Convex Hull

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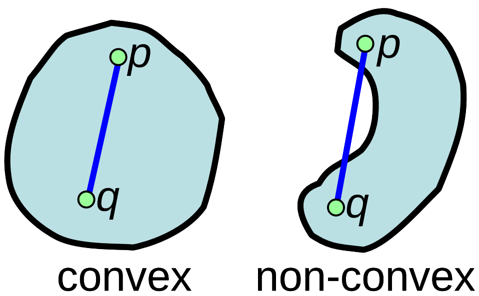
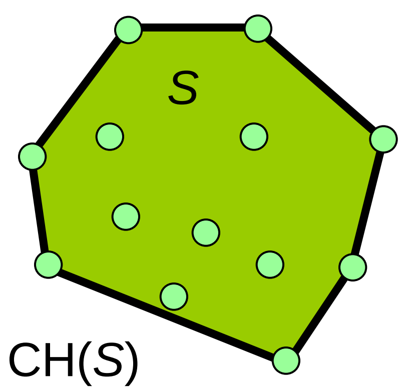
Convexity and Convex Hull

- A set S is *convex* if any pair of points $p,q \in S$ satisfy $pq \subseteq S$.



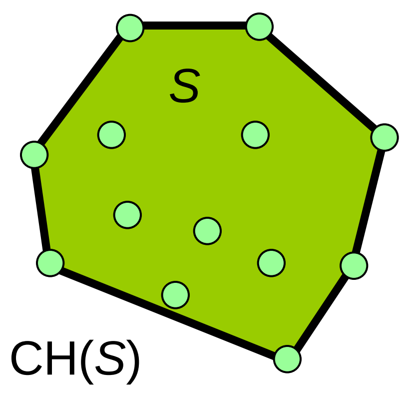
- The *convex hull* of a set S is:
  - The minimal convex set that contains S, i.e. any convex set C such that $S \subseteq C$ satisfies $\mathrm{CH}(S) \subseteq C$.
  - The intersection of all convex sets that contain S.
  - The set of all convex combinations of $p_i \in S$, i.e. all points of the form:

$$\sum_{i=1}^{n} \alpha_i p_i \ , \qquad \alpha_i \geq 0, \ \sum_{i=1}^{n} \alpha_i = 1$$



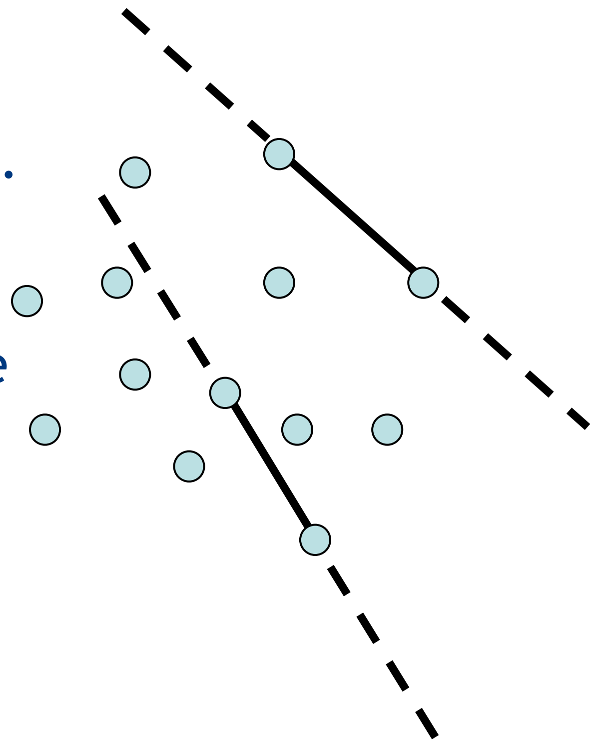
Convex Hulls – Some Facts

- The convex hull of a set is unique (up to colinearities).
- The boundary of the convex hull of a point set is a polygon on a subset of the points.



Convex Hull – Naive Algorithm

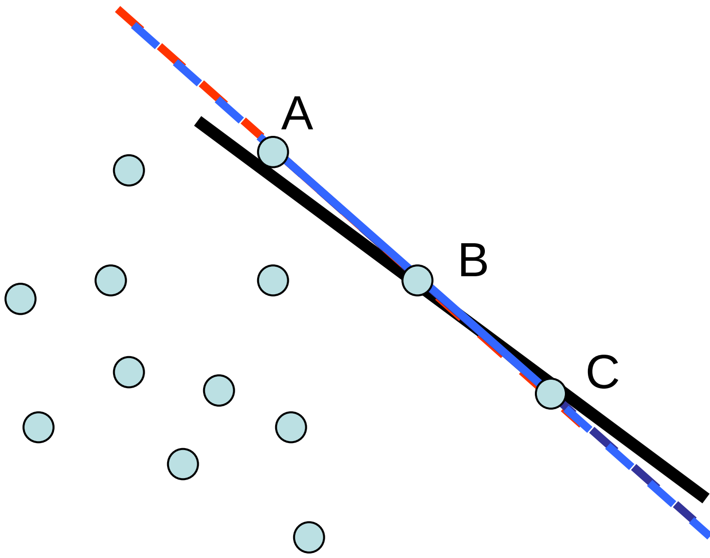
- Description:
  - For each pair of points construct its connecting segment and *supporting line*.
  - Find all the segments whose supporting lines divide the plane into two halves, such that one half plane contains *all* the other points.
  - Construct the convex hull out of these segments.



- Time complexity:
  - All pairs: $$O\left(\binom{n}{2}\right) = O\left(\frac{n(n-1)}{2}\right) = O(n^2)$$
  - Check all points for each pair: O(n)
  - Total: O(n^3)

Possible Pitfalls

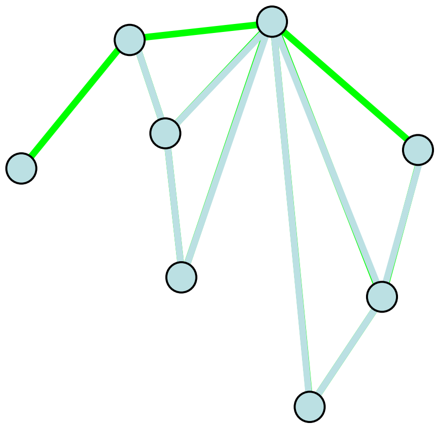
- Degenerate cases – e.g. 3 collinear points. Might harm the correctness of the algorithm. Segments AB, BC and AC will *all* be included in the convex hull.



- Numerical problems – We might conclude that *none* of the three segments belongs to the convex hull.

