Stochastic Fractional Diffusion Equations

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Context

Fractional diffusion equations arise in many domains to model complex diffusion phenomena (pollutant dispersion in the ocean, flows in fractured porous media, dispersion in random media, …). Spatial fractional diffusion models are needed to handle scale dependencies and correlations absent in standard Fickian models. Time fractional models are required for diffusion processes involving memory effects.

Besides their mathematical analysis, the practical use of fractional models for simulations faces several difficulties. One difficulty concerns the development of accurate numerical schemes adapted to the non-locality of the fractional operators (space-, time-convolutions) [1,2]. The non-locality also translates into dense discrete problems, with substantial computational challenges when applied to problems in 2 and 3 spatial dimensions. Further, applications require the calibration of the fractional and diffusion coefficients against experimental observations. Finally, as the coefficients are not perfectly known, they should be treated as random quantities, and efficient stochastic procedures are needed to assess the uncertainty in the fractional model predictions.

Objectives

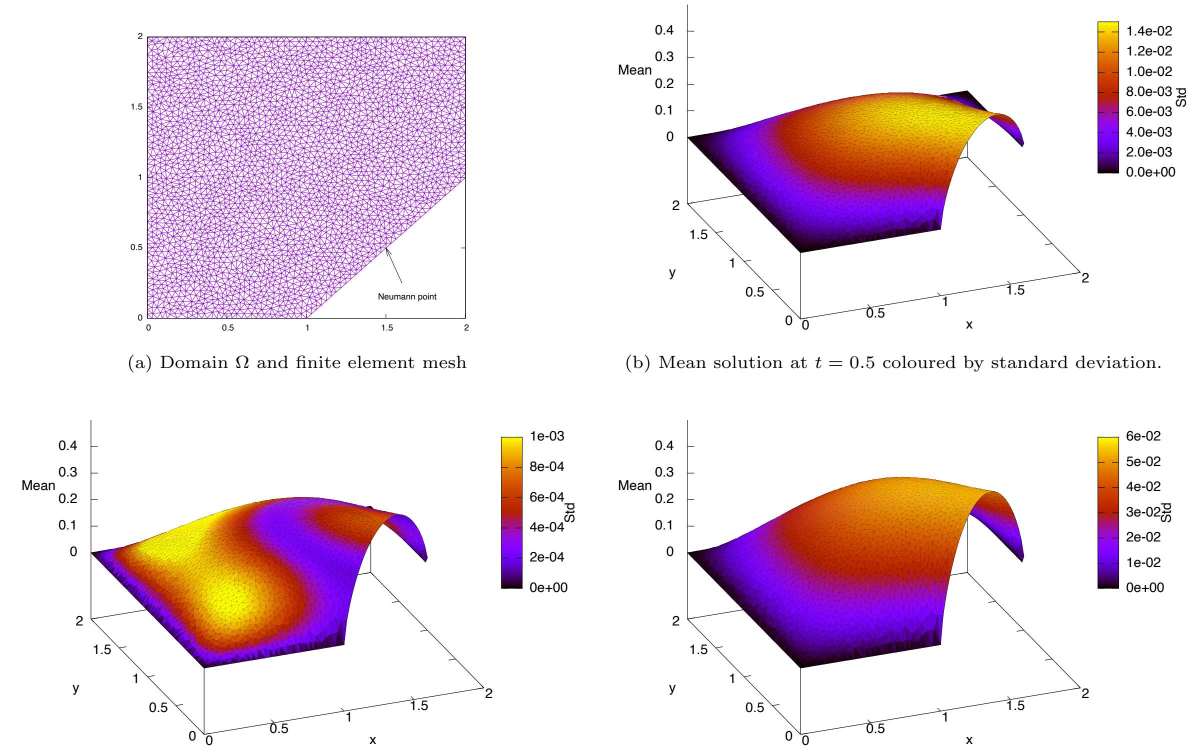
Depending on the interest and skills of the research intern, the work proposed will consider the following:

- **Galerkin solvers for stochastic fractional diffusion equations**. The intern will develop advanced numerical methods relying on Polynomial Chaos expansions [3] to account for random fractional coefficients. The computational complexity of the convolutions inherent to the fractional diffusion operators will call for hierarchical matrix and sparse approximation techniques [4] that will have to be extended to the stochastic case.

Or

- **The Bayesian Calibration fractional diffusion models**. The intern will work on advanced inference methods for calibrating the (space or time) fractional and diffusion coefficients [5]. They will develop a) surrogate-based approximations of the Bayesian posterior of the coefficients and its sampling and b) design experiments to prevent confusion between fractional and diffusion coefficients (identifiability problem).

This work will be held in the Platon Team at CMAP. O. Le Maître and P.M. Congedo will supervise it. For further details, don't hesitate to get in touch with O. Le Maître or P.M. Congedo. (olivier.le-maitre@polytechnique.edu pietro.congedo@inria.fr).



(a) Domain Ω and finite element mesh

(b) Mean solution at $t = 0.5$ coloured by standard deviation.

(c) Mean solution at $t = 1.1$ coloured by standard deviation.

(d) Mean solution at $t = 5$ coloured by standard deviation.

Figure 5: Computational domain, finite element mesh and mean of the PC solution coloured by standard deviation value at different times as indicated. Case of $\boldsymbol{\alpha} \sim \mathcal{U}[0.75, 1]$, with PC expansions using $\mathrm{n_o} = 10$.

Figure 1: Illustration of the Stochastic Galerkin solution of a time-fractional diffusion problem [6].

References

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[4] W. Boukaram, M. Lucchesi, G. Turkiyyah, O. Le Maître, O. Knio, and D. Keyes, Hierarchical Matrix Approximation for Space-Fractional Diffusion Equations, *Computer Methods in Applied Mechanics and Engineering*, 369, pp.113191, (2020).

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