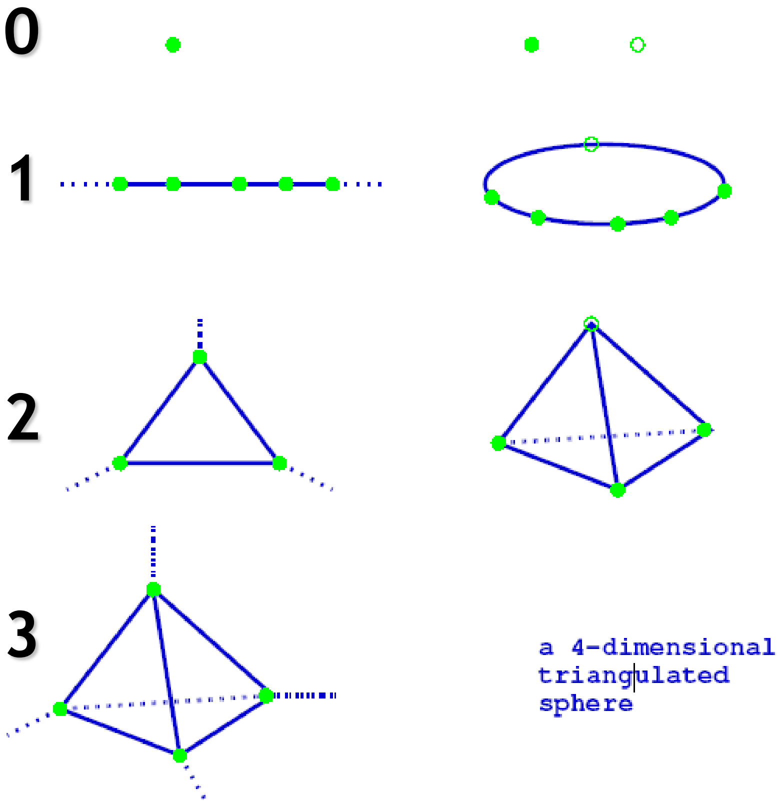


Triangulations as a Set of Faces

- All triangulations in CGAL **tile the convex hull** of their vertices. Triangulated polygonal regions can be obtained through constrained triangulations.
- An imaginary vertex (so-called **infinite vertex** is added).
  - Any face is a triangle.
  - Any edge is incident to two exactly 2 faces.
  - The set of faces is equivalent to a **2D topological sphere**.

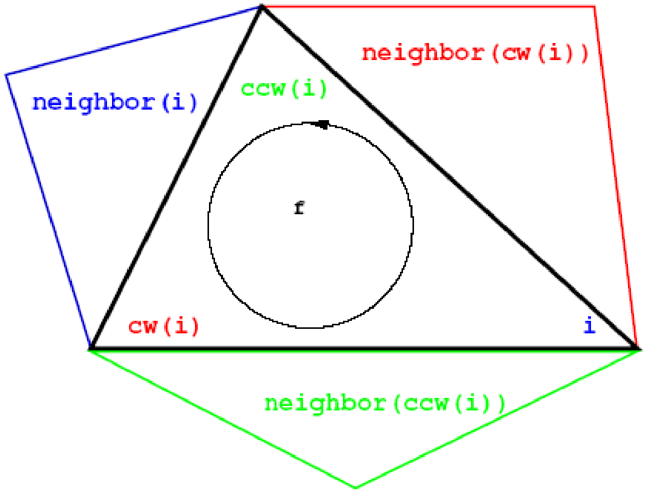
Triangulations as a Set of Faces

In any dimension, the set of faces is combinatorically equivalent to a **triangulated sphere.**