Convex Hull – Graham's Scan

- Algorithm:
  - Sort the points according to their *x* coordinates.
  - Construct the upper boundary by scanning the points in the sorted order and performing only "right turns".
  - Construct the lower boundary (with "left turns").
  - Concatenate the two boundaries.



- Time Complexity: O(*n*log*n*)
- May be implemented using a stack
- **Question:** How do we check for "right turn" ?

The Algorithm

- Sort the points in increasing order of x-coord:

  $$p_1, \ldots, p_n.$$

- Push(S,p_1); Push(S,p_2);
- For i = 3 to n do
  - While Size(S) $\geq$ 2 and Orient(p_i,top(S),second(S)) $\leq$ 0 do Pop(S);
  - Push(S,p_i);
- Print(S);

Graham's Scan – Time Complexity

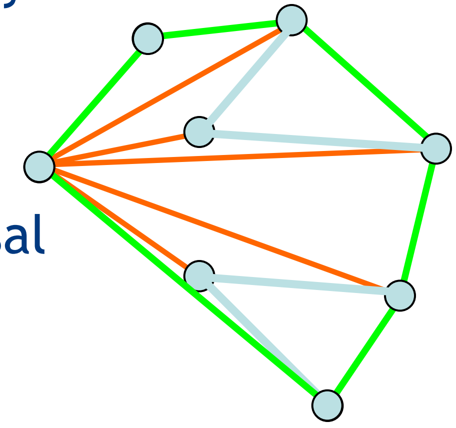
- Sorting – O(n log n)
- If D_i is number of points popped on processing p_i,

$$\text{time} = \sum_{i=1}^{n} (D_i + 1) = n + \sum_{i=1}^{n} D_i$$

- Each point is pushed on the stack only once.
- Once a point is popped – it cannot be popped again.
- Hence $$\sum_{i=1}^{n} D_i \leq n$$

Graham’s Scan– a Variant

- Algorithm:
  - Find one point, p_0, which must be on the convex hull.
  - Sort the other points by the *angle* of the rays to them from p_0.
  - **Question:** Is it necessary to compute the actual angles ?
  - Construct the convex hull using one traversal of the points.



- Time Complexity: O(*n* log *n*)

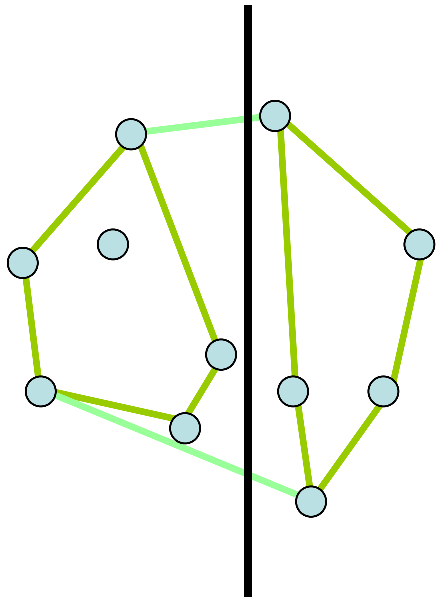
- **Question:** What are the pros and cons of this algorithm relative to the previous ?

Convex Hull - Divide and Conquer

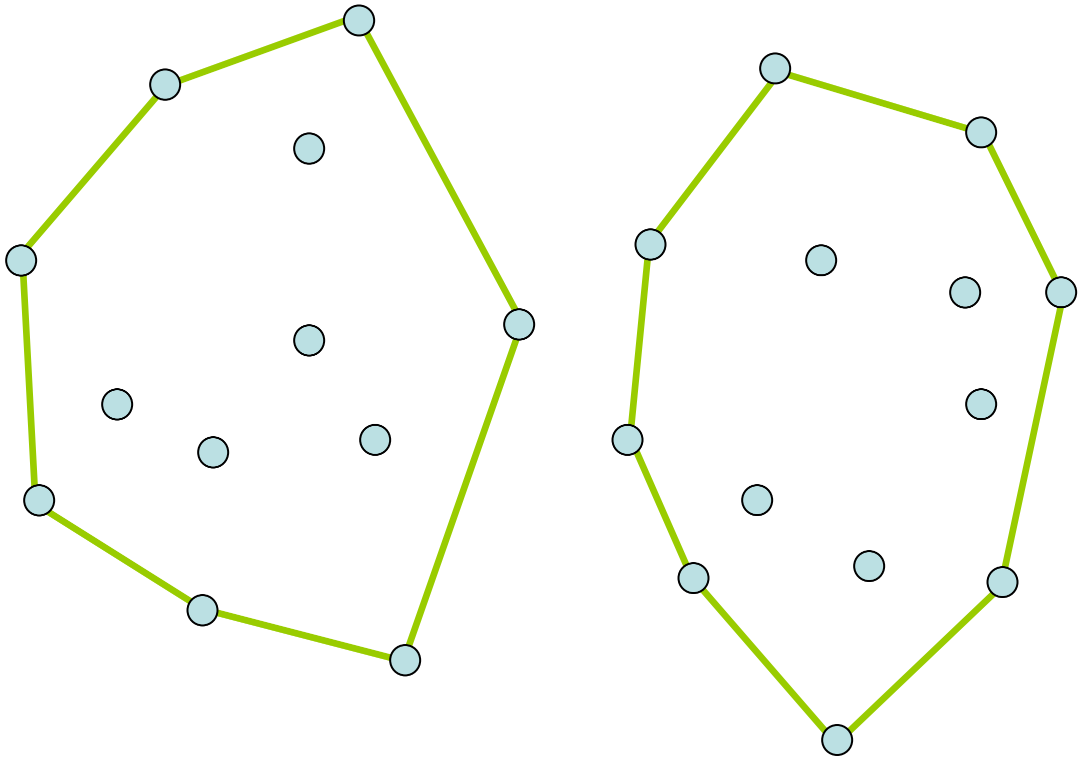
- Algorithm:
  - Find a point with a median *x* coordinate (time: O(*n*))
  - Compute the convex hull of each half (recursive execution)
  - Combine the two convex hulls by finding common *tangents*.

    This can be done in O(*n*).

