Voronoi diagrams
Delaunay Triangulations

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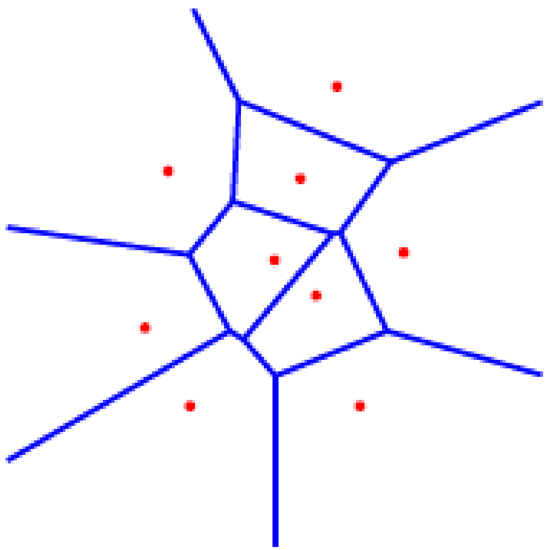
Voronoi Diagram

Voronoi Diagram

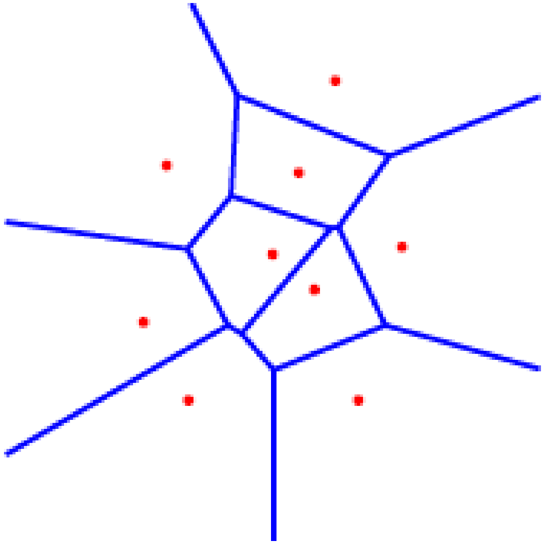
Let $\mathcal{E} = \{\mathbf{p_1}, \ldots, \mathbf{p_n}\}$ be a set of points (so-called sites) in $\mathbb{R}^d$. We associate to each site $\mathbf{p_i}$ its Voronoi region $V(\mathbf{p_i})$ such that:

$$V(\mathbf{p_i}) = \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x} - \mathbf{p_i}\| \leq \|\mathbf{x} - \mathbf{p_j}\|, \forall j \leq n\}.$$



Voronoi Diagram

- The collection of the non-empty Voronoi regions and their faces, together with their incidence relations, constitute a cell complex called the **Voronoi diagram** of E.
- The locus of points which are equidistant to two sites pi and pj is called a **bisector**, all bisectors being affine subspaces of IR^d (lines in 2D).



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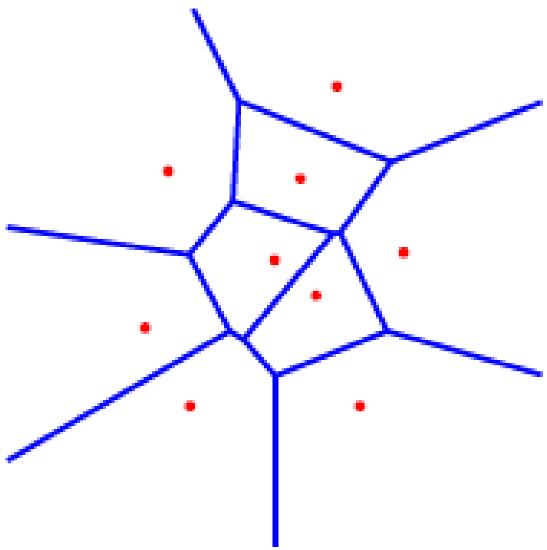
Voronoi Diagram

- A Voronoi cell of a site *pi* defined as the intersection of closed half-spaces bounded by bisectors. Implies: All Voronoi cells are **convex**.



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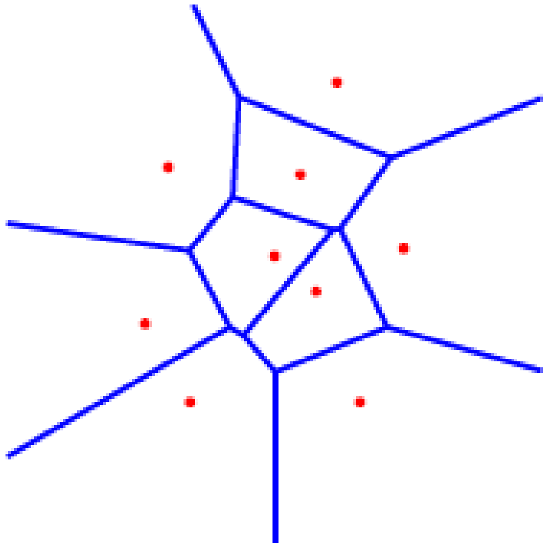
Voronoi Diagram

- Voronoi cells may be **unbounded** with unbounded bisectors. Happens when a site pi is on the boundary of the convex hull of E.



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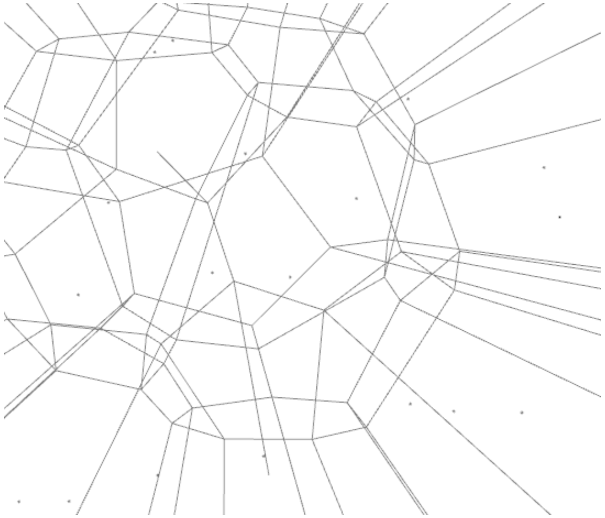
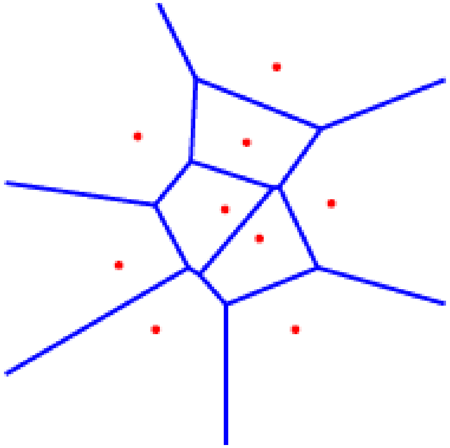
Voronoi Diagram

- **Voronoi cells** have **faces** of different dimensions.
- In 2D, a face of dimension k is the intersection of 3 - k Voronoi cells. A **Voronoi vertex** is generically equidistant from three points, and a **Voronoi edge** is equidistant from two points.