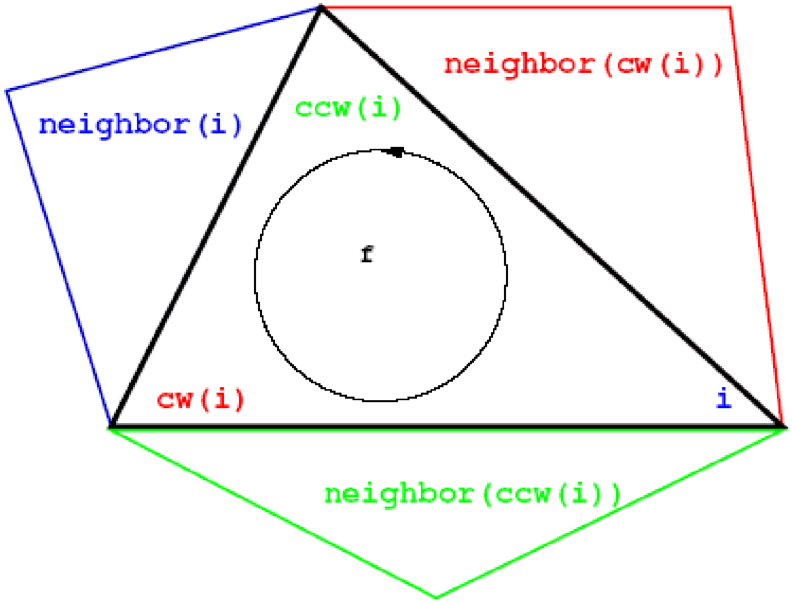
Representation

- The internal representation is based on **faces** and **vertices.**
- Edges are **implicitly** represented
- **Vertex**
  - `Face* v_face`
- **Face**
  - `Vertex* vertex[3]`
  - `Face* neighbor[3]`



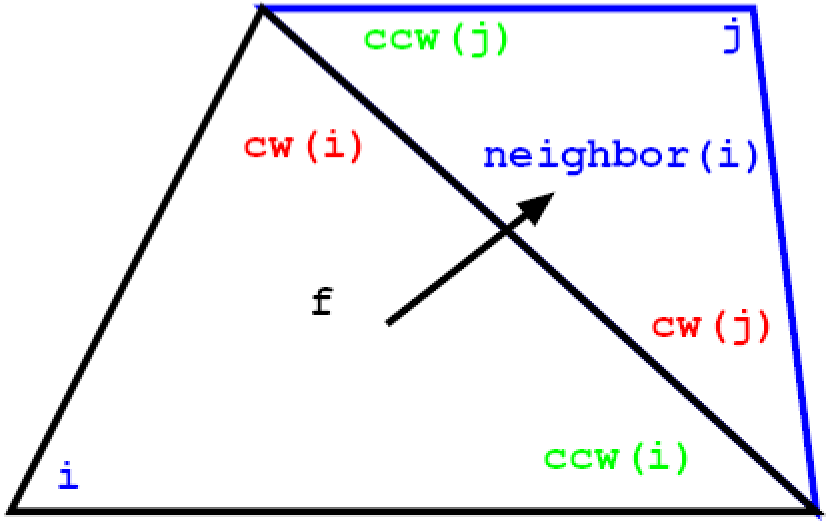
Representation

- functions cw(i) & ccw(i)



Representation

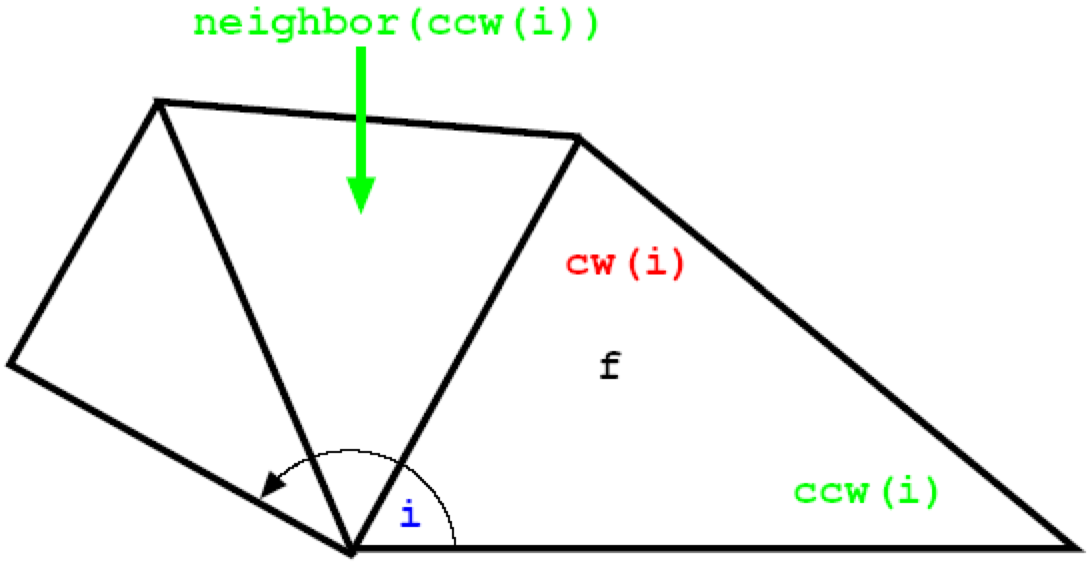
- From one Face to Another



```
n = f->neighbor(i)
j = n->index(f)
```