  - Complexity: O(*n*log*n*)



Finding Common Tangents



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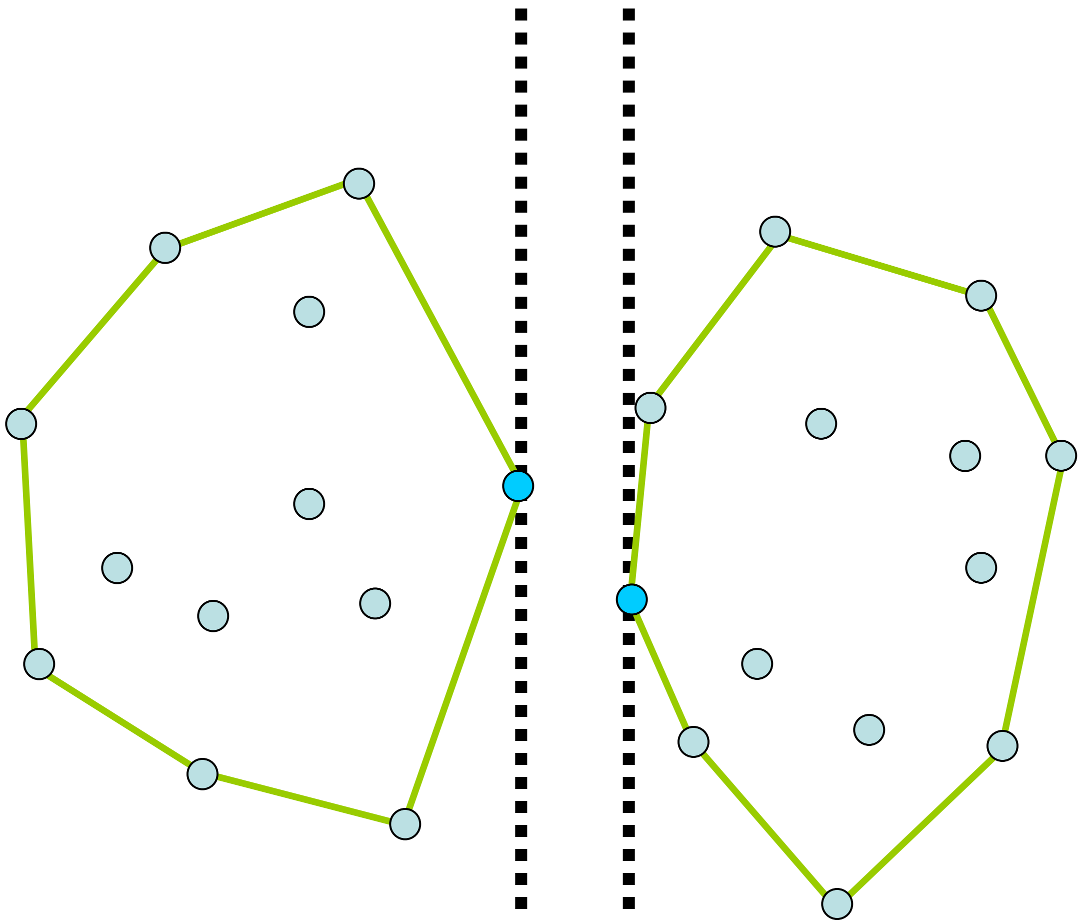

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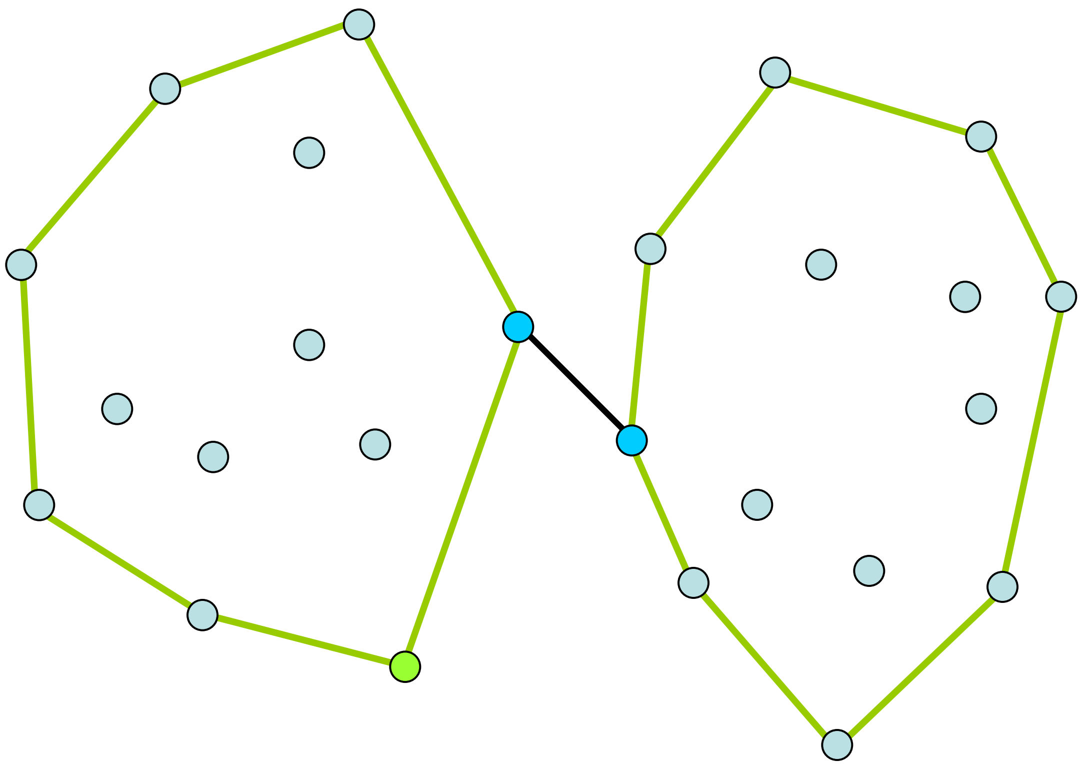