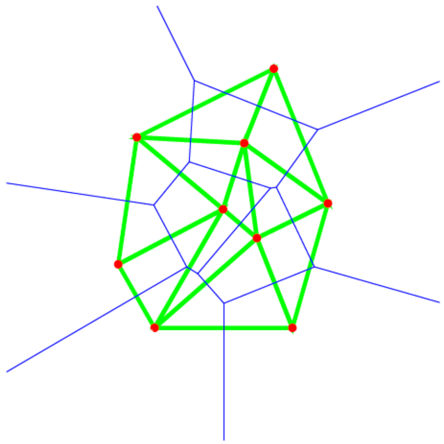




Delaunay Triangulation

Delaunay Triangulation

- Dual structure of the Voronoi diagram.
- The Delaunay triangulation of a set of sites E is a simplicial complex such that k+1 points in E form a Delaunay simplex if their Voronoi cells have nonempty intersection



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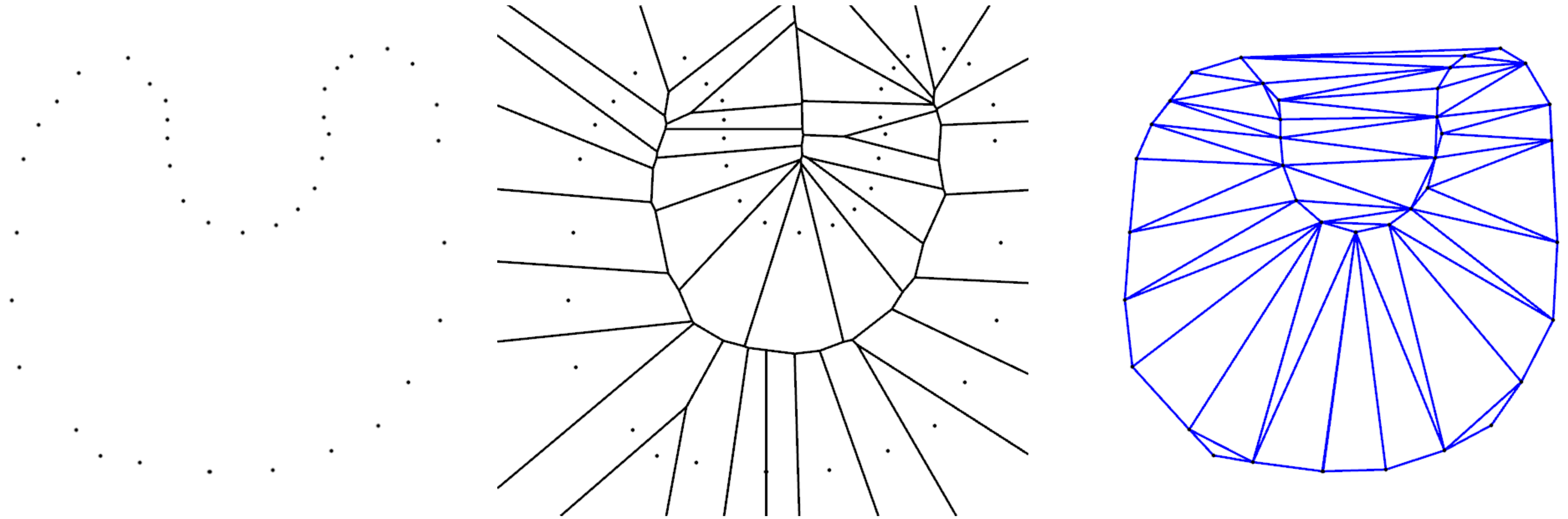
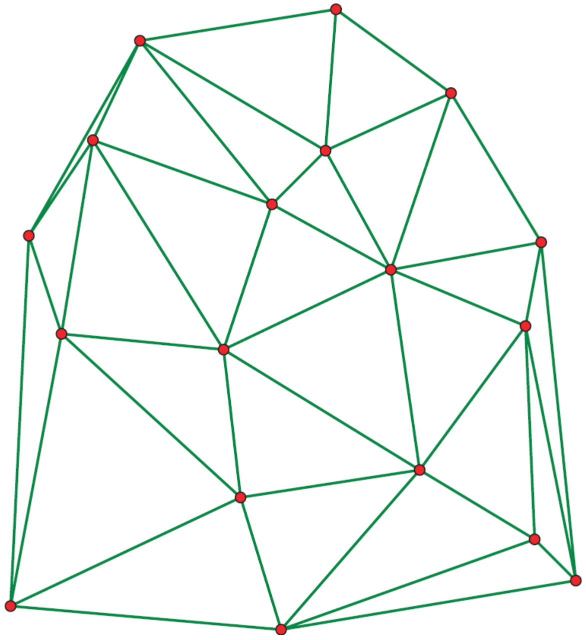
Delaunay Triangulation

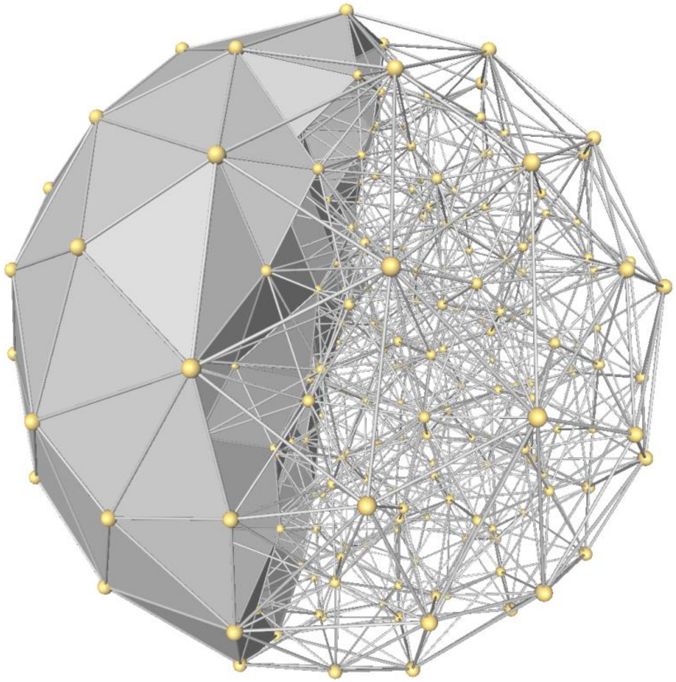
- The Delaunay triangulation of a point set E covers the convex hull of E.