Representation

- Around a vertex



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Software Design

- Many **CGAL** classes are parameterized by one or more **template parameters:**
  - `Polygon_2<Traits, Container>`
  - `Polyhedron_3<Traits, HDS>`
  - `Planar_map_2<Dcel,Traits>`
  - `Arrangement_2<Dcel,Traits,Base node>`
  - `Min_circle_2<Traits>`
  - `Point_set_2<Traits>`
  - `Range_tree_k<Traits>`

Triangulation Classes

`Triangulation_2<Traits, TDS>`

`Triangulation_3<Traits, TDS>`

- **Traits**
  - Geometric traits
- **TDS**
  - Triangulation Data Structure

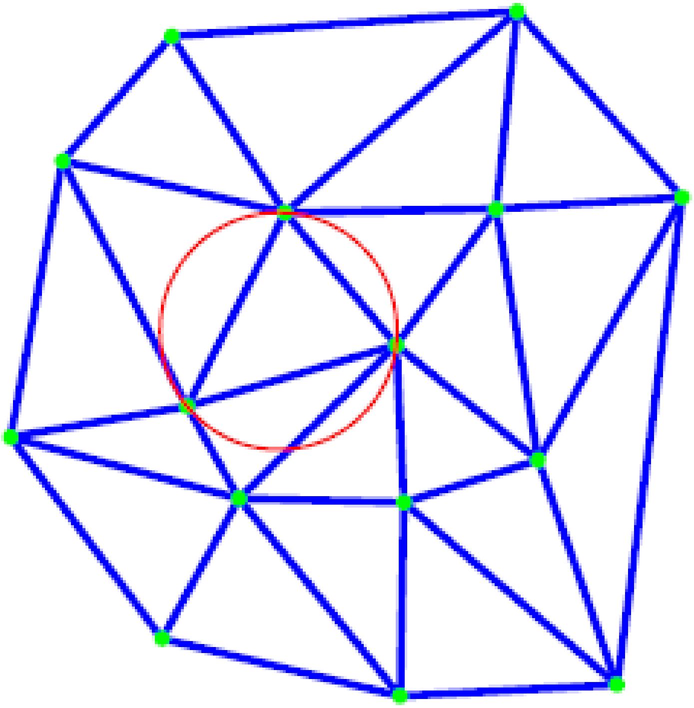
Geometric Traits

- Geometric traits classes provide :
  - Basic geometric objects
  - Predicates and Constructors
- Requirements for traits are documented
  - basic library data structures and algorithms can be used with user-defined objects
- Default traits classes are provided

Traits Class for Delaunay Triangulation

- Requirements:
  - Point
  - Segment
  - Triangle
  - Line
  - Ray

  - orientation test
  - in circle test
  - circumcenter
  - bisector



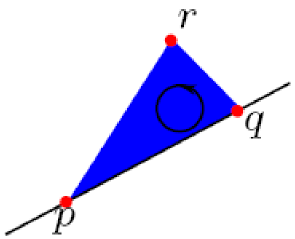
Traits Class for Delaunay Triangulation

- Default traits class:
  - Triangulation_euclidean_traits_2<Kernel>

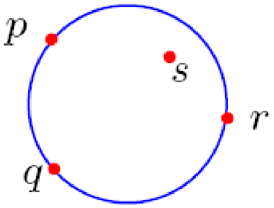
- Delaunay triangulation of 2D points:
  - typedef Cartesian<double> Kernel;
  - typedef Triangulation_euclidean_traits_2<Kernel> Traits;
  - typedef Delaunay_triangulation_2<Traits> Triangulation;