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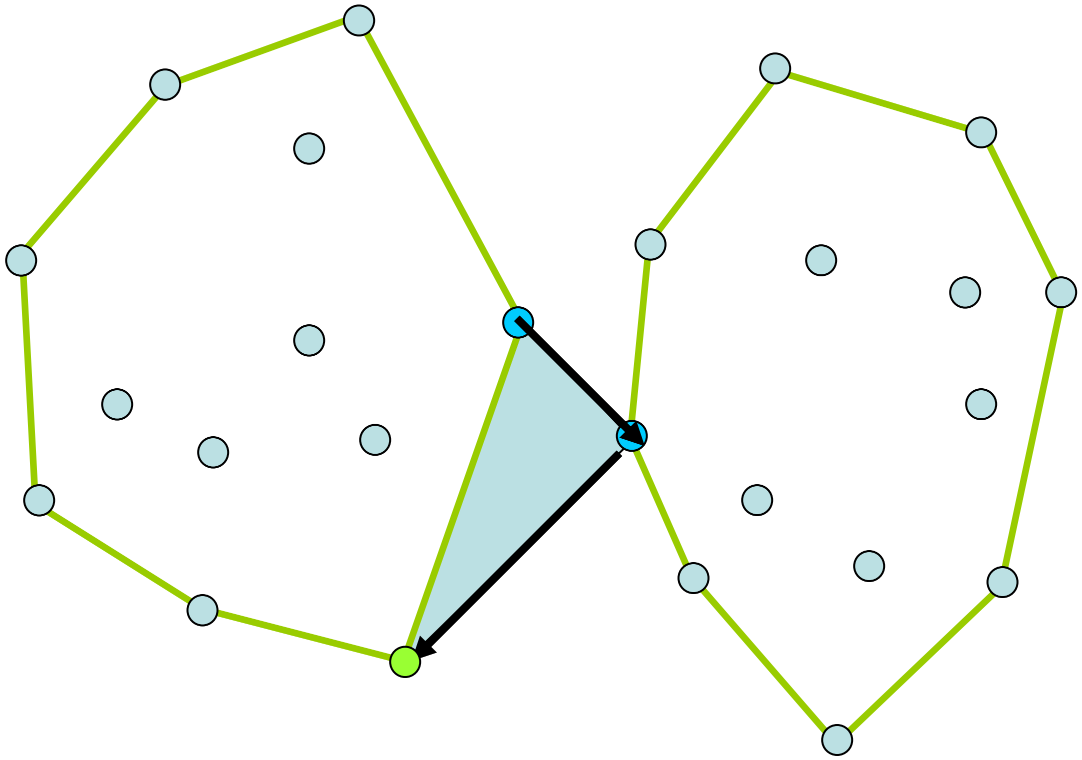

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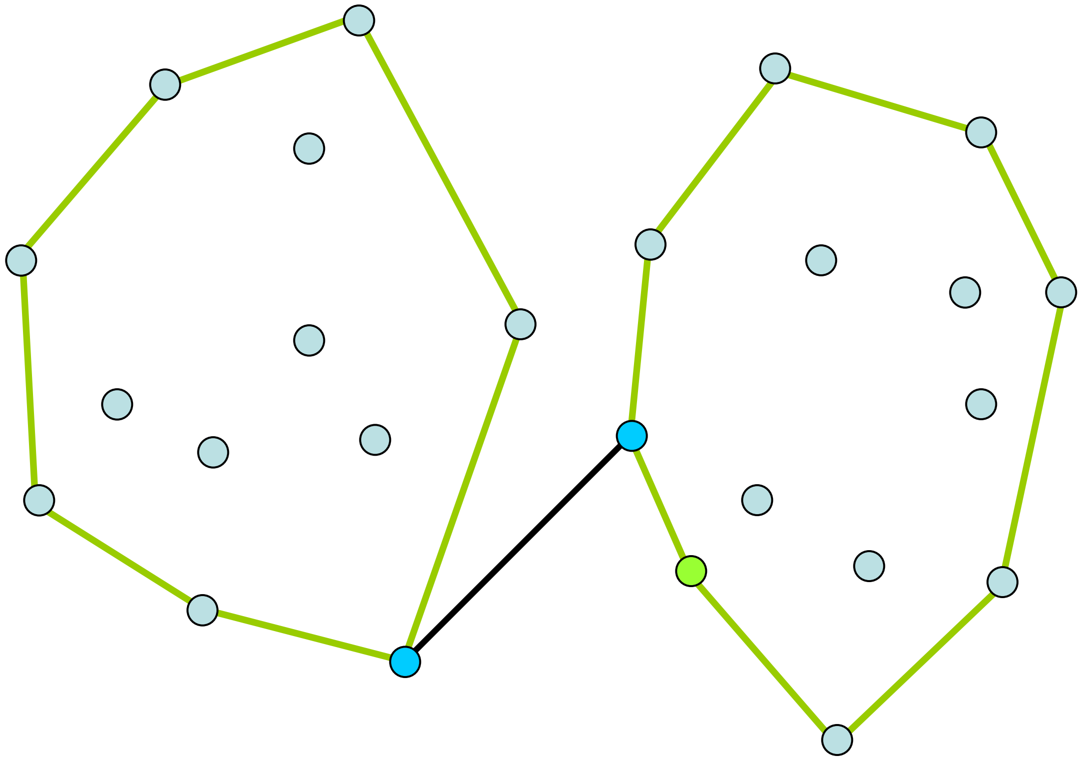

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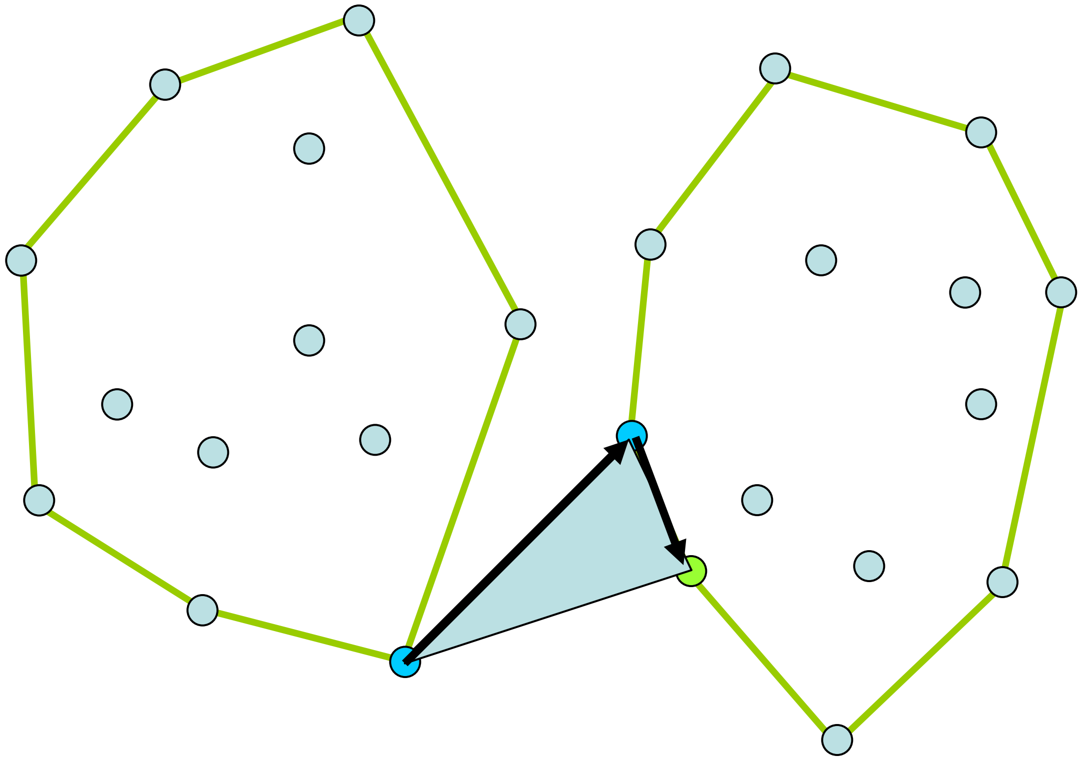

