

Vekua approximations and applications to soundfield sampling and source localisation

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Context

Joint work with Laurent Daudet and Albert Cohen



Project ANR ECHANGE (ECHANtillonnage Acoustique Nouvelle GÉnÉration) :

- Institut Langevin (acoustics - signal processing)
- INRIA METISS (signal processing)
- LJLL (UPMC) (numerical analysis)
- IJLRA (UPMC) (experimental acoustics)

Acoustic signal processing

Signals = Spatial behavior of the acoustic pressure

In the frequency domain: easier and more difficult

Inverse problems:

- inversion of a (ill-conditioned) physical process
- regularised with a signal model

A classical inverse problem in acoustics

Nearfield Acoustical Holography (NAH)



MPIA (IJLRA)

A classical inverse problem in acoustics

Direct problem: computing the radiated p from the velocity w :

$$p = g \star w + n$$

Inverse problem: recovering the velocity from measurements of the pressure. It is a deconvolution problem for complex images:

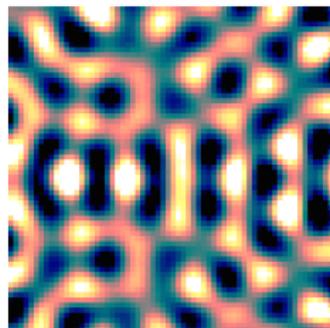
$$\hat{w} = g^{-1} \star p$$

Tikhonov regularisation:

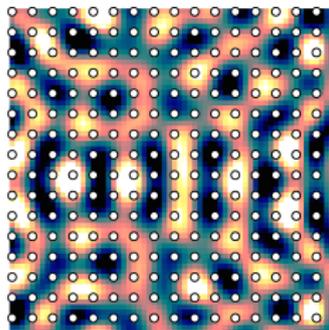
$$\hat{w} = \underset{w}{\operatorname{argmin}} \underbrace{\|p - g \star w\|_2^2}_{\text{physical model}} + \underbrace{\lambda \|w\|_2^2}_{\text{signal model}}$$

Acoustical field sampling

Signal model: low-pass image, Shannon-Nyquist-etc. sampling theorem

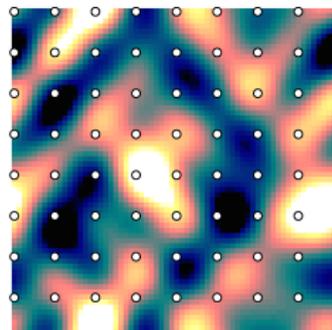


Acoustical field



Fine regular sampling

OK

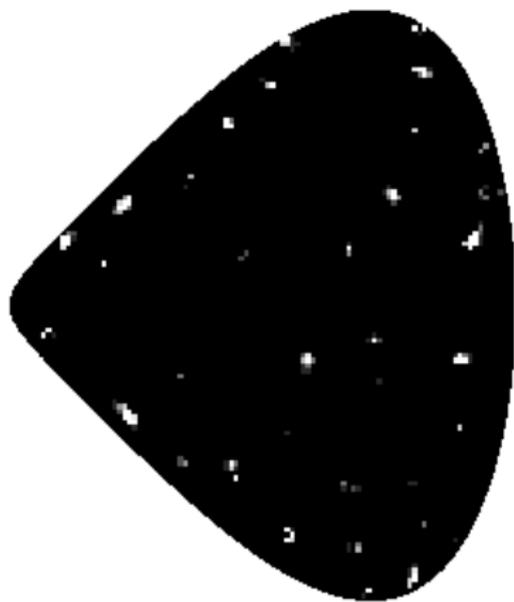


Coarse regular sampling

Fail

Source localisation in a room

Signal model: spatial sparsity of the sources



These simple signal models are not enough

We want to measure physical quantities

We use physical models (PDEs) to build better signal models

Outline

- 1 Sparse models for waves : Vekua theory
- 2 Measuring solutions to the Helmholtz equation
- 3 Source localisation
- 4 Global conclusion

Sparse models

A vector x is said to be sparse if only a few coefficients of its expansion in a fixed dictionary D are nonzero:

$$x = Ds \text{ with } \|s\|_0 \text{ small}$$

D is usually obtained through heuristics, learning algorithms, etc.

Can be used as a prior for inverse problems, compressed sensing, etc.

Here, we will use physics to build dictionaries.