Predicates for Delaunay Triangulation

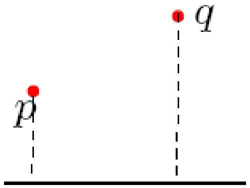
- Orientation test



- Incircle test



- Comparison of coordinates



Traits Class for Terrains

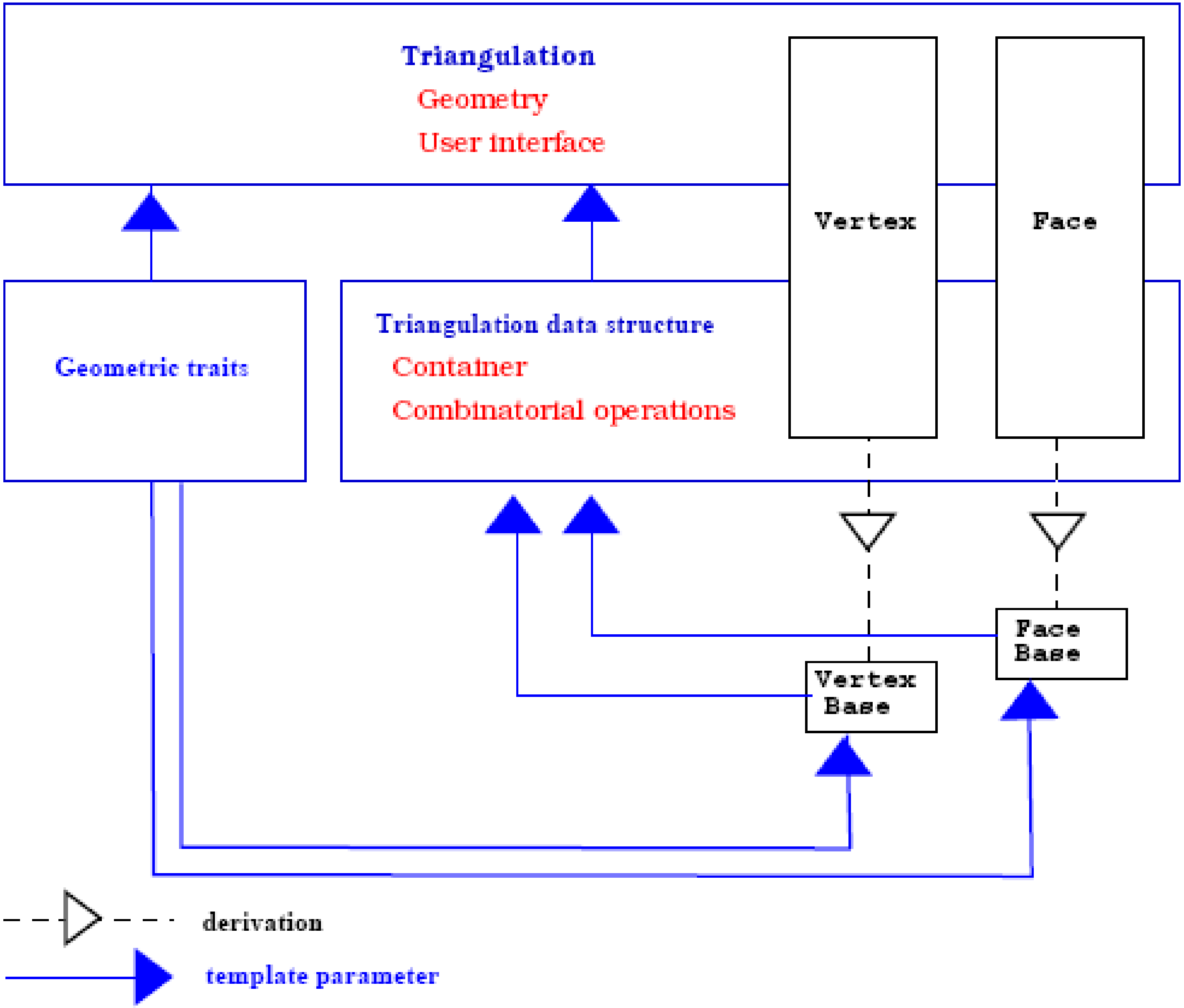
Needs

- 3D points
- orientation
- in circle
- on x and y coordinates

Triangulation_euclidean_traits_xy_3<kernel>

Definition:

```
typedef Cartesian<double> kernel;
typedef Triangulation euclidean traits xy 3<kernel> Traits;
typedef Delaunay triangulation 2<Traits> Triangulation;
```



Triangulation
Geometry
User interface

Vertex

Face

Geometric traits

Triangulation data structure
Container
Combinatorial operations

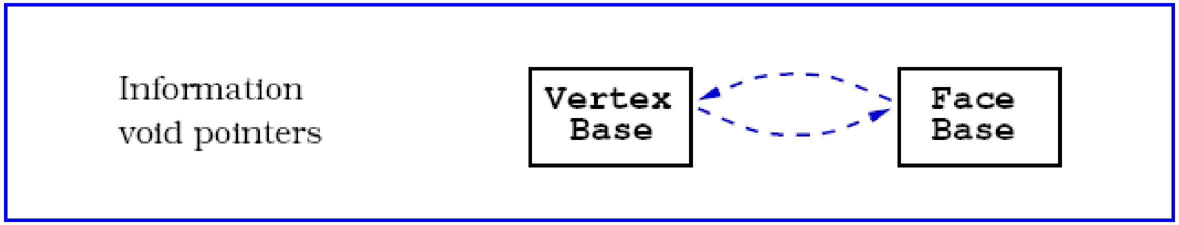
Vertex Base

Face Base

derivation