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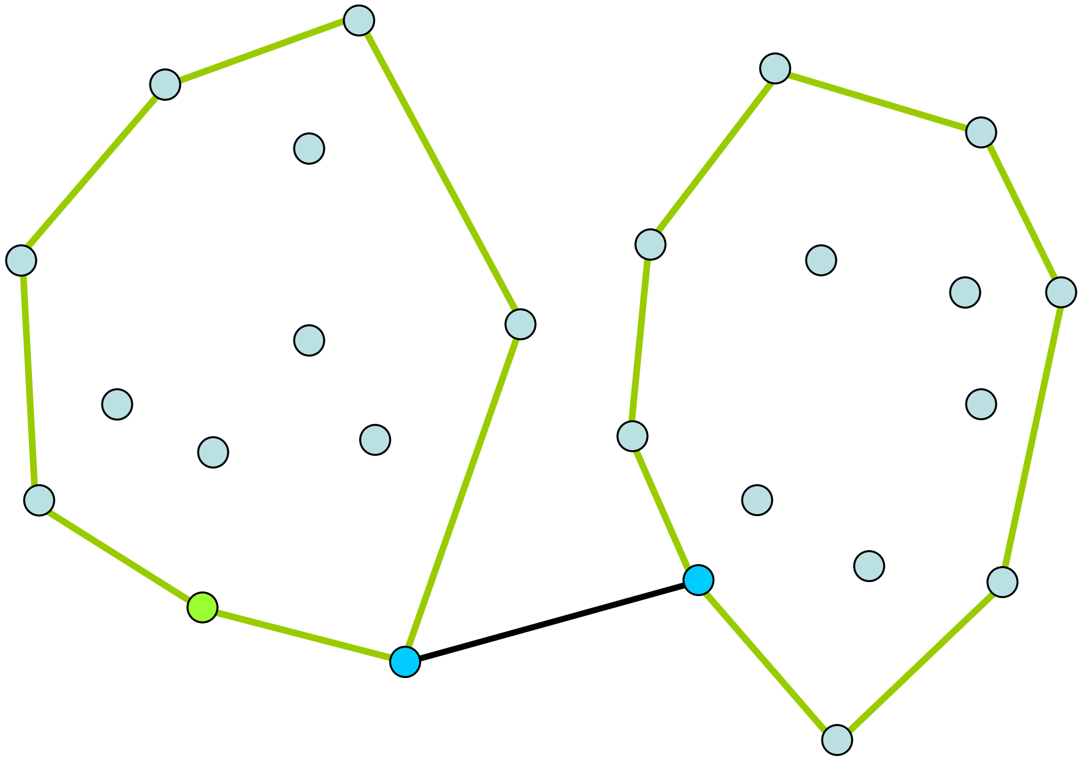

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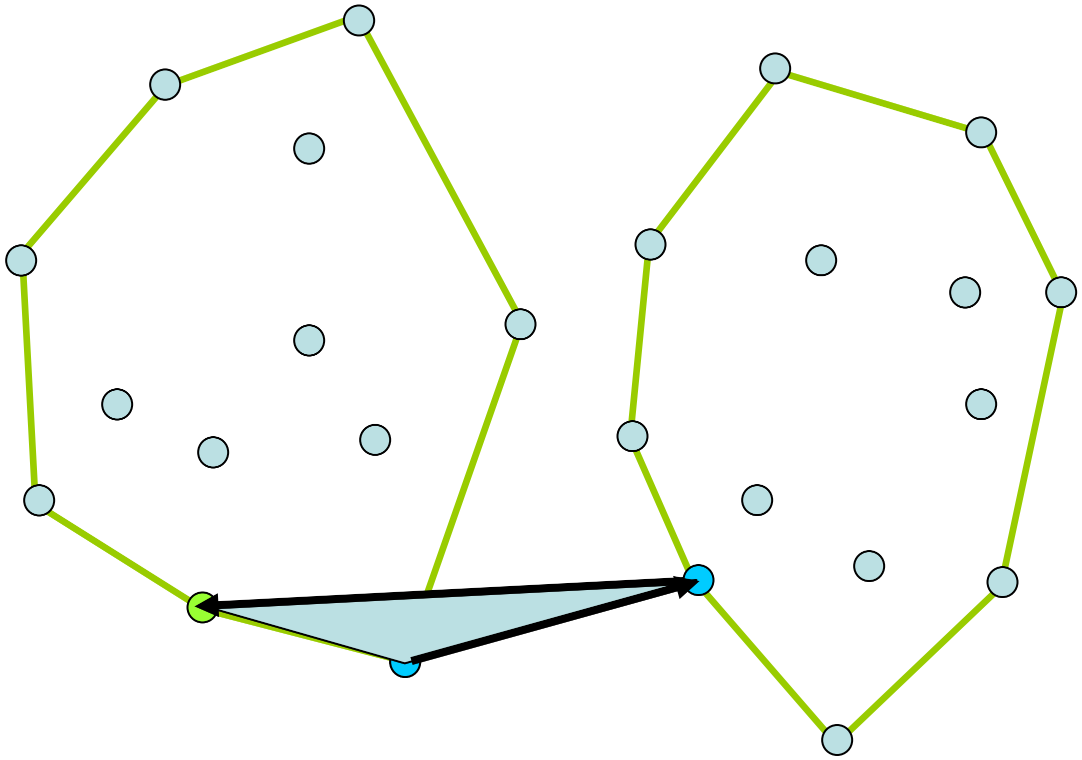