Physical model

Helmholtz equation

$$\Delta u + k_0^2 u = 0 \quad (\text{H})$$

used to model a large class of physical phenomena:

- **acoustics**
- electromagnetics
- quantum physics

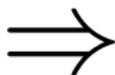
Main tool : Vekua theory

Vekua theory

Holomorphic world

Helmholtz world

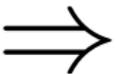
$$\frac{\partial f}{\partial \bar{z}} = 0$$



$$f \approx \sum_{n=0}^L a_n z^n$$

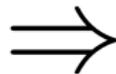


$$\Delta u + k_0^2 u = 0$$



$$u \approx \sum_{n=-L}^L \alpha_n e^{in\theta} J_n(k_0 r)$$

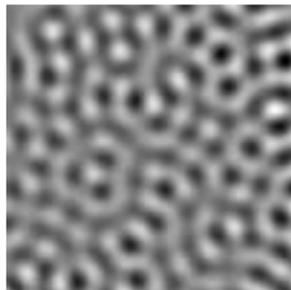
Fourier-Bessel functions



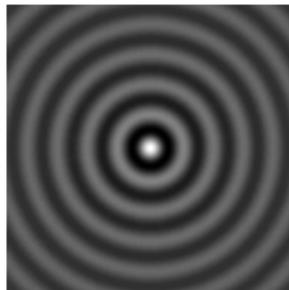
$$u \approx \sum_{n=-L}^L \beta_n e^{i\vec{k}_n \cdot \vec{x}}$$

Plane waves

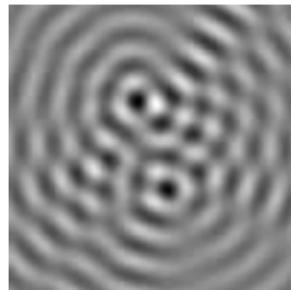
$$\Delta u + k^2 u = 0$$



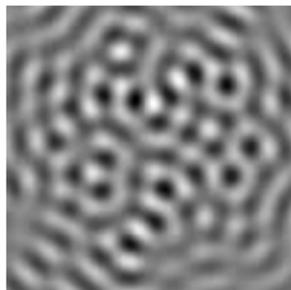
$L = 0$



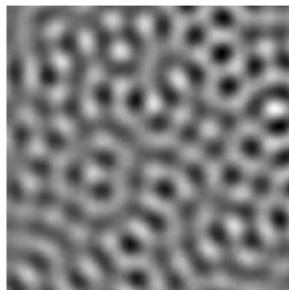
$L = 10$



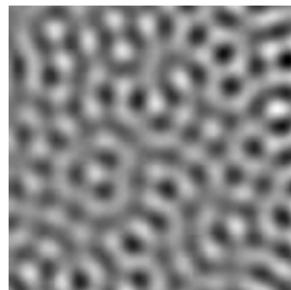
$L = 20$



$L = 30$



$L = 40$



Approximation error

If $p \leq q$ (Melenk 1999, Moiola 2011), then, for an approximation \tilde{u}_L at order L :

$$\|u - \tilde{u}_L\|_{H^p} \leq C L^{q-p} \|u\|_{H^q}$$

If u and its derivatives up to order q have finite energies, then u and its derivatives up to order p can be approximated.

$L \propto kD$ seems to be sufficient, but this problem remains open (in maths).

Extension to various wave models

Similar approximation schemes for other physical models can be obtained using these results (cf Moiola et al., GC-LD):

- electromagnetism (Maxwell equations)
- linear elasticity
- linear plate models (Kirchhoff-Love, Reissner-Mindlin)

Extension to thin plates (Kirchhoff-Love model)

To approximate plate vibrations, we extend the previous results to the Kirchhoff-Love model.

The normal displacement for a modal shape of a plate is solution to:

$$\Delta^2 w - k^4 w = 0$$

The approximation scheme is extended by completing it with modified Fourier-Bessel functions $e^{in\theta} I_n(kr)$ or exponential functions $e^{\vec{k}_J \cdot \vec{x}}$:

