# Markov Decision Processes and Reinforcement Learning

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# Markov Decision Processes and Reinforcement Learning

- Markov Decision Process (compute optimal decisions offline)
- Reinforcement Learning (learn optimal decisions online)
    - Q-Learning
    - Upper Confidence Reinforcement Learning

# Example 1: Chess Players

$B$ often plays chess against $C$ with the following odds.

When $B$ plays agressive (A): 55 % losses and 45 % wins.
When $B$ plays defensive (D): 15 % losses, 75 % draws, 10 % wins.

$B$ challenges $C$ over a two game match.

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All *strategies* $AA$, $AD$,$DA$, $DD$ are losing. Respective expected gain $-10\%, -11\%, -11\%$ or $-8\%$ (for DD).

# Example 1: Chess Players (II)

$B$ uses the catenaccio strategy:
*play $A$ in the first game then play $D$ after a win or $A$ after a defeat*

let us compute the winning probabilities:
Victory: WW ou WD: $0.45 \times (0.10 + 0.75) = 0.3825$
Draw: WL ou LW : $0.45 \times 0.15 + 0.55 \times 0.45 = 0.315$
Loss: LL: $0.55 \times 0.55 = 0.3025$

Expected gain: $+8\%$.

# Example 2: wheel of fortune

You have 4 tries at the wheel of fortune: (with gains 1,2,3,4,5,6,7,8,9,10)
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Backward Induction

Last try: expected gain: 5.5

Last two tries: optimal strategy is to *stop after first try if gain is* $>$ *5.5*
Expected gain: $1/2 \times 5.5 + 1/2 \times 8 = 6.75$

Last 3 tries: optimal strategy is to *stop after first try if gain is* $>$ *6.75*
expected gain: $8.5 \times 4/10 + 6.75 \times 6/10 = 7.45$

4 tries: optimal strategy is to *stop after first try if gain is* $>$ *7.45*
expected gain: 7.915.

# Markov Decision Process (MDP)

- States $x \in \mathcal{X}$ (finite)
- Horizon: $T$ (finite) or discounted or average.
- Rewards: $r_t(x)$
- Actions: $a \in \mathcal{A}$
- Transitions $P_t(x, a, y)$
- strategy: $\pi : X \to A$

The goal of the controler is to maximize the expected gain over a finite Horizon $T$

$$J(x) = \mathbb{E} \sum_{t=0}^{T} r_t(X_t, A_t)$$

with $X_0 = x$ et $X_{t+1}$ the next state according to the transition matrix $P_t(X_t, A_t, \cdot)$.

# Bellman Equation

Let us denote by $J_t^\pi(x)$, the expected gain under $\pi$ from $t$ to $T$, starting in $x$ at $t$.

$$
\begin{aligned}
J_t^\pi(x) &= \mathbb{E}\sum_{u=t}^{T} r_u(X_u, \pi_u(X_u)) \\
&= r_t(x, \pi_t(x)) + \mathbb{E}\sum_{u=t+1}^{T} r_u(X_u, \pi_u(X_u))
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&= r_t(x, \pi_t(x)) + \sum_y P_t(x, \pi_t(x), y) J_{t+1}^\pi(y).
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$$J_t^*(x) \quad = \quad \sup_a \left( r_t(x,a) + \sum_y P_t(x,a,y) J_{t+1}^*(y) \right).$$

When $\sup$ is reached ($\mathcal{A}$ compact, $P$ et $r$ sont continuous en $a$), then the optimal strategy exists:

$$\pi_t^*(x) \quad = \quad \arg\max_a \left( r_t(x,a) + \sum_y P_t(x,a,y) J_{t+1}^*(y) \right).$$

## Bellman Optimality Equation

Let $J^*_t(x) = \sup_\pi J^\pi_t(x)$

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Computation using Dynamic Programming

1. $J^*_T(x) = 0, \forall x \in \mathcal{X}$
2. for $t$ from $T-1$ to $0$ compute $J^*_t$ using BOE.

# Infinite Horizon

Discounted gain: $J(x) = \mathbb{E} \sum_{t=0}^{\infty} \lambda^t r(X_t, A_t)$.

Transition matrices $P(\cdot, a, \cdot)$ and rewards $r(\cdot, a)$ are homogeneous in time

Let us define $J_t^T(x) = \mathbb{E} \sum_{u=t}^{T} \lambda^{u-t} r(X_u, A_u)$.

