Traffic light control on road networks

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Outline

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1. Macroscopic Traffic Flow Model including Traffic Lights
2. Optimal Traffic Light Setting
3. Tuning the Optimization Process

Modeling traffic lights

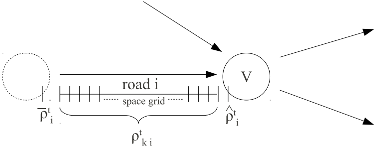
- **Motivation:** Minimize travel times or maximize the traffic flow inside the road network
- **Literature:** Broad variety of literature devoted to optimal signal timing of traffic lights, e.g. cellular automaton (Schadschneider et al., 2001), mixed integer optimization (Lin et al., 2010), fluid dynamic models (Chtitour, Piccoli, 2005)
- **Goal:** Find a traffic light setting that enables the traffic participants to drive smoothly through the road network encountering as few congestion as possible, i.e.

$$\max \sum_{i \in E} \Big(\int_0^T \int_0^{L_i} f_i(\rho_i(x,t))dxdt + \int_0^T \hat{\gamma}_i(t)dt \Big)$$

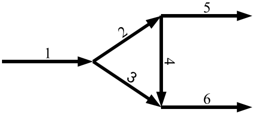
and add the modified traffic flow network model as constraints.

Traffic network

- Directed graph $G = (V, E)$ where each edge $i \in E$ represents a road



- Coupling density/flow $\hat{\rho}_i, \hat{\gamma}_i$ (ingoing) and $\bar{\rho}_i, \bar{\gamma}_i$ (outgoing)
- Flow distribution parameters $0 \leq d_{ij} \leq 1$ which prescribe the percentage of flow going from edge i to edge j.



$$\longrightarrow \quad \mathbf{d} = \begin{pmatrix} 0 & 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 0.7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- Distribution of flow $\bar{\gamma}_j(t) = \sum_{j \in \delta_v^{in}} d_{ij} \cdot \hat{\gamma}_i(t)$

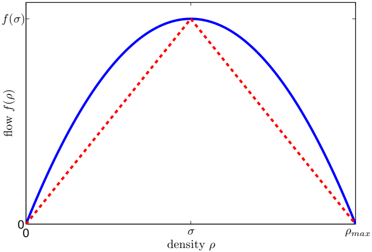
Evolution of traffic density on roads

- Conservation law is used to describe flow behavior:

$$\begin{cases} \partial_t \rho_i + \partial_x f(\rho_i) = 0, \\ \rho_i(x,0) = \rho_{0,i}(x) \end{cases}$$

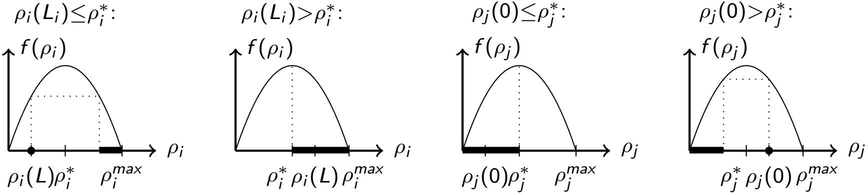
Lighthill-Whitham-Richards Model

- f is strictly concave flow function depending on the traffic density, e.g.



Deriving admissible coupling densities

- Solving Riemann problems at junctions (conservation of cars)



- Incoming roads:

$$\hat{\rho}_i \in \begin{cases} \{\rho_i(L_i)\}\cup]\tau(\rho_i(L_i)),\rho_i^{max}], & \text{if } 0\leq \rho_i(L_i)\leq \rho_i^* \\ [\rho_i^*,\rho_i^{max}], & \text{else.} \end{cases}$$

- Outgoing roads:

$$\bar{\rho}_j \in \begin{cases} [0,\rho_j^*], & \text{if } 0\leq \rho_j(0)\leq \rho_j^*, \\ \{\rho_j(0)\}\cup[0,\tau(\rho_j(0))[, & \text{else.} \end{cases}$$

Modeling of traffic lights

- Traffic light parameter $A_i : [0, T] \to \mathbb{B}$.

$$A_i = 1 \Leftrightarrow \text{ light is green}, \; A_i = 0 \Leftrightarrow \text{ light is red}$$

- Solve linear program for the maximization of fluxes through nodes:

$$\max \sum_{i \in \delta^{in}} \hat{\gamma}_i(t)$$

$$\text{such that}$$

$$\bar{\gamma}_j(t) = \sum_{j \in \delta_v^{in}} d_{ij}(t) \hat{\gamma}_i(t),$$

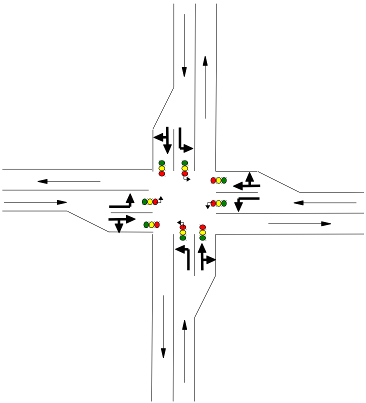
$$0 \leq \hat{\gamma}_i(t) \leq A_i(t) \cdot \hat{F}_i(t)$$