template parameter

Triangulation Design



Vertex base

Vertex base :: Point

Vertex base(Point p, void* f)

Point point();

void* face();

void* set point();

void* set face();

Face base

Face base(void* v0, void* v1, void* v2,

void* n0, void* n1, void* n2)

void* vertex(int i);

void* neighbor(int i);

void* set vertex(int i, void* v);

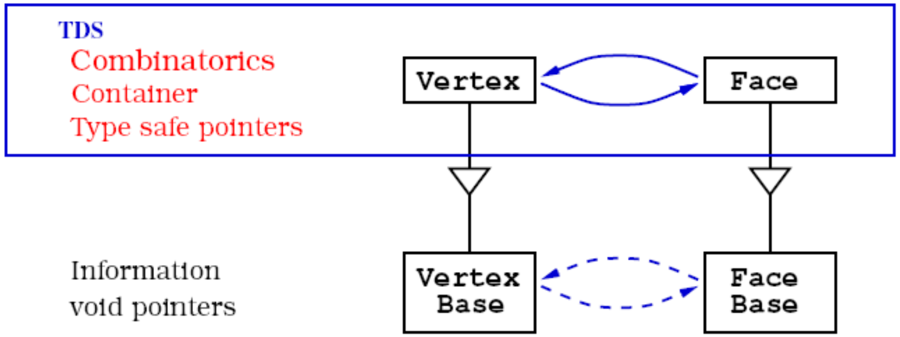
void* set neighbor(int i, void* f);

Triangulation Data Structure

Tds<Vb,Fb>

Types:

- Tds<Vb,Fb>::Vertex inherits from Vb
- Tds<Vb,Fb>::Face inherits from Fb
- Tds<Vb,Fb>::Face iterator
- Tds<Vb,Fb>::Edge iterator
- Tds<Vb,Fb>::Vertex iterator
- Tds<Vb,Fb>::Face circulator
- Tds<Vb,Fb>::Edge circulator
- Tds<Vb,Fb>::Vertex circulator