Delaunay Triangulation

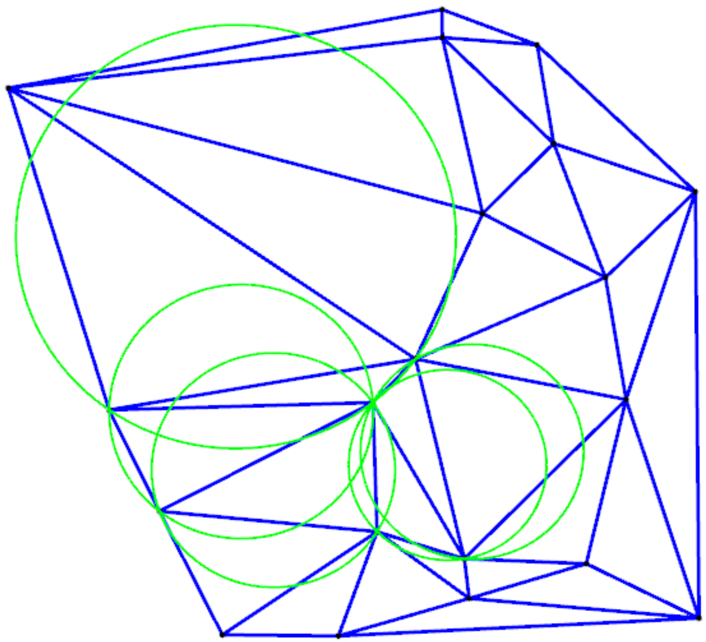
- canonical triangulation associated to any point set





Delaunay Triangulation: Local Property

- **Empty circle**: A triangulation T of a point set E such that any d-simplex of T has a circumsphere that does not enclose any point of E is a Delaunay triangulation of E. Conversely, any k-simplex with vertices in E that can be circumscribed by a hypersphere that does not enclose any point of E is a face of the Delaunay triangulation of E.



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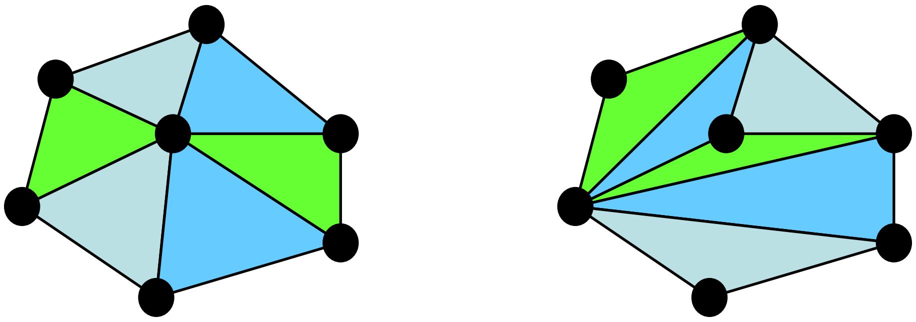
Delaunay Triangulation

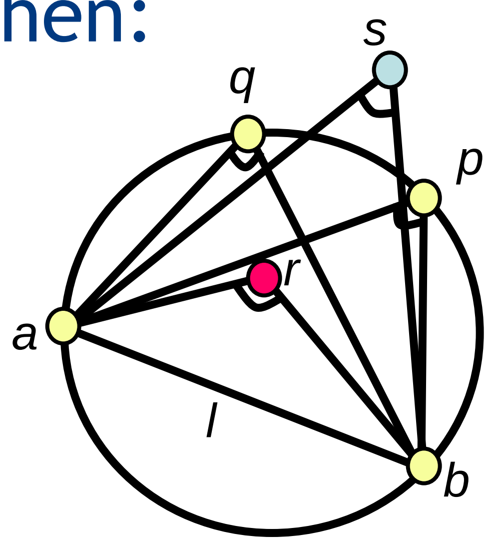
- **In 2D: « quality » triangulation**
  - Smallest triangle angle: The Delaunay triangulation of a point set E is the triangulation of E which **maximizes the smallest angle**.
  - Even stronger: The triangulation of E whose **angular vector** is **maximal** for the lexicographic order is the Delaunay triangulation of E.



good bad

Delaunay Triangulation

- **Thales' Theorem**: Let *C* be a circle, and *l* a line intersecting C at points *a* and *b*. Let *p*, *q*, *r* and *s* be points lying on the same side of *l*, where *p* and *q* are on *C*, *r* inside *C* and *s* outside *C*. Then:



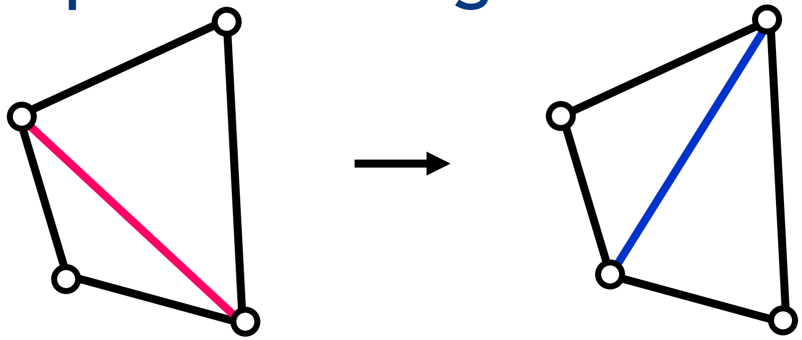
$$\angle arb = 2\angle apb$$

$$\angle aqb > \angle asb$$

Delaunay Triangulation

Improving a triangulation:

- In any convex quadrangle, an *edge flip* is possible. If this flip *improves* the triangulation locally, it also improves the global triangulation.

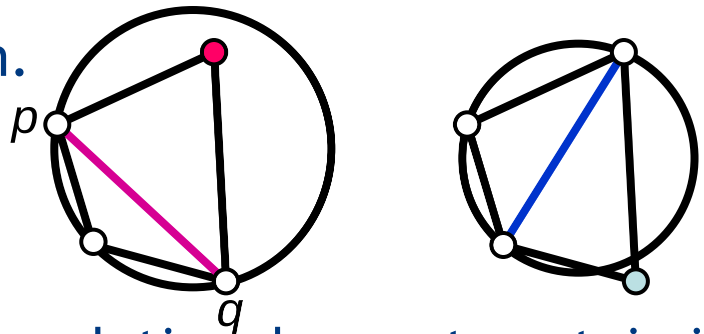


- If an edge flip improves the triangulation, the first edge is called **illegal**.