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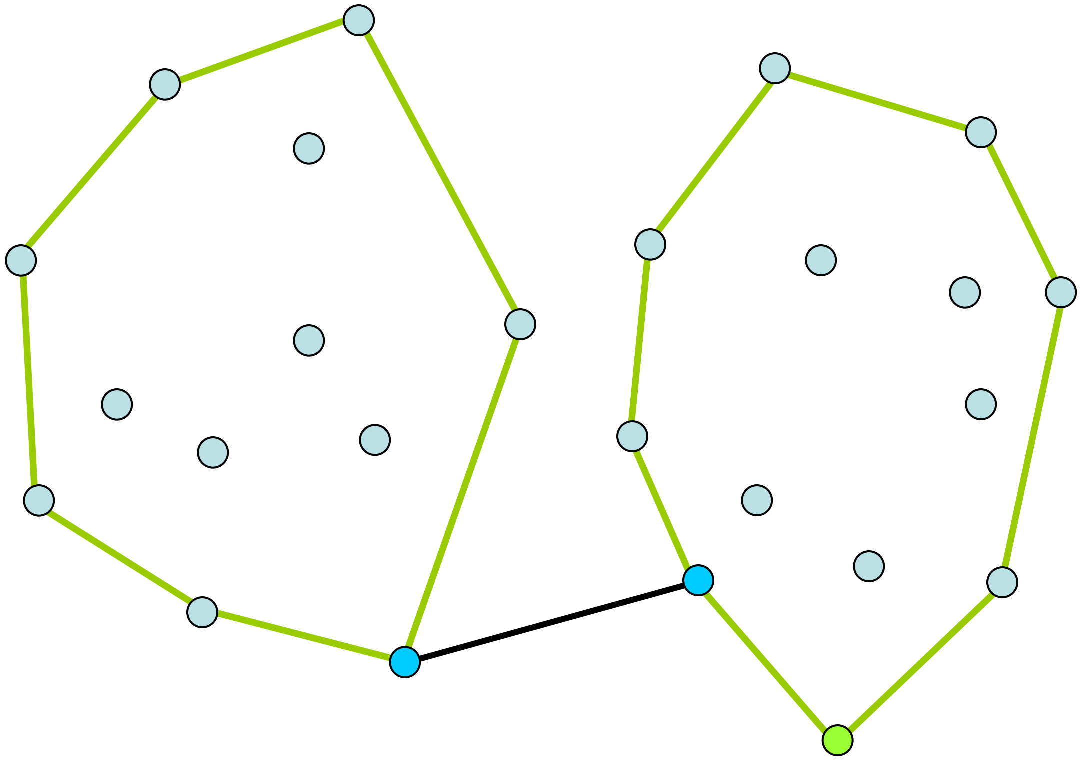

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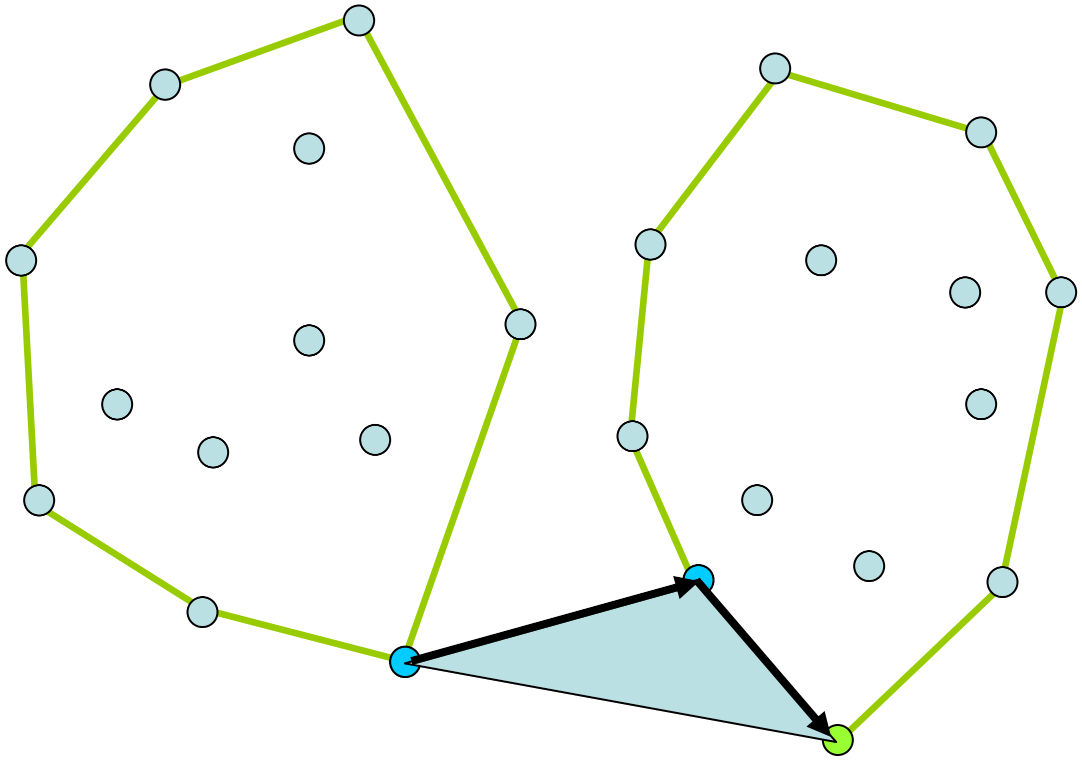

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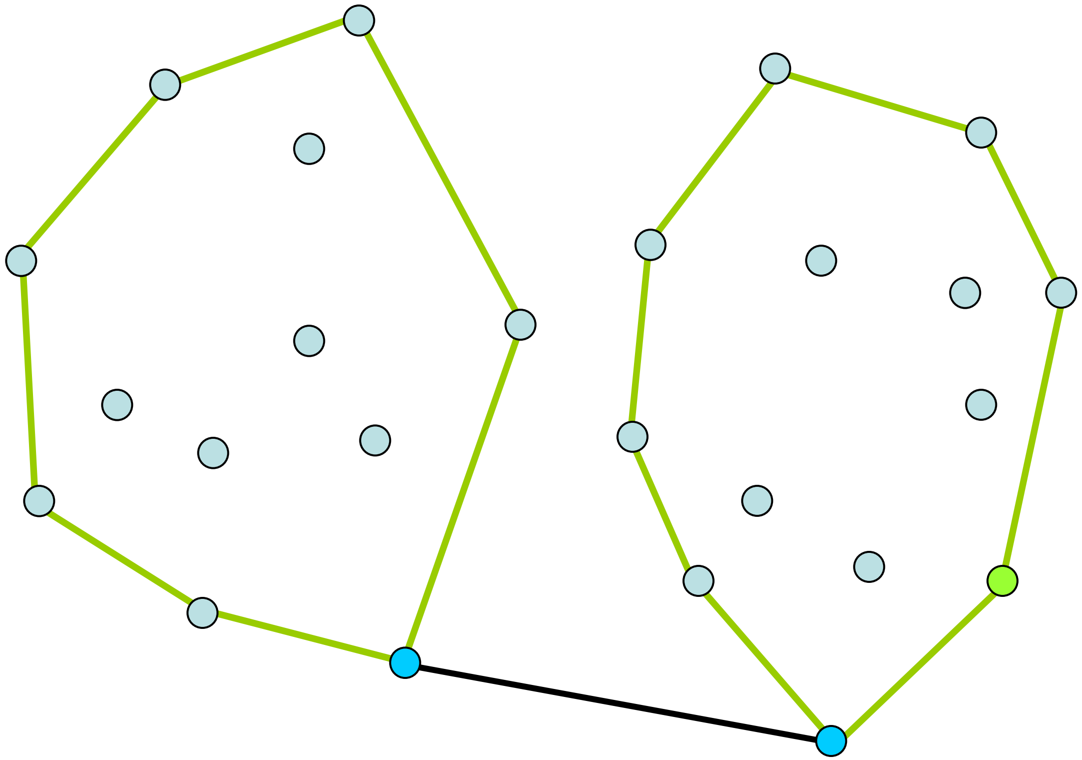