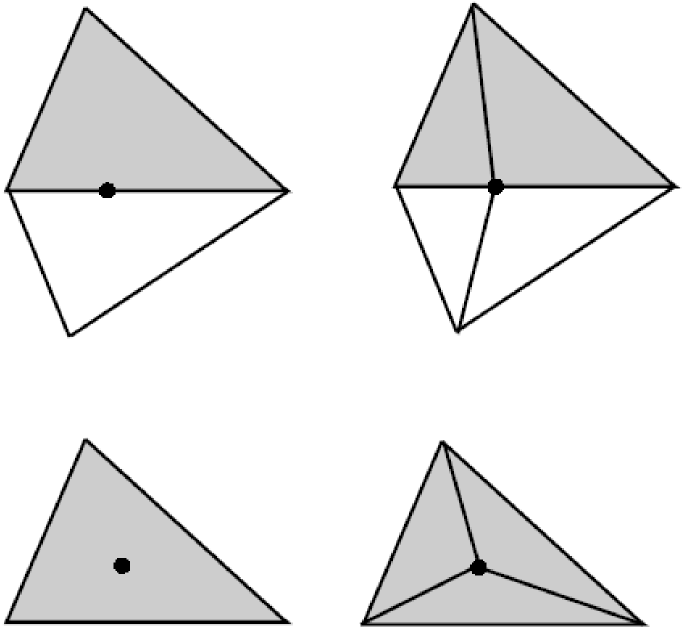


TDS
Combinatorics
Container
Type safe pointers

Information
void pointers

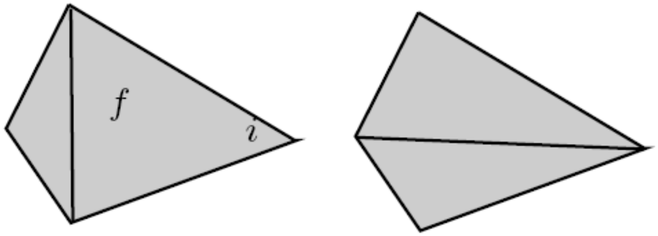
Combinatorial Operations

```
void insert_in_face (Vertex* v,Face* f)
void insert_in_edge (Vertex* v, Face* f, int i)
void remove_degre_3 (Vertex* v);
```



Combinatorial Operations

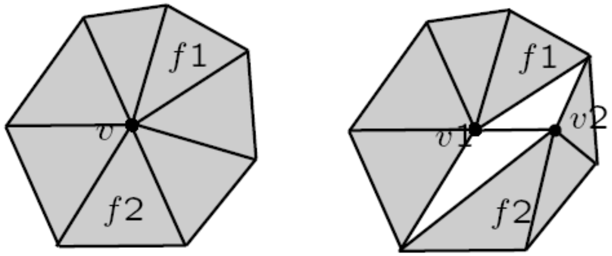
void flip (Face* f, int i);



(on-going work)

void split vertex (Vertex*, Face* f1, Face* f2)

void join vertices (Vertex* v1, vertex* v2)



The Triangulation Class

CGAL::Triangulation 2<Gt, Tds >
typedef Gt geometric_traits;
typedef Tds Triangulation_data_structure;
typedef Triangulation_2<Gt, Tds > Triangulation;

Types
Gt::Point_2
Gt::Segment_2
Gt::Triangle_2
Triangulation::Vertex inherits from Tds::Vertex
Triangulation::Face inherits from Tds::Face
Triangulation::Vertex_handle
Triangulation::Face_handle

typedef pair<Face handle, int> Edge ;
Triangulation::Face_iterator
Triangulation::Edge_iterator
Triangulation::Vertex_iterator
Triangulation::Line_face_circulator
Triangulation::Face_circulator
Triangulation::Edge_circulator
Triangulation::Vertex_circulator