Delaunay Triangulation

Illegal edges:

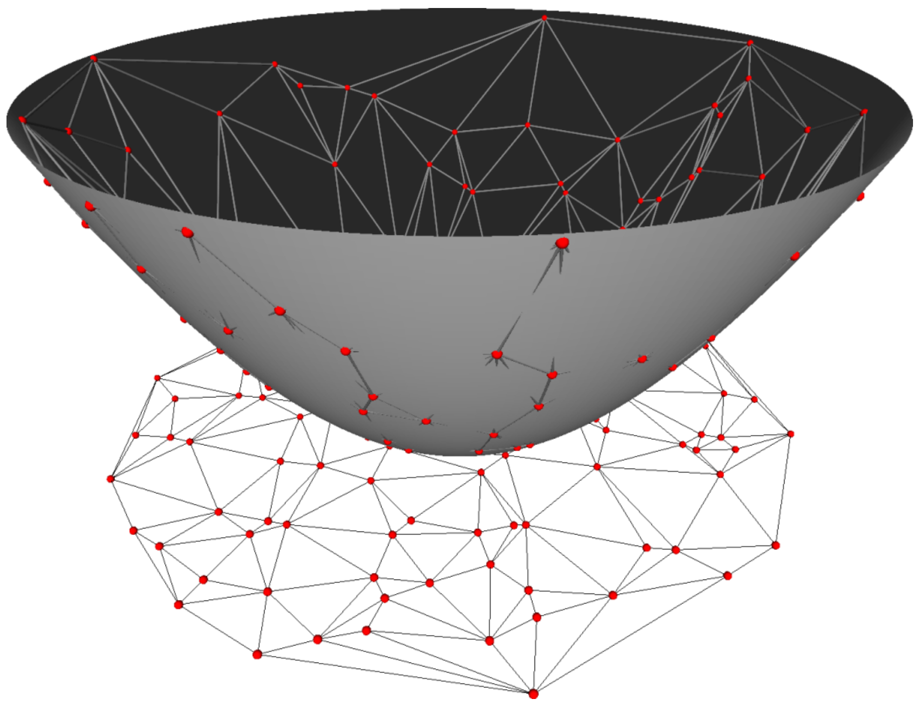
- **Lemma:** An edge *pq* is illegal iff one of its opposite vertices is inside the circle defined by the other three vertices.
- **Proof:** By Thales' theorem.



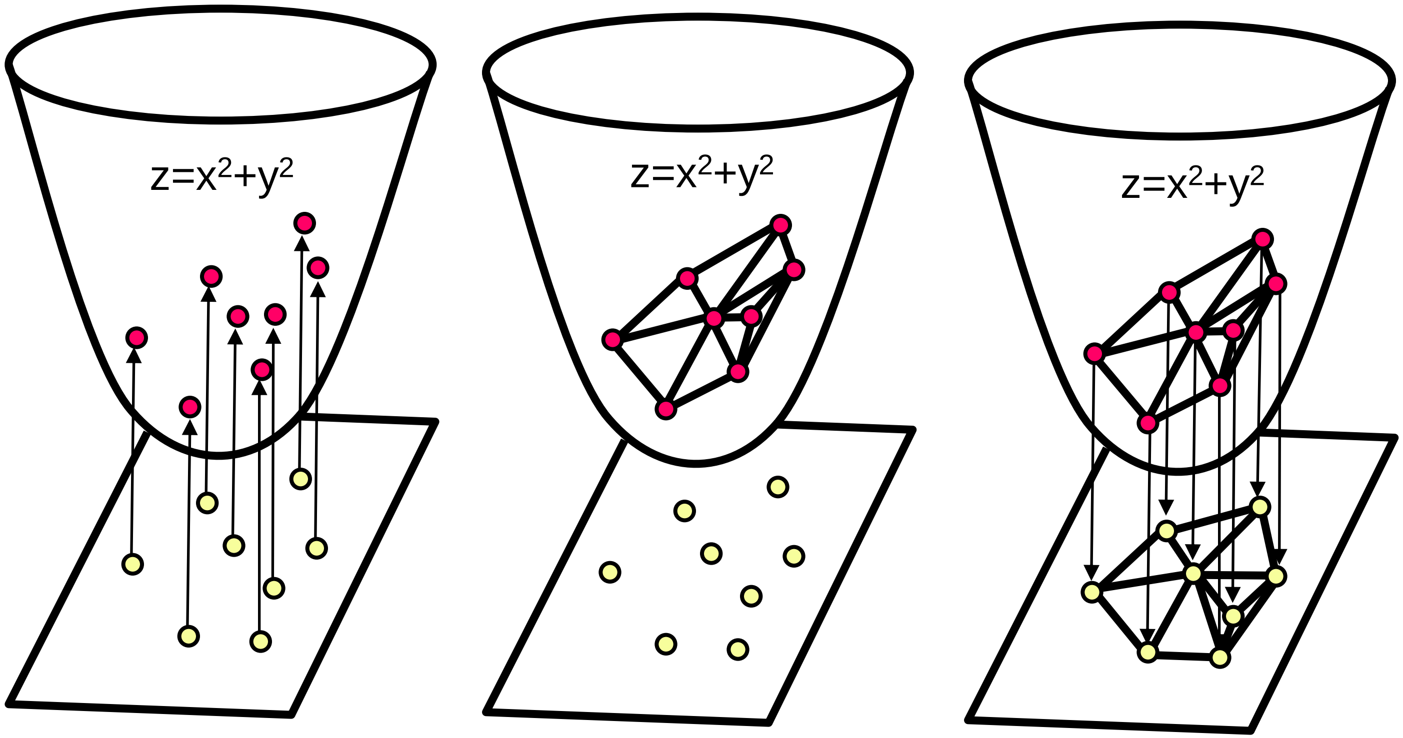
- **Theorem:** A Delaunay triangulation does not contain illegal edges.
- **Corollary:** A triangle is Delaunay iff the circle through its vertices is empty of other sites (the *empty-circle* condition).
- **Corollary:** The Delaunay triangulation is not unique if more than three sites are co-circular.

Delaunay Triangulation

- **Duality on the paraboloid**: Delaunay triangulation obtained by projecting the lower part of the convex hull.



Delaunay Triangulation



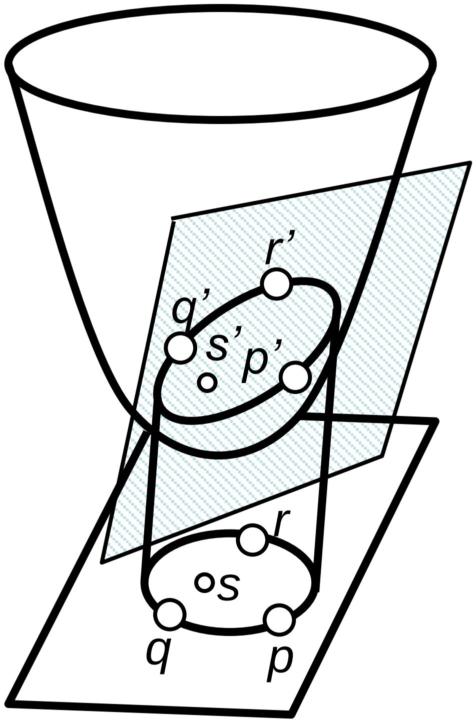
Project the 2D point set onto the 3D paraboloid → Compute the 3D lower convex hull → Project the 3D facets back to the plane.