$$\|w - \hat{w}_L\|_{H^p} \leq CL^{q-p} \|w\|_{H^{q+2}}$$

Sparse models for waves - conclusion

(synthesis) sparsity models for various elliptic PDEs in the harmonic regime

Analysis sparsity for the time domain (wave equation) cf. Nancy

Similar results in 3D

Plan

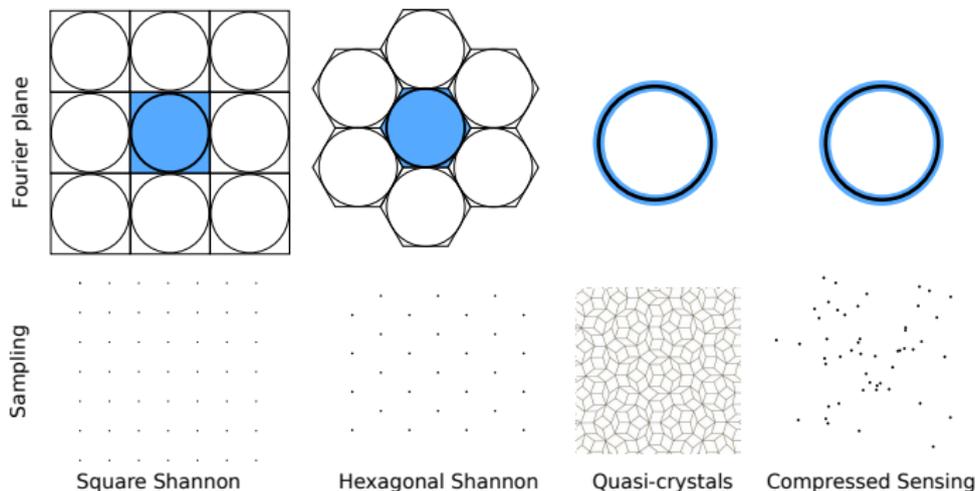
- 1 Sparse models for waves : Vekua theory
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Measuring solutions to the Helmholtz equation

If it exists, the Fourier transform of u verifies

$$-k^2 \hat{u} + k_0^2 \hat{u} = 0 \quad (\hat{H})$$

\hat{u} contained in the circle of radius k_0



Here, none of these methods, if applicable, yields good results

Least-squares approximation from samples

We choose an order of approximation L

Using n ($> L$) samples, we find a least-squares approximation of the soundfield in the space $V_L = \text{span}_{l=-L\dots L} e^{il\theta} J_l(kr)$.

Questions:

- Which order L to choose ?
- With which density ρ should we draw the measurement points ?

Cohen-Davenport-Leviatan (2011) give results allowing us to answer these questions.

collab. A. Cohen

Flatness of an orthogonal basis

We define K , the flatness of an orthogonal basis (e_j) of V_L , by

$$K(L) = \max \sum |e_j(x)|^2$$

If

$$K(L) < \frac{n}{\log n}$$

the reconstruction error is of the same order as the best approximation error in V_L .

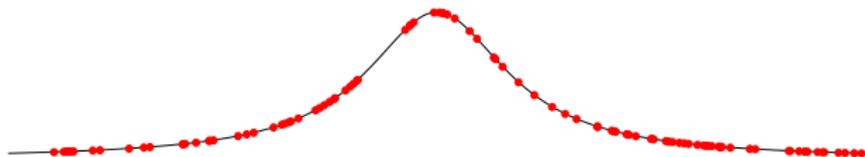
Important : $K(L)$ also depends on the density of the samples

Example in one dimension

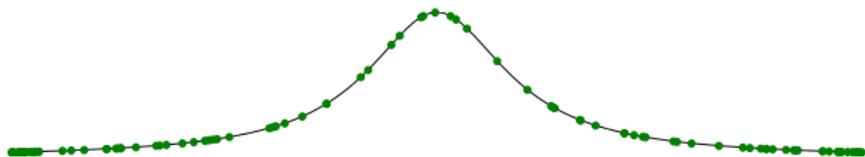
Example : functions defined on $] - 1, 1[$

$$V_L = \text{span}_{0 \leq k \leq L-1} x^k$$

Samples uniformly drawn : $K(L) = L^2$



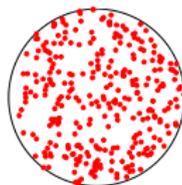
Samples drawn using $d\rho = \frac{dx}{\pi\sqrt{1-x^2}}$: $K(L) = 2L - 1$



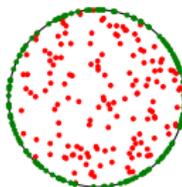
Stability of the chosen sampling methods