Bellman Equation in finite time is

$$J_t^{T*}(x) = \sup_a \left( r(x,a) + \lambda \sum_y P(x,a,y) J_{t+1}^{T*}(y) \right).$$

when $T$ goes to $\infty$

$$J^*(x) = \sup_a \left( r(x,a) + \lambda \sum_y P(x,a,y) J^*(y) \right).$$

# Dynamic Programming Operators

Operators $\mathcal{T}^\pi$ et $\mathcal{T}$:

$$\begin{aligned} L^\pi : \mathbb{R}^{\mathcal{X}} &\rightarrow \mathbb{R}^{\mathcal{X}} \\ J(x) &\rightarrow r(x, \pi(x)) + \lambda \sum_y P(x, \pi(x), y) J(y). \end{aligned}$$

$$\begin{aligned} \mathcal{L} : \mathbb{R}^{\mathcal{X}} &\rightarrow \mathbb{R}^{\mathcal{X}} \\ J(x) &\rightarrow \sup_a \left( r(x, a) + \lambda \sum_y P(x, a, y) J(y) \right), \end{aligned}$$

# Value/Policy Iteration

Value Iteration

1. $J(x) = 0 \, \forall x \in \mathcal{X}$
2. **Repeat** until $\varepsilon$-convergence $J_{n+1} := \mathcal{L} J_n$
3. $\pi := \arg\max \mathcal{T} J$;

Policy Iteration

1. $\pi$ arbitrary policy.
2. **Repeat** until convergence of $\pi$:
3. $(J_{n+1} = L^{\pi_n} J_{n+1})$;
4. $\pi_{n+1} := \arg\max \mathcal{L} J_{n+1}$;

# Average Reward

Average reward : $g(x) = \mathbb{E} \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{\infty} r(X_t, A_t)$.

Under some mild conditions (weakly communicating MDP) the limit exists and the optimal average reward does not depend on $x$.

$g^*$ is the solution of the non discounted Bellman equation:

$$g^* + h(i) = \max_a \left( r(x, a) + \sum_j P(i, a, j) h(j) \right)$$

Can be computed using value iteration, policy iteration or linear programming.

# No Free Lunch...

MDP suffers from two main weaknesses:

- Curse of dimensionality: the state space of Markovian models are large even for moderate size systems.
- Curse of Information: The transition probabilities $P(x, a, y)$ must be known for all $x, a, y$.

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Actually there are close relations between these two problems: Deep Reinforcement Learning is an efficient way to address both drawbacks at the same time.

# Stochastic Approximation

(Also called the ODE Method)
A stochastic dynamical system of the form

$$X_{n+1} = X_n + \gamma_n (f(X_n) + M_{n+1})$$

is a stochastic approximation of the o.d.e. $\dot{x} = f(x)$
if $\mathbb{E}(M_{n+1}|\mathcal{F}_n) = 0$, $\mathbb{E}(||M_{n+1}||^2|\mathcal{F}_n) \leq K(X_n + C)$ and $\gamma_n$ is in $L^2 \setminus L^1$.
If $\sup ||X_n||$ is bounded, then $X_n$ is an APT (Asymptotic Pseudo Trajectory) of the o.d.e, (Benaim):

$$\lim_{s\to\infty} \sup_{t\in[s,s+T]} ||X(t) - x_s(t)|| = 0.$$

Furthermore, $X_n$ converges to an internally chain transitive invariant set of the o.d.e..

**Corollary**

*If the o.d.e. has a unique global attractor, then* $\lim_{n\to\infty} f(X_n) = 0$ *a.s..*

# Q-value function

Consider a discounted MDP $(r, A, P, \beta)$. By definition, the Q-value function $q^*(x, a)$ is the optimal gain starting in state $x$ and using action $a$. The optimal reward is related to $q^*$ by $J^*(x) = \max_a q^*(x, a)$.

The BOE for $Q$ is:

$$\begin{aligned} q^*(x, a) &= r(x, a) + \beta \sum_y P(x, a, y) J^*(y) \\ &= \sum_y P(x, a, y) \left[ r(x, a) + \beta \max_b q^*(y, b) \right] \end{aligned}$$

In vector form, $q^* = Fq^*$.
Value Iteration for $Q$-value function: $q_{n+1} = Fq_n$ converges to the optimal $q^*$ because $F$ is $\beta$-contracting.

## Q-learning Algorithm (PhD of Watkins, 1989)

Learning the optimal actions in each state by monitoring the system on-line:

In state $X_n$, choose an action $a_n$ according to some selection policy.

The system gives you an instant reward $r$ and moves to state $X_{n+1}$ according to an (unknown) Markov transition.