Proof

- The intersection of a plane with the paraboloid is an ellipse whose projection to the plane is a circle.
- *s* lies within the circumcircle of *p*, *q*, *r* iff *s*' lies on the lower side of the plane passing through *p*', *q*', *r*'.

$\downarrow$

- *p*, *q*, *r* $\in$ *S* form a Delaunay triangle iff *p*', *q*', *r*' form a face of the convex hull of *S*'.

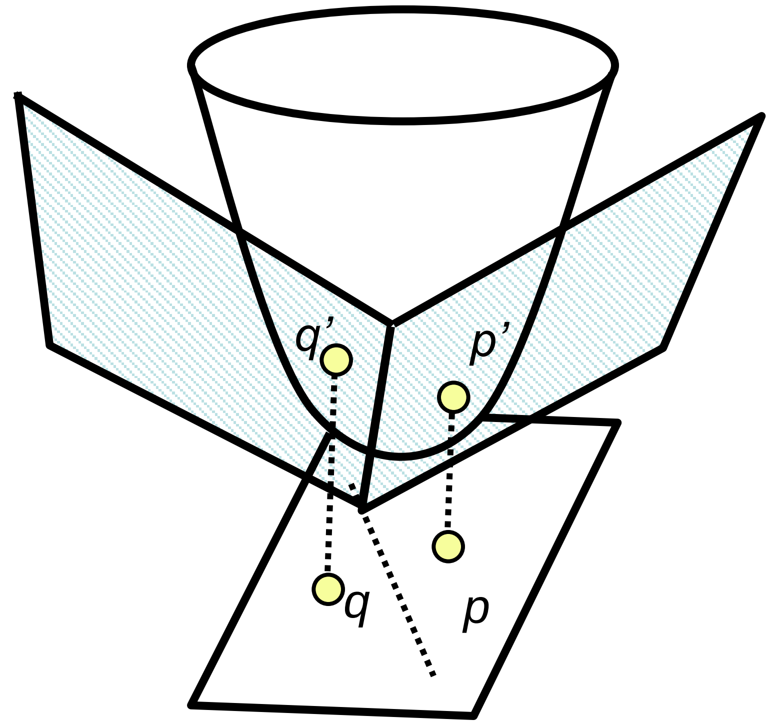


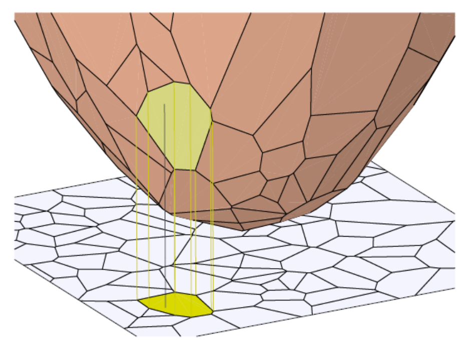
Voronoi Diagram

- Given a set S of points in the plane, associate with each point $p=(a,b) \in S$ the plane tangent to the paraboloid at p:

$$z = 2ax+2by-(a2+b2).$$

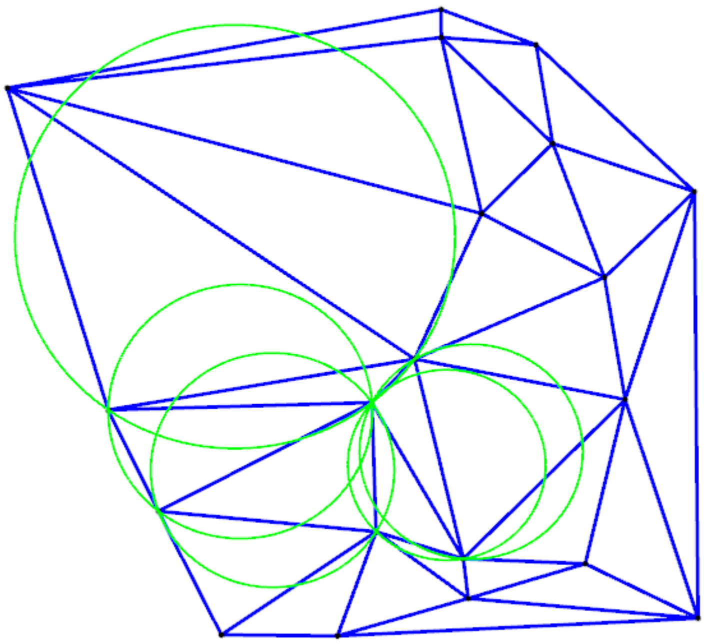
- VD(S) is the projection to the (x,y) plane of the 1-skeleton of the convex polyhedron formed from the intersection of the halfspaces above these planes.





Delaunay Triangulation

Naïve $O(n^4)$ Construction Algorithm?



Delaunay Triangulation

Naïve O(n^4) Construction Algorithm

- **Repeat until impossible:**
  - Select a triple of sites.
  - If the circle through them is empty of other sites, keep the triangle whose vertices are the triple.

The In-Circle Test

Theorem: If *a*,*b*,*c*,*d* form a CCW convex polygon, then *d* lies in the circle determined by *a*, *b* and *c* iff:

$$\det\begin{pmatrix} a_x & a_y & a_x^2+a_y^2 & 1 \\ b_x & b_y & b_x^2+b_y^2 & 1 \\ c_x & c_y & c_x^2+c_y^2 & 1 \\ d_x & d_y & d_x^2+d_y^2 & 1 \end{pmatrix} > 0$$

Proof: We prove that equality holds if the points are co-circular. There exists a center *q* and radius *r* such that:

$$(a_x - q_x)^2 + (a_y - q_y)^2 = r^2$$

and similarly for *b*, *c*, *d*:

$$\begin{pmatrix} a_x^2+a_y^2 \\ b_x^2+b_y^2 \\ c_x^2+c_y^2 \\ d_x^2+d_y^2 \end{pmatrix} - 2q_x\begin{pmatrix} a_x \\ b_x \\ c_x \\ d_x \end{pmatrix} - 2q_y\begin{pmatrix} a_y \\ b_y \\ c_y \\ d_y \end{pmatrix} + (q_x^2 + q_y^2 - r^2)\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 0$$