$$V_L = \text{vect}_{-L \leq l \leq L} e^{il\theta} J_l(kr)$$

Sampling with uniform density on the disk : $K(L) > CL^2$

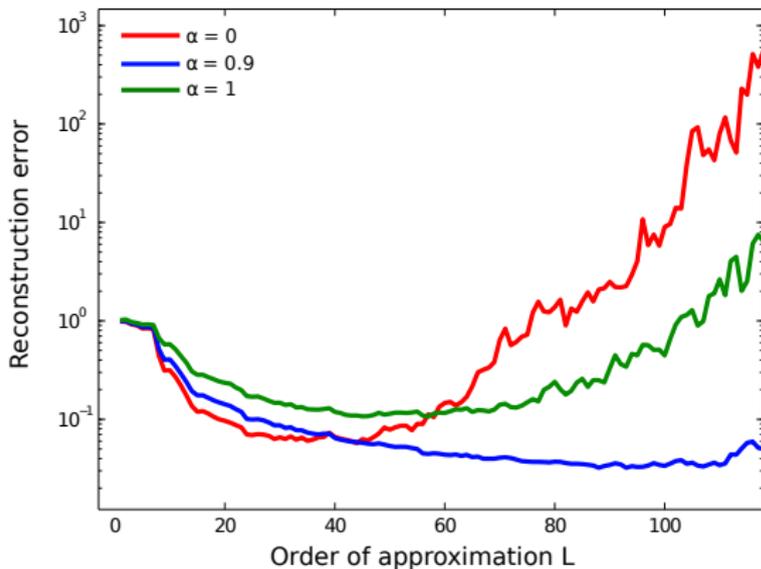


Proportion α of samples on the border, the remainder inside: $K(L) \approx \frac{2}{\alpha} L$



Numerical simulations

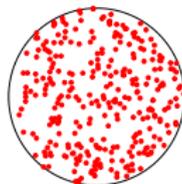
Approximation error in function of dimension L and proportion α of samples on the border (400 samples)



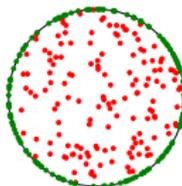
Sampling in a ball

$$V_L = \text{span}_{l=0 \dots L, m=-l \dots l} Y_{lm}(\theta, \phi) j_l(kr)$$

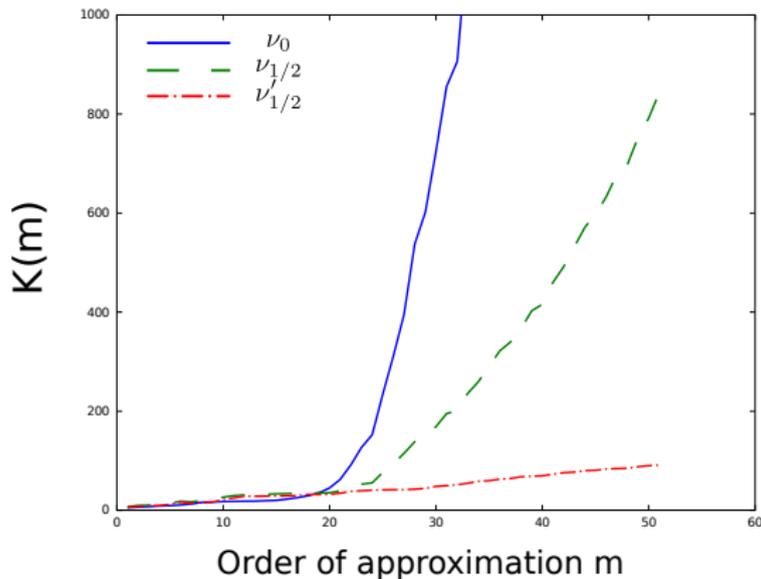
Sampling with uniform density in the ball : $K(L) > CL^3$



Proportion α of samples on the sphere, the remainder inside:
 $K(L) \approx \frac{1}{\alpha} L^2$