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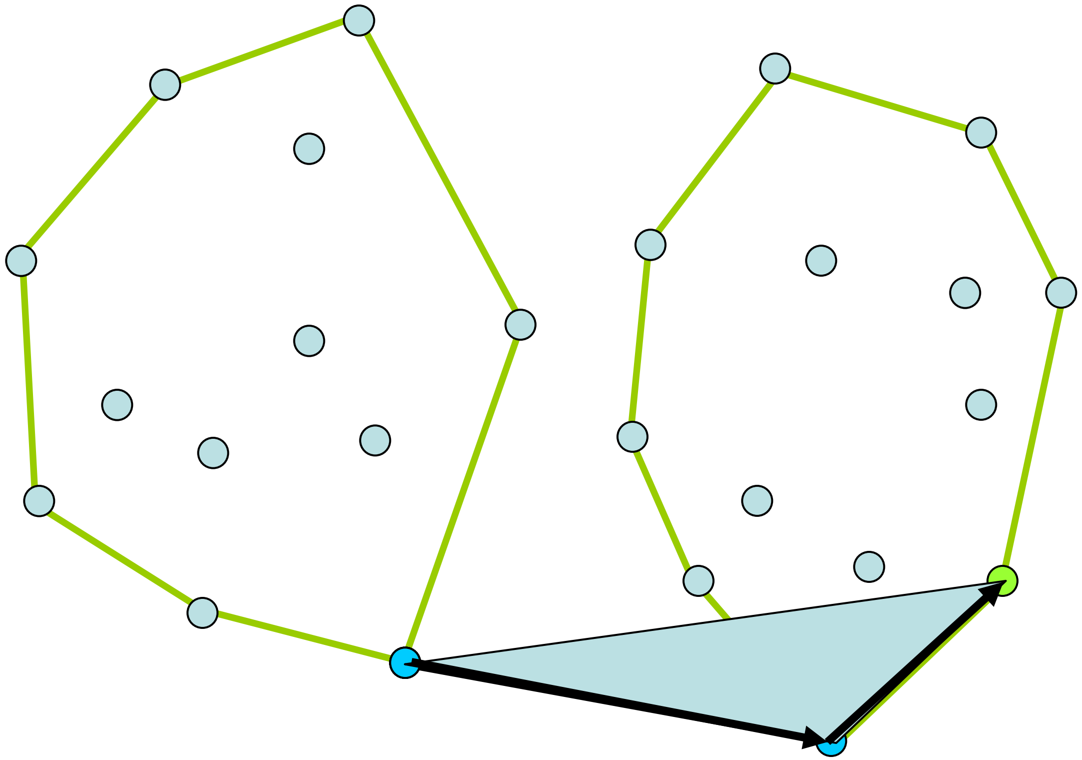

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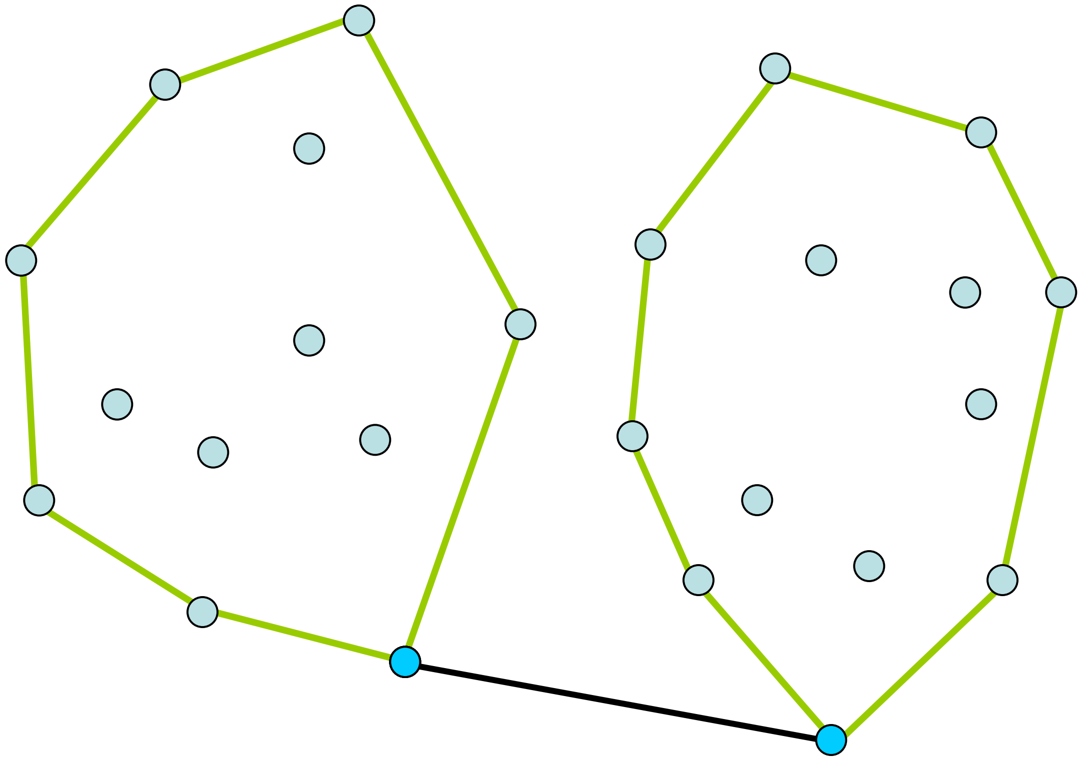