High Level Functions

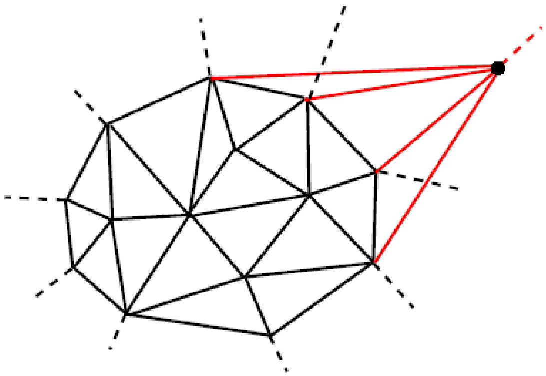
```
enum Locate_type {      VERTEX=0, EDGE, FACE,
                        OUTSIDE_CONVEX_HULL,
                        OUTSIDE_AFFINE_HULL}

Face_handle locate(     Point query,
                        Locate_type& lt,
                        int& li,
                        Face_handle h =Face_handle() );


Vertex_handle insert(Point p)


void remove(Vertex_handle v)
```

Insertion



```
Vertex handle insert (Point p)
{
      Locate type lt; int li;
      Face handle loc = locate(p, lt, li);
      switch(lt){
      case VERTEX : return f->vertex(li);
      case EDGE :return insert_in_edge( p, loc,li);
      case FACE :return insert_in_face(v,loc);
      case OUTSIDE CH :return insert_outside ch(p,loc);
      case OUTSIDE AH :return insert_outside ah(p);
}}
```

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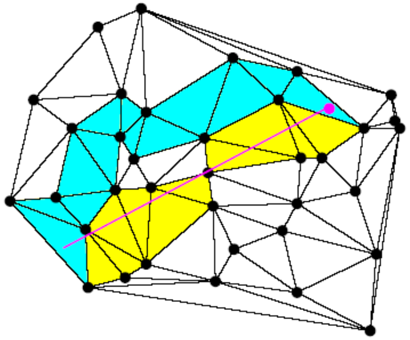
Algorithms for Triangulation

- All CGAL triangulations are built through **incremental** on-line insertion of vertices.
- The main algorithmic issue is therefore to deal with **point location.**

- **CGAL** offers different algorithms :
  - linewalk 🟡
  - Zigzag walk 🟢
  - jump and walk strategy
  - the Delaunay hierarchy



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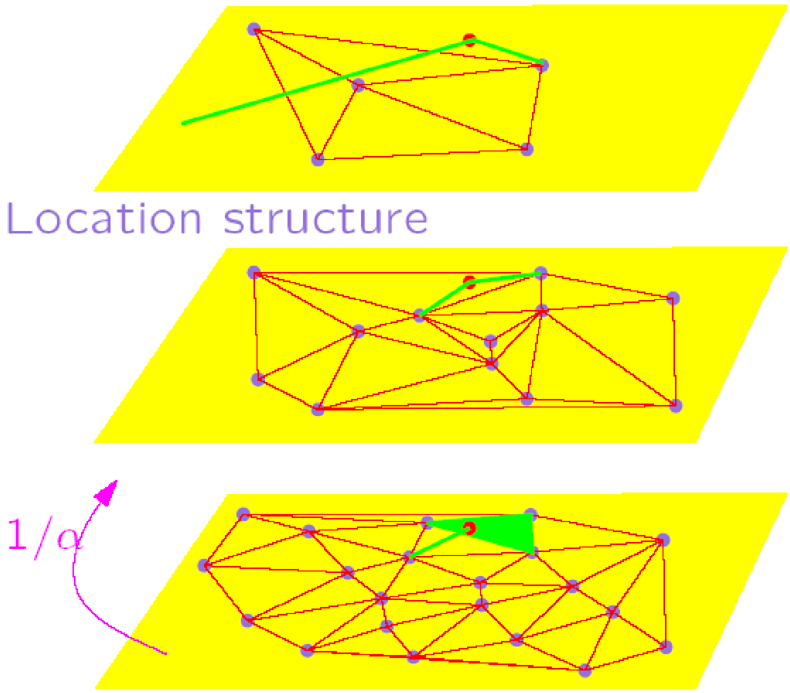
Efficient Localization

