Internship Proposal
Reoptimization Methods in Convex Conic Optimization

Keywords: Conic Optimization, Convex Optimization, Julia Programming Language

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1 Background

In several contexts, we need to optimize multiple optimization problems for which only a limited part of the parameters changes, typically in the objective. This occurs for instance when solving subproblems within a decomposition algorithm (Benders or Dantzig-Wolfe decompositions), when there is uncertainty in the problem specification as in stochastic and robust optimization, or when optimizing sequences of policies in Markov decision processes or subproblems in the context of differentiable optimization layers in machine learning pipelines [2, 1].

When all problems in the series are linear, the simplex algorithm is the typical method of choice, with powerful hot-start and warm-start capabilities using the primal or dual simplex, depending on whether the objective or right-hand side vector change. When the problem is convex but nonlinear, the typically used algorithms (interior point methods, sequential quadratic programming) are notoriously harder to warm-start, which means that every new problem has to be solved from scratch. The question is then whether we can design methods that leverage the fact that multiple similar problems are solved to avoid using the more expensive solver on every one of them.

2 Method and goals

We consider convex conic optimization problems:

$$
\begin{aligned}
(P)\quad & \min_{x}\ c^\top x \\
& \text{s.t. } Ax = b \\
& \qquad x \in \mathcal{K},
\end{aligned}
$$

with $\mathcal{K}$ a Cartesian product of convex proper cones in the sense of [3]. We aim to design efficient methods to optimize sequences of problems (P) with a varying objective c (or equivalently the right-hand side b by duality). The dual of (P) is:

$$
\begin{aligned}
(D)\quad & \min_{y,s}\ b^\top y \\
& \text{s.t. } A^\top y + s = c \\
& \qquad s \in \mathcal{K}^*
\end{aligned}
$$

with $\mathcal{K}^*$ the dual cone of $\mathcal{K}$. The optimality of a primal-dual triplet (x, y, s) is expressed as:

$$
\begin{aligned}
(KKT)\quad & A^\top y + s = c \\
& Ax = b \\
& (x, s) \in \mathcal{K} \times \mathcal{K}^* \\
& x \circ s = 0.
\end{aligned}
$$

If after optimizing the problem for given objectives $\{c_1, c_2, \dots c_{t-1}\}$, one stores the corresponding triplets $\{(x_1, y_1, s_1), (x_2, y_2, s_2), \dots (x_{t-1}, y_{t-1}, s_{t-1})\}$, we can evaluate the quality of the k-th solution $c_t^\top x_k$ for the new objective by finding a corresponding dual vector y, which solves the linear system:

$$A^\top y = b - s_k.$$

If the system admits a solution $\hat{y}$, then (KKT) is satisfied by $(x_k, \hat{y}, s_k)$ and the primal point x is optimal for the new objective. The two main directions of the project will be to:

- define new methods to provide dual bounds from inexact dual certificates, i.e. compute an upper bound on the gap between the optimal value $c^\top x$ and that of the solution $c^\top x_k$,
- design algorithms to reoptimize jointly (y, s) for a given x.

The goal will then be to design and implement algorithms to optimize sequences of conic optimization problems and adaptively call an underlying solver providing exact (but expensive) solutions, only when the previously-collected solutions cannot provide a satisfactory objective value. We will compare the new method against:

1. Standard solvers solving the problem from scratch at every iteration
2. Solving methods capable of warm-starting from a single solution, such as ADMM-based methods.[1]

3 Candidate Profile

We are looking for a Master's student in applied mathematics, computer science, statistics, operations research, or related fields. The candidate should be familiar with:

1. constrained convex optimization (elements of linear and conic optimization, Lagrangian duality, modeling in convex optimization)
2. basics of scientific programming (solving optimization problems and linear systems, building problem using modeling language such as JuMP or CVXPy).

4 Additional information

Context
The internship may be continued as a PhD.

Contact
For more information, please contact `mathieu.besancon@polymtl.ca` and `nicolas.gast@inria.fr`.

Location
The intern will be hosted in the POLARIS Inria team which is joint between Inria and the LIG (Grenoble Computer Science Laboratory) and is located on the Grenoble University main campus (`https://batiment.imag.fr/`).

References

[1] B. Amos and J. Z. Kolter. "Optnet: Differentiable optimization as a layer in neural networks". In: *International Conference on Machine Learning*. PMLR. 2017, pp. 136–145.

[2] M. Besançon, J. D. Garcia, B. Legat, and A. Sharma. "Flexible differentiable optimization via model transformations". In: *arXiv preprint arXiv:2206.06135* (2022).

[3] S. P. Boyd and L. Vandenberghe. *Convex optimization*. Cambridge university press, 2004.

[1]See for instance the `COSMO.jl` solver `https://oxfordcontrol.github.io/COSMO.jl/stable/getting_started/#Warm-starting`.