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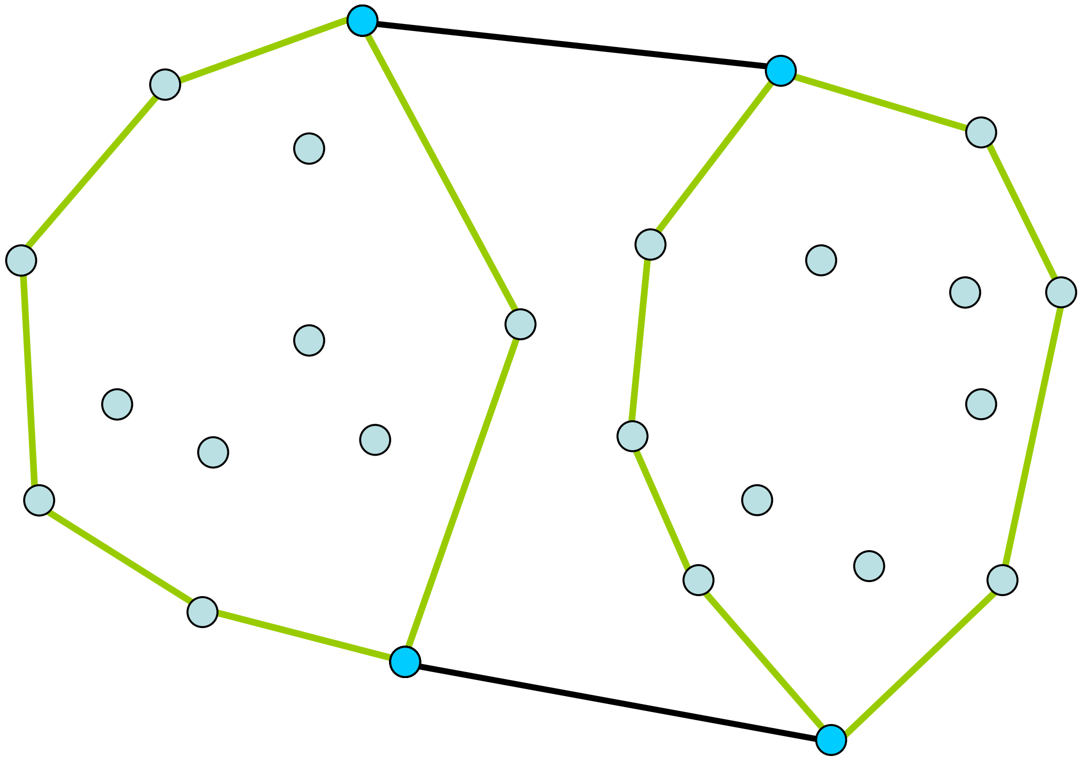


Finding Common Tangents

To find lower tangent:

- Find *a* - the rightmost point of H_A
- Find *b* – the leftmost point of H_B

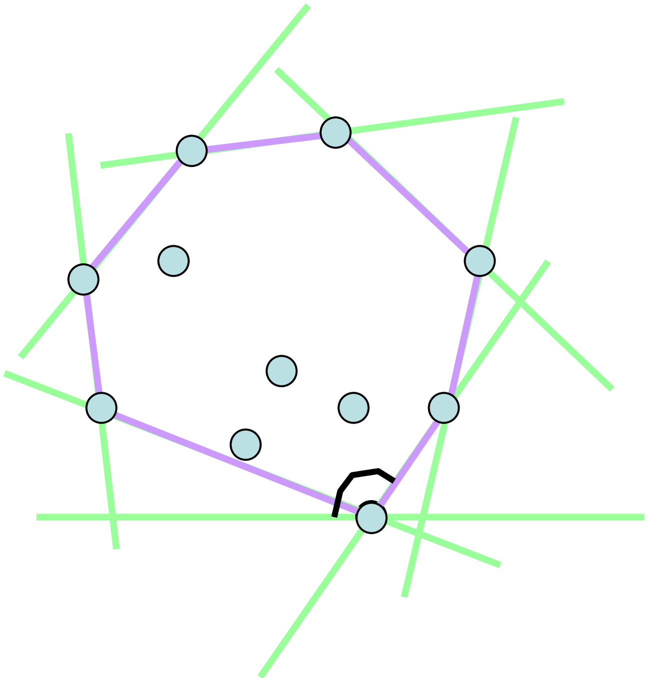
O(n)

- While *ab* is not a lower tangent for H_A and H_B, do:
  - If *ab* is not a lower tangent to H_A do $a = a$-1
  - If *ab* is not a lower tangent to H_B do $b = b$-1

Output-Sensitive Convex Hull Gift Wrapping

- Algorithm:
  - Find a point p_1 on the convex hull (e.g. the lowest point).
  - Rotate counterclockwise a line through p_1 until it touches one of the other points (start from a horizontal orientation).

    Question: How is this done ?



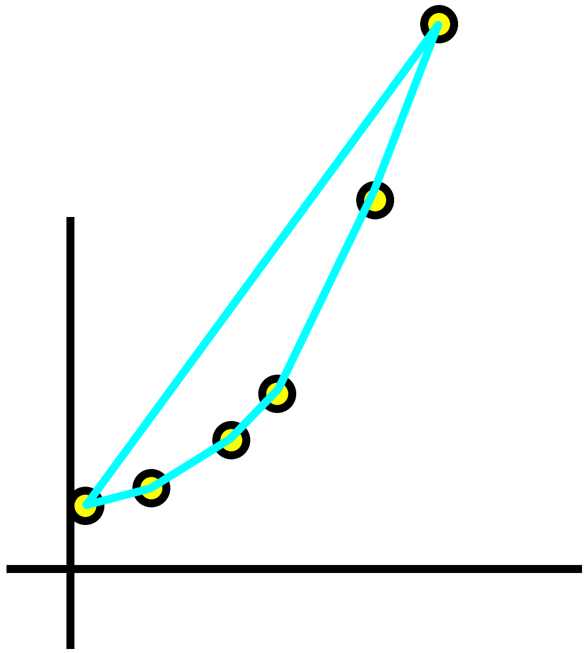
  - Repeat the last step for the new point.
  - Stop when p_1 is reached again.
- Time Complexity: O(nh), where n is the input size and h is the output (hull) size.

General Position

- When designing a geometric algorithm, we first make some simplifying assumptions, e.g.
  - No 3 collinear points.
  - No two points with the same *x* coordinate.
  - etc.
- Later, we consider the general case:
  - How should the algorithm react to degenerate cases ?
  - Will the correctness be preserved ?
  - Will the runtime remain the same ?

Lower Bound for Convex Hull

- A reduction from sorting to convex hull is:
  - Given n real values x_i, generate n 2D points on the graph of a convex function, e.g. (x_i, x_i^2).
  - Compute the (ordered) convex hull of the points.
  - The order of the convex hull points is the numerical order of the x_i.



- So CH=$\Omega(n \lg n)$