Numerical evaluation of $K(L)$ for the square

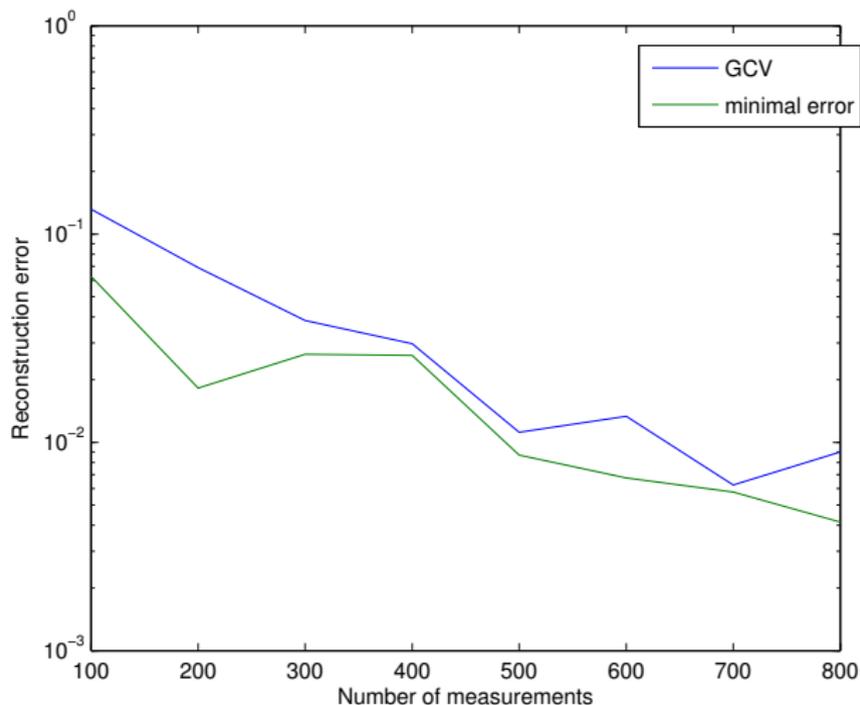


ν_0 : uniform density

$\nu_{1/2}$: 50% inside, 50% on the border

$\nu'_{1/2}$: 50% inside, 50% on the border, denser near the corners

Estimation of the model order (cross-validation)



Sampling - conclusion

Using the physical model, we reduce the numbers of samples needed to measure an acoustical field

The placement of the samples is important:

Sampling on the border seems necessary, but not sufficient

It is possible to estimate k (cf. Laurent)

Plan

- 1 Sparse models for waves : Vekua theory
- 2 Measuring solutions to the Helmholtz equation
- 3 Source localisation**
- 4 Global conclusion

Source localisation

Localisation in free-field is a standard problem in signal processing.
Beamforming, MUSIC, sparsity (Malioutov)

First problem: known reverberant room
cf. Dokmanic-Vetterli, time reversal

Second problem: unknown reverberant room

Alternative models, measurements and algorithms are needed.

Source localisation in free-field

A simple sparse model is directly obtained for the free-field propagation:

$$p = \sum a_j y_0(k \| r - r_j \|)$$

$$y_0(kr) = \frac{e^{ikr}}{4\pi r}$$

A dictionary is build using these functions:

- columns = possible source positions
- rows = measurement points

Source localisation in a known room

In a reverberating room, the radiated field is solution to

$$\begin{cases} \Delta p - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = s(x, t) \\ \frac{\partial p}{\partial n} = 0 \text{ on the boundaries} \end{cases}$$

Dokmanic and Vetterli suggest to replace the free-field dictionary by a reverberated dictionary computed using the FEM.

Problems:

- the room must be perfectly known
- the dictionary does not have good properties

Direct problem in a unknown room

Compute the field radiated to the microphones by known sources *without knowing the boundary conditions*

$$\begin{cases} \Delta p - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = s(x, t) \\ \text{no boundary conditions} \end{cases}$$

The direct problem is unsolvable!

A simple trick (and a specific measurement method) makes the localisation possible.

Physical model

The sound propagation is modeled by the wave equation, with e.g. Neumann boundary conditions:

$$\left\{ \begin{array}{l} \Delta p - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = s(x, t) \\ \text{boundary conditions (ex.: } \frac{\partial p}{\partial n} = 0 \text{)} \end{array} \right.$$

Sources are assumed sparse in space

$$s = \sum s_j(t) \delta_j(x)$$

In harmonic regime, we use the Helmholtz equation

$$\Delta \hat{p} + k^2 \hat{p} = \sum \hat{s}_j(\omega) \delta_j(x)$$

Acoustical field decomposition

The acoustic field \hat{p} can be decomposed as the sum of

- a particular solution \hat{p}_s (source field)
- an homogeneous solution \hat{p}_0 (diffuse field)

$$\hat{p} = \hat{p}_s + \hat{p}_0$$

We choose $Y_0(kr)$ (second-kind Bessel function fo order 0) as the fundamental solution :

$$p_s = \sum s_j Y_0(k\|x - x_j\|)$$

Acoustic field decomposition

\hat{p}_0 is solution of

$$\begin{cases} \Delta \hat{p}_0 + k^2 \hat{p}_0 = 0 \\ \text{b.c. (e.g.: } \frac{\partial \hat{p}_0}{\partial n} = -\frac{\partial \hat{p}_s}{\partial n} \text{)} \end{cases}$$

\hat{p}_0 can be approximated by sums of plane waves

Sparse model for the complete field

$$\begin{aligned}\hat{p} &= \hat{p}_0 + \hat{p}_s \\ &\approx \sum \alpha_l e^{i\vec{k}_l \cdot x} + \sum s_j Y_0(k\|x - x_j\|)\end{aligned}$$

After sampling

$$\mathbf{p} \approx \mathbf{W}\alpha + \mathbf{S}\beta$$

where

- \mathbf{W} is a plane wave dictionary with $\|\vec{k}_l\| = k$ (or Fourier-Bessel functions)
- α is low-dimensional
- \mathbf{S} is a dictionary of sources $Y_0(k\|x - x_m\|)$
- β is sparse, its support is what we are looking for.