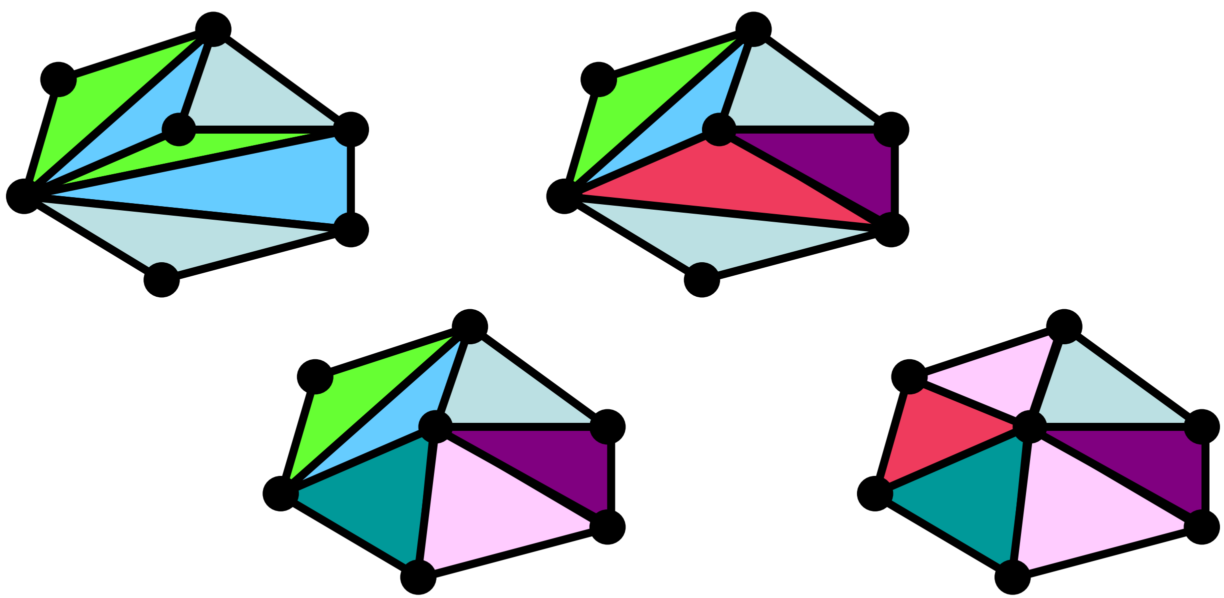
So these four vectors are linearly dependent, hence their det vanishes.

Corollary: $d\in$circle(a,b,c) iff $b\in$circle(c,d,a) iff $c\notin$circle(d,a,b) iff $a\notin$circle(b,c,d)

Delaunay Triangulation

Another naive construction:

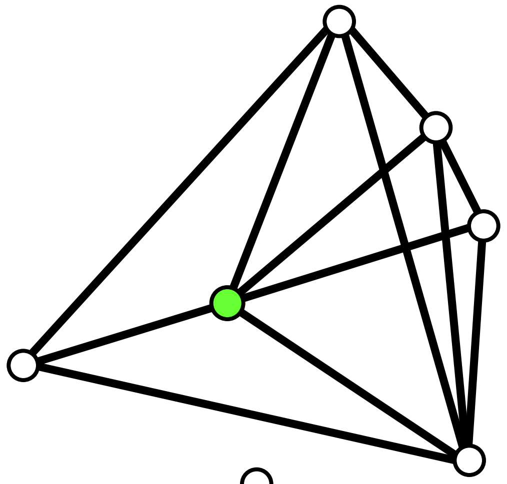
- Start with an arbitrary triangulation. Flip any illegal edge until no more exist.
- Requires proof that there are no local minima.
- Could take a long time to terminate.

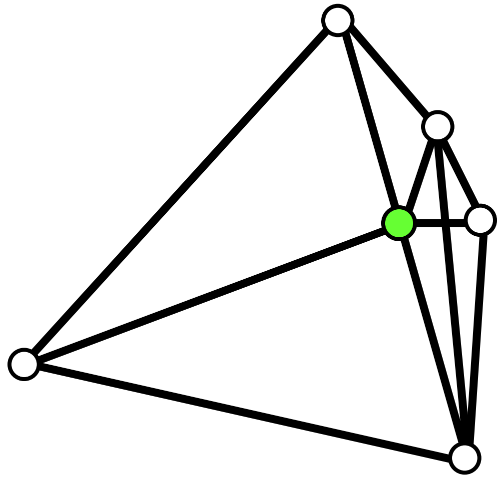


O(*n*log*n*) Delaunay Triangulation Algorithm

Incremental algorithm:

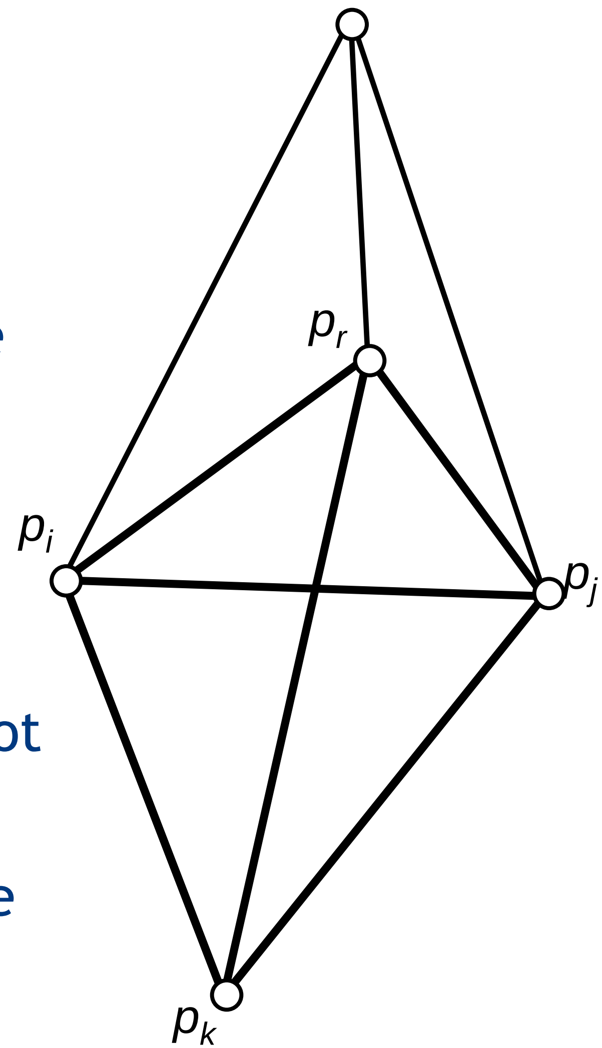
- Form bounding triangle which encloses all the sites.
- Add the sites one after another in random order and update triangulation.
- **If the site is inside an existing triangle:**
  - Connect site to triangle vertices.
  - Check if a 'flip' can be performed on one of the triangle edges. If so – check recursively the neighboring edges.
- **If the site is on an existing edge:**
  - Replace edge with four new edges.
  - Check if a 'flip' can be performed on one of the opposite edges. If so – check recursively the neighboring edges.