- Delaunay Hierarchy



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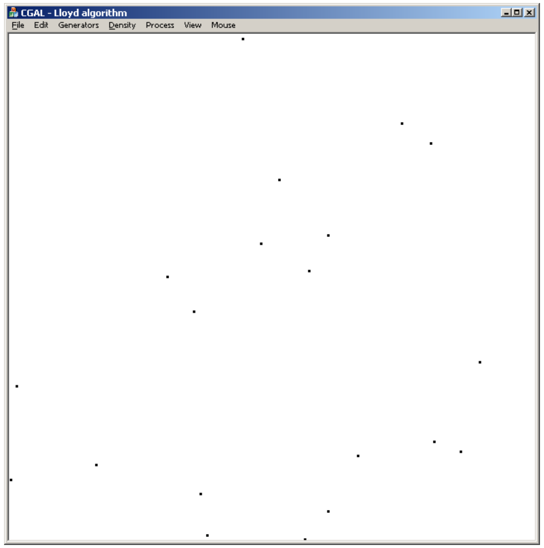
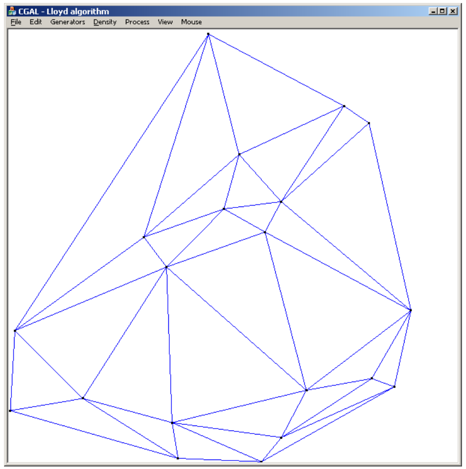
```
#include <CGAL/Cartesian.h>
#include <CGAL/Triangulation_2.h>

using namespace CGAL;
using namespace std;

typedef Cartesian<double> Kernel;
typedef Triangulation_2<Kernel> Triangulation;
typedef Triangulation::Vertex_circulator Vertex_circulator;
typedef Kernel::Point_2 Point;


   Triangulation t;
   Point p;
   while(cin >> p) t.insert(p);
   Vertex_circulator vc = t.incident_vertices(t.infinite_vertex());
   Vertex_circulator done(vc);
   do
        cout << vc->point();
   while(++vc != done);
}
```

Drawing Generators…

```
template <class kernel, class TDS>
class DT2 : public CGAL::Delaunay_triangulation_2<kernel,TDS>
{
    public:
    void gl_draw_generators()
    {
        ::glBegin(GL_POINTS);
        Point_iterator it;
        for(it = points_begin();
            it != points_end();
            it++)
        {
            const Point& p = *it;
            ::glVertex2d(p.x(),p.y());
        }
        ::glEnd();
    }
}
```



Drawing Delaunay Edges...



```
void gl_draw_delaunay_edges()
{
        ::glBegin(GL_LINES);
        Edge_iterator it;
        for(it = edges_begin();
             it != edges_end();
             it++)
        {
                    // edge = std::pair<Face_handle,int>
                    Edge& edge = *it;
                    const Point& p1 = edge.first->vertex(ccw(edge.second))->point();
                    const Point& p2 = edge.first->vertex(cw(edge.second))->point();
                    ::glVertex2f(p1.x(), p1.y());
                    ::glVertex2f(p2.x(), p2.y());
        }
        ::glEnd();
}
```

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Drawing Voronoi Edges

```
void gl_draw_voronoi_edges()  {
        ::glBegin(GL_LINES);
        Edge_iterator hEdge;
        for(hEdge  = edges_begin();  hEdge != edges_end(); hEdge++)
        {
                CGAL::Object object = dual(hEdge);
                Segment segment;
                Ray ray;
                Point source, target;
                if(CGAL::assign(segment,object))
                {
                        source = segment.source();
                        target = segment.target();
                }
                else if(CGAL::assign(ray,object))
                {
                        source = ray.source();
                        target = ray.point(1);
                }
                ::glVertex2f(source.x(),source.y());
                ::glVertex2f(target.x(),target.y());
        }
        ::glEnd(); }
```