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Given this sequence $(X_n, a_n)_{n\in\mathbb{N}}$ of state-action pairs obtained by monitoring the system, the Q-learning algorithm computes the following on-line sequence of values for all states and actions:

$$\begin{aligned} Q_{n+1}(X_n, a_n) &= (1-\gamma_n)Q_n(X_n, a_n) + \gamma_n(r(X_n, a_n) + \beta \max_b Q_n(X_{n+1}, b)) \\ Q_{n+1}(x, a) &= Q_n(x, a) \quad \text{if } (x, a) \neq (X_n, a_n) \end{aligned}$$

# Convergence of Q-learning Algorithm

**Theorem**

*If each state-action couple $(x, a)$ appears an infinite number of times in the sequence $(X_n, a_n)$, then $\lim_{n \to \infty} Q_n(x, a) = q^*(x, a)$.*

The stochastic sequence

$$Q_{n+1}(x, a) = Q_n(x, a) + \gamma_n \big[ r(x, a) + \beta \max_b Q_n(Y, b) - Q_n(x, a) \big]$$

is a stochastic approximation of

$$\dot{q}(x, a) = \mathbb{E}\big[ r(x, a) + \beta \max_b q(Y, b) - q(x, a) \big| (x, a) \big]$$

The operator $F$ is contracting, therefore, $Q_{n+1}$ converges to the unique point $q^\infty$ s.t. $\mathbb{E}\big[ r(x, a) + \beta \max_b q^\infty(Y, b) - q^\infty(x, a) \big| (x, a) \big] = 0$.
so $q^\infty = q*$.

## Speed of Convergence: Theoretical Bounds

Several variants of the algorithm exist according to the choice of $a_n$.
The classical $\varepsilon$ greedy selection: current best action w. p. $1-\epsilon$ (other choices such as exp-weight).
The convergence rate of Q-learning has been investigated by Even-Dar and Mansour, 2003.

**Theorem**

*For any $\epsilon > 0$, after*

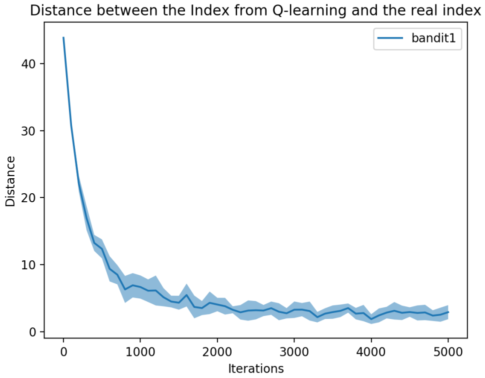
$$T = O\left(\frac{R^2 \log \frac{N}{\delta}}{\epsilon^2(1-\beta)^4}\right)$$

*steps of QL, the uniform approximation error $||q* - Q_T|| \leq \epsilon$, with probability larger than $1-\delta$.*

Essential points: convergence in $\sqrt{T}$, in $\log(N)$, in $(1-\beta)^4$.

# Speed of Convergence: Numerical Experiments

In most experiments we tried, convergence is slow. Example from Kimang's work on restful bandits ($4$ states, 2 actions, $\beta = 0.5$)




Many improvements have been proposed: ZAP Q-learning, speedy Q-learning, Approximate/Projected Q-learning (Berstekas)...

# Upper Confidence Reinforcement Learning

Principle: Use UCB ideas to learn the optimal policy.

# Upper Confidence Reinforcement Learning Auer & Ortner, 2008

Let us consider a non-discounted unichain MDP ($M$),

**Repeat**

1. estimate: $\hat{r}_t(x,a) = \dfrac{R_t(x,a)}{N_t(x,a)}$ and $\hat{p}_t(x,a,y) = \dfrac{N_t(x,a,y)}{N_t(x,a)}$
2. Compute new policy: $\tilde{\pi}_k := \arg\max_{\pi}(g(M,\pi), M' \in \mathcal{M}_t)$ (L.P.)
   over all $|r'(x,a) - \hat{r}_t(x,a)| \leq UCB_r$ and
   $|p'(x,a,y) - \hat{p}_t(x,a,y)| \leq UCB_p$
3. Use policy $\tilde{\pi}_k$ until $UCB_r$ or $UCB_p$ decreases by half.

with $UCB_r(t,x,a) = \sqrt{\dfrac{\log(2t^2NA)}{2N_t(x,a)}}$. (similar definition for $UCB_p$).

# Speed of Convergence

A direct consequence of Chernoff-Hoeffding inequality is that $M$ belongs to $\mathcal{M}_t$ with probability higher than $1 - 1/t^2$.

Expected regret: $R_T = \mathbb{E} \sum_{t=1}^{T} r_t - Tg^*$

$$R_T \leq C \frac{AN^5 T_M K_M^2}{\Delta^2} \log T$$

where $T_M$ is the max first passage time of the MDP $M$
$K_M$ is the ratio of first passage time to return time.
and $\Delta$ is the min cost difference over all suboptimal policies.

That's all Folks!