Algorithms

Iterative algorithm based on OMP:

Removal of the diffuse field and iterative localisation of the sources.

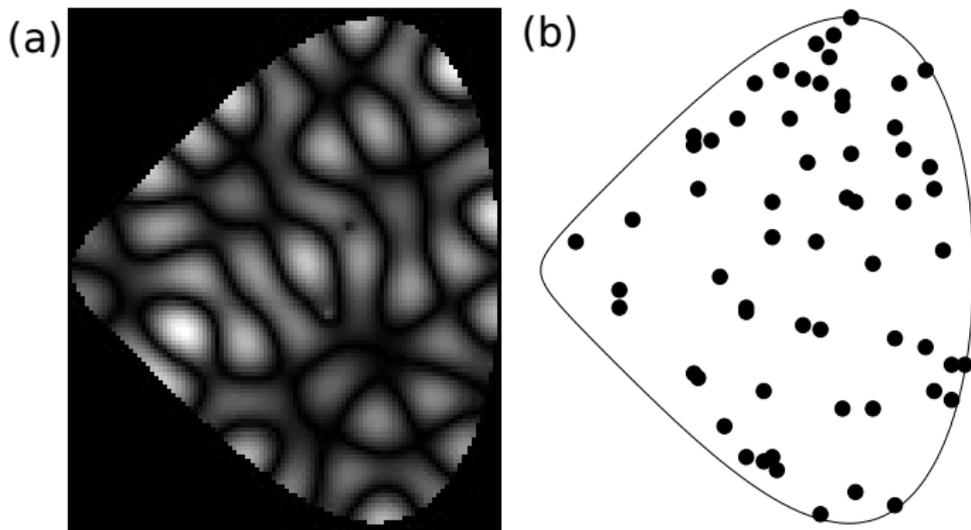
Basis Pursuit method: $\ell_1 + \ell_2$ minimisation

$$(\hat{\alpha}, \hat{\beta}) = \underset{\alpha, \beta}{\operatorname{argmin}} \|\alpha\|_2 + \|\beta\|_1 \quad \text{s.t.} \quad \|\mathbf{W}\alpha + \mathbf{S}\beta - \mathbf{p}\| < \epsilon$$

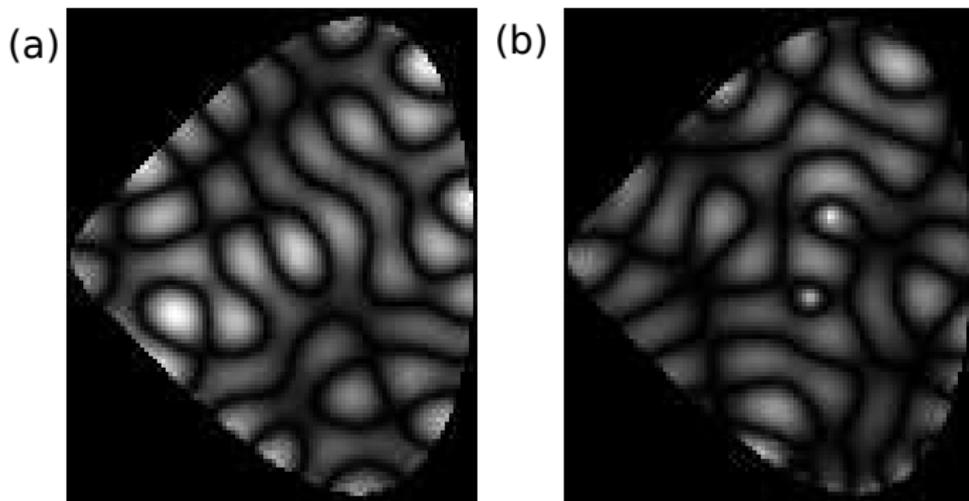
- No priors on the plane waves coefficients $\alpha \rightarrow$ least squares
- Sources coefficients β sparse $\rightarrow \ell_1$ norm

Simulations

Localisation in 2D. Field simulated using FEM (FreeFem++)
2 sources, 60 measurements