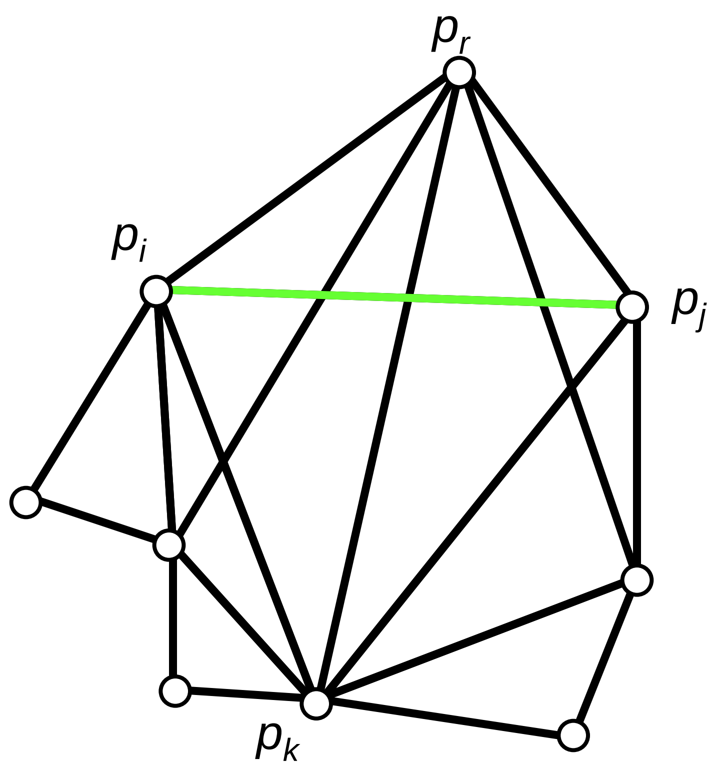


Flipping Edges

- A new vertex p_r is added, causing the creation of edges.
- The legality of the edge $p_i p_j$ (with opposite vertex) p_k is checked.
- If $p_i p_j$ is illegal, perform a flip, and recursively check edges $p_i p_k$ and $p_j p_k$, the new edges opposite p_r.
- Notice that the recursive call for $p_i p_k$ cannot eliminate the edge $p_r\, p_k$.
- **Note:** All edge flips replace edges opposite the new vertex by edges incident to it!



Flipping Edges - Example



Number of Flips

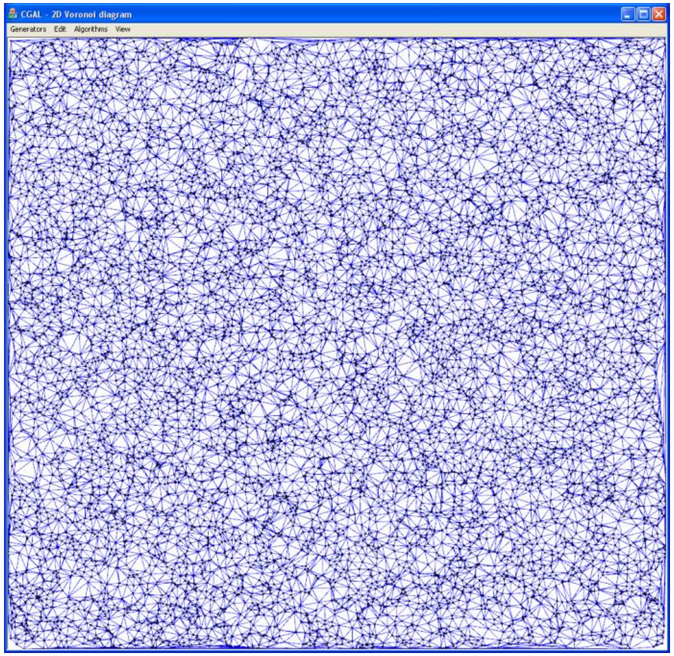
- **Theorem:** The expected number of edges flips made in the course of the algorithm (some of which also disappear later) is at most $6n$.

- **Proof:** During insertion of vertex p_i, k_i new edges are created: 3 new initial edges, and k_i-3 due to flips.

    Backward analysis: $E[k_i]$ = the expected degree of p_i after the insertion is complete = 6 (Euler).

Algorithm Complexity

- **Point location for every point:** O(log n) time.
- **Flips:** Θ(n) expected time in total (for all steps).
- **Total expected time:** O(n log n).
- **Space:** $\Theta(n)$.



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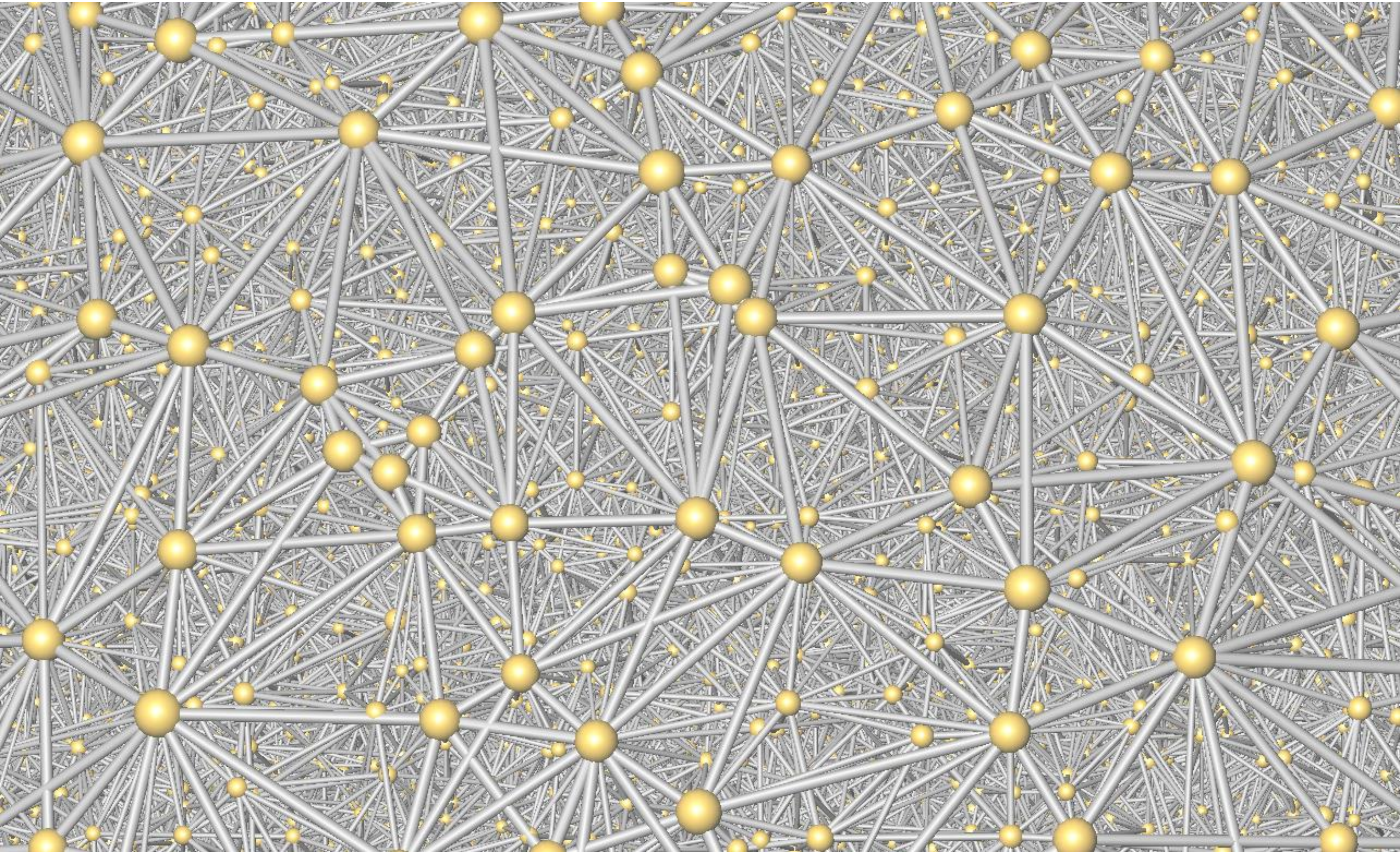
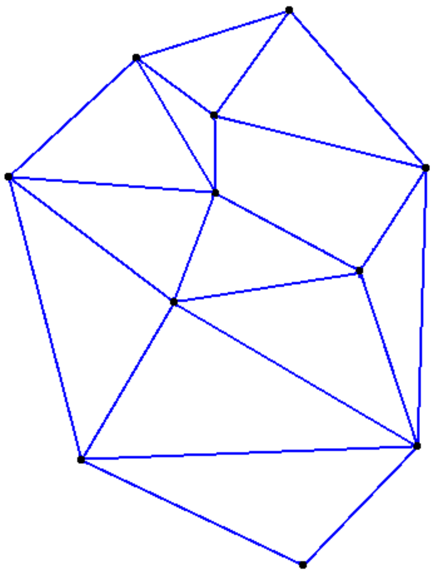
3D Delaunay Triangulation