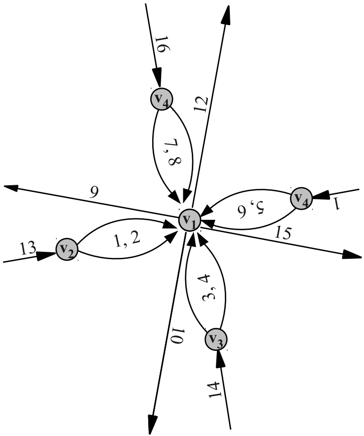
$$0 \leq \bar{\gamma}_j(t) \leq \bar{F}_j(t).$$

- $\hat{F}_i$ and $\bar{F}_j$ are upper bounds for the possible fluxes.

Outline

A traffic cross





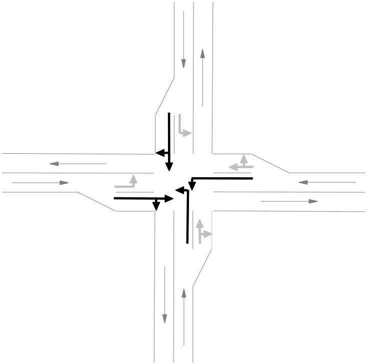
- **Question:** How to route the traffic in a safe way?

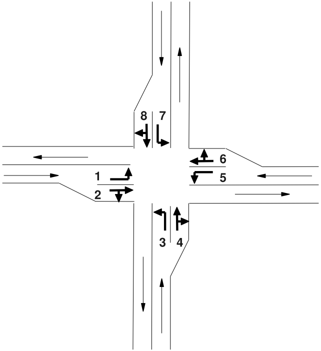
A traffic cross

- **Answer:** Define secure sets as additional constraints.

Definition

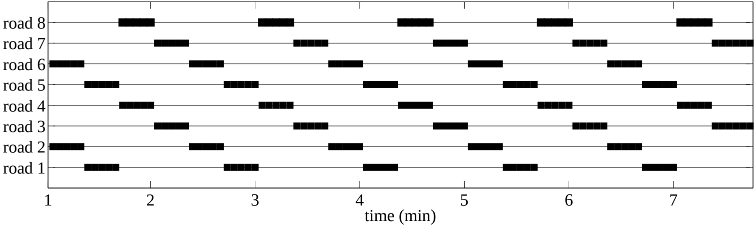
A **secure set** $S \subset E$ is the index set of traffic lights, that cannot be green at the same time.





Default traffic light settings

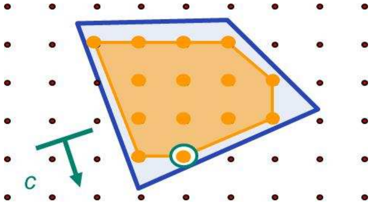
- A feasible solution might be

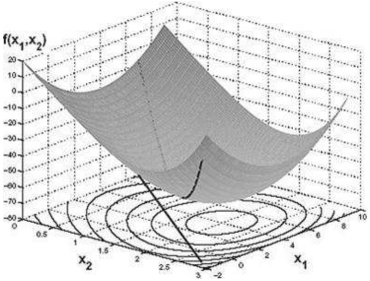


- The objective function value of is 66.84.

Optimization approaches for PDE-constraint problems

- Classical optimization procedures: e.g. gradient descent methods
- **Problem:** Many discrete decisions!
- **Way out:** Use an alternative approach: full discretization and linearization of the whole problem
- Interpretation as a linear mixed integer optimization problem (MIP)
- Solvers like CPLEX using Branch and Cut procedures are applicable





Derivation of the mixed integer model

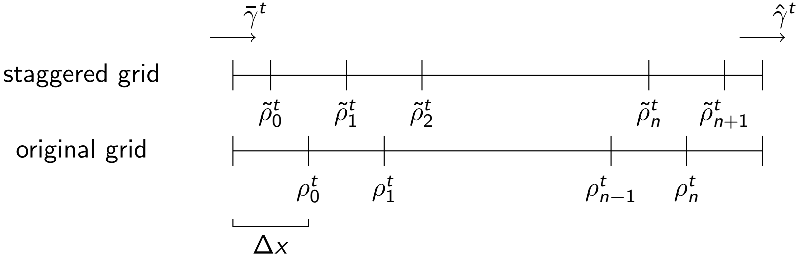