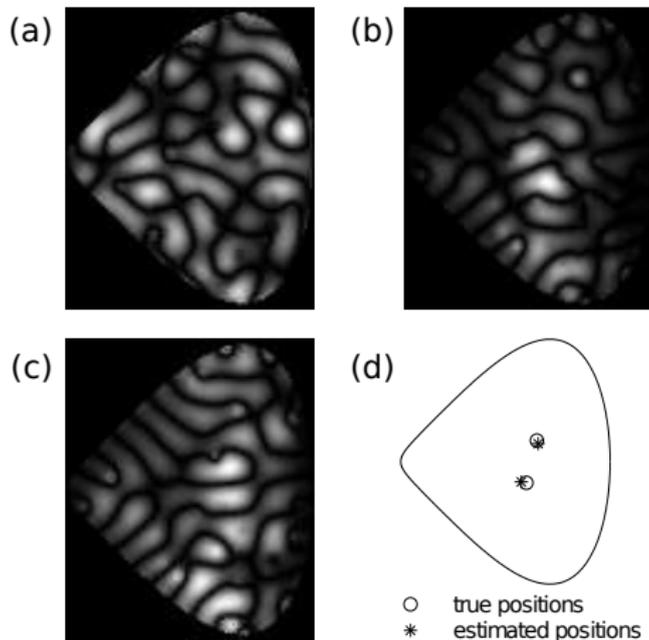


Field decomposition



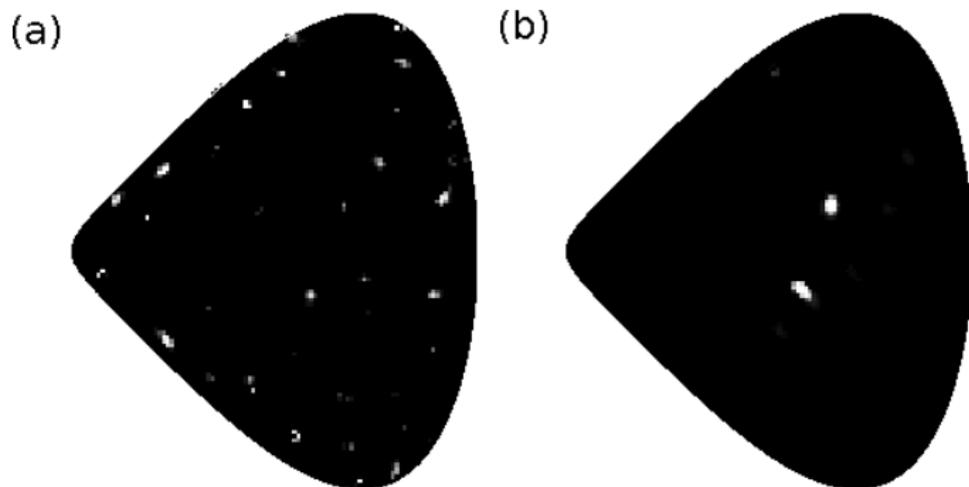
(a) diffuse field p_0 (b) source field p_s

Source localisation - iterative algorithm



(a) Standard beamforming (b) Correlations after removal of the diffuse field

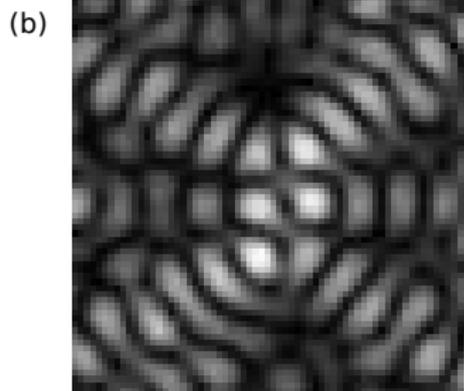
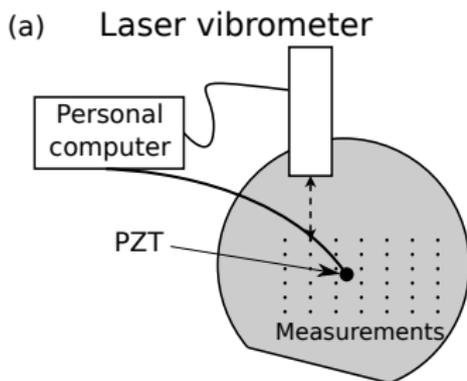
(c) Correlations at the second step (d) Results