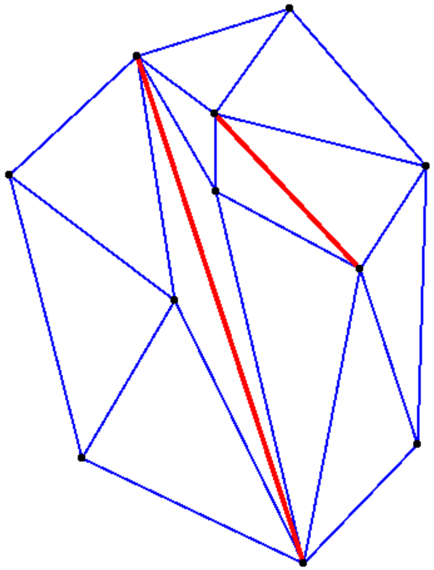


Constrained Delaunay Triangulation

2D Constrained Delaunay Triangulation

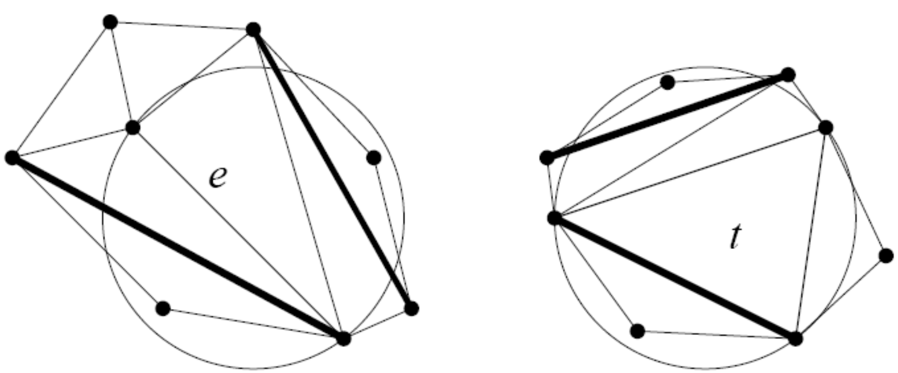
- **Definition 1** : Let (P, S) be a PSLG. The constrained triangulation T(P, S) is constrained Delaunay iff the circumcircle of any triangle t of T encloses no vertex visible from a point in the relative interior of t.





2D Constrained Delaunay Triangulation

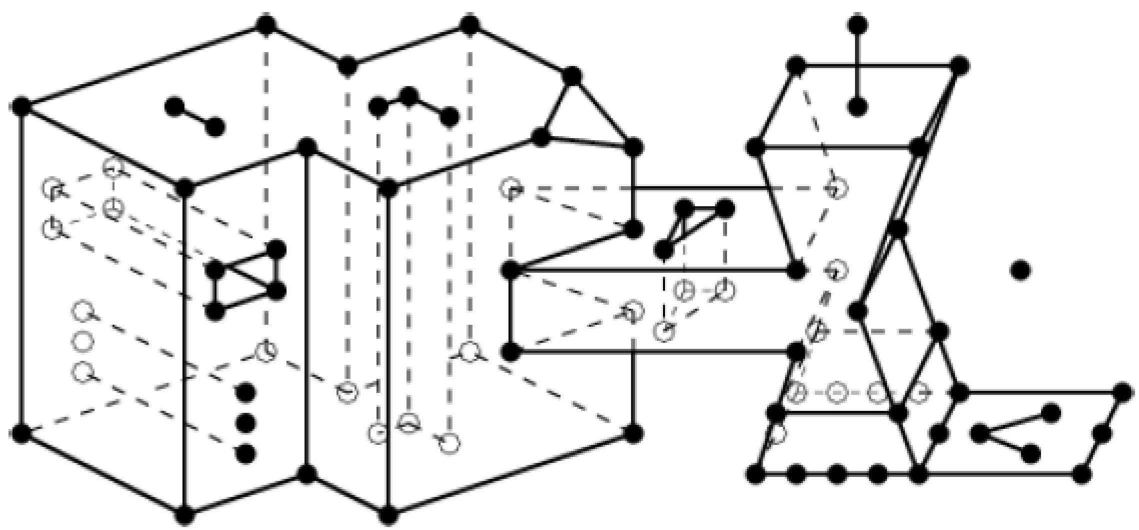
- **Definition 2** : Let (P, S) be a PSLG. The constrained triangulation T(P, S) is constrained Delaunay iff any edge e of T is either a segment of S or is constrained Delaunay.
- Simplex e constrained Delaunay with respect to the PSLG (P, S) iff: int(e) $\cap$ S = 0
- There exists a circumcircle of e that encloses no vertex visible from a point in the relative interior of e.



2D Constrained Delaunay Triangulation

- Any PSLG (P, S) has a constrained Delaunay triangulation. If (P, S) has no degeneracy , this triangulation is unique.

3D Constrained Delaunay Triangulation



[Shewchuk, Si]