Source localisation - $\ell_1 + \ell_2$ minimisation

(a) ℓ_1 with free-field dictionary (b) $\ell_1 + \ell_2$

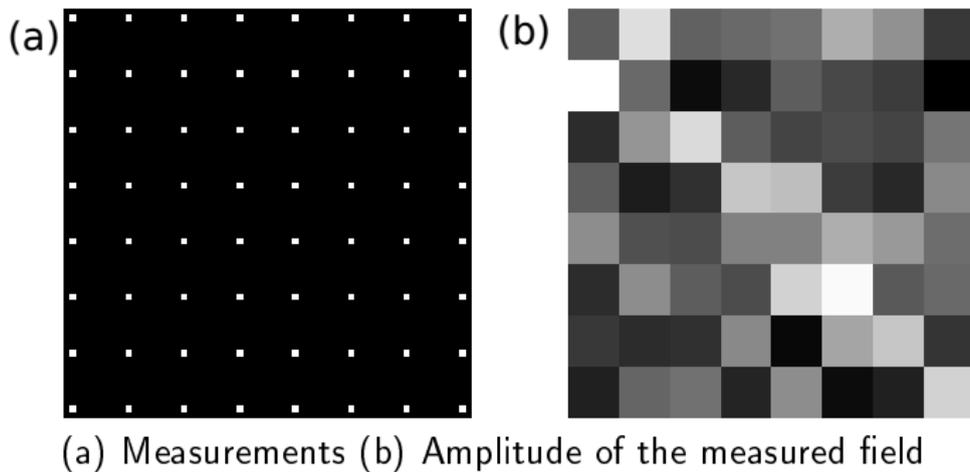
Experimental results

Measurements on a chaotic plate at 30 631 Hz
Sub-Nyquist regular sampling

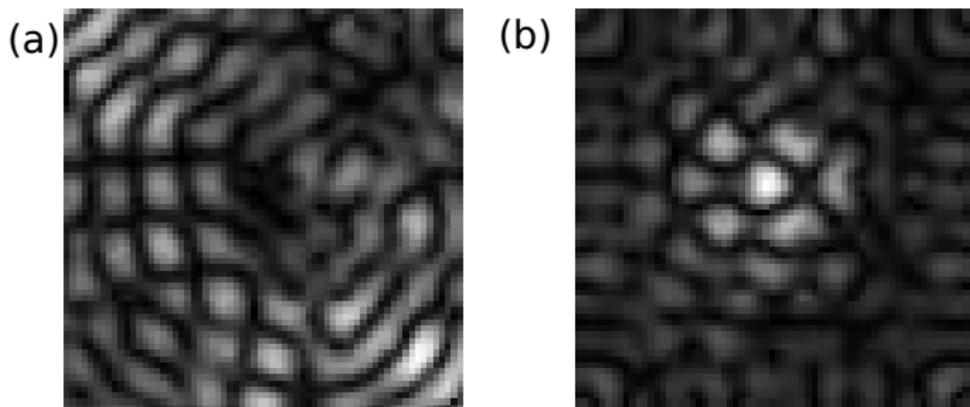


(a) setup (b) radiated field

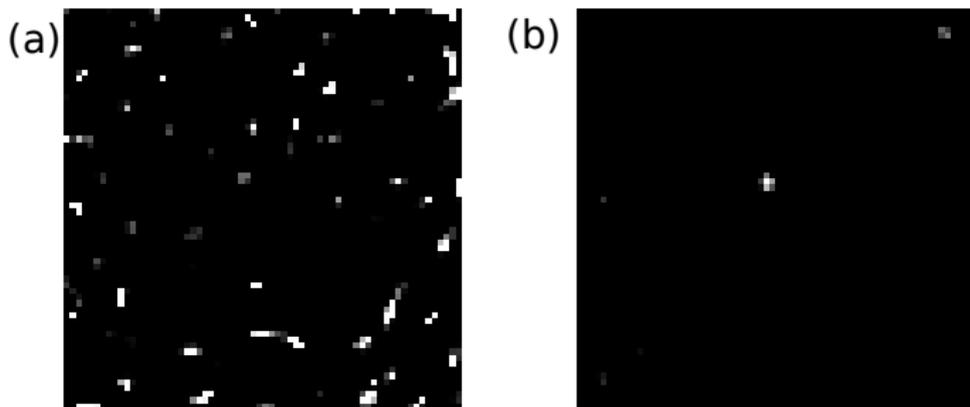
Sampling



Iterative algorithm - results



(a) Correlations with complete field (b) Correlations after decomposition

$l_1 + l_2$ minimisation - results

(a) l_1 with free-field dictionary (b) $l_1 + l_2$

Source localisation - Conclusion

Source localisation in an unknown reverberating room is possible.

How many measurements are necessary ?

Where should we sample the acoustic field ?

Perspectives :

- localisation of directive sources (block sparsity)
- hyperspectral localisation (joint sparsity)
- experiments in 3D acoustics

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Global conclusion

In these various problems sparsity helped us to

- extract informations from our measurements
- design new measurements methods
- compute solutions of PDEs in a efficient way

Sparsity can translate physics in a way that can be exploited by signal processing algorithms.

Perspectives

Theoretical perspectives:

- analysis of the approximations
- analysis of the algorithms

Experimental perspectives:

- source localisation
- source separation
- electromagnetism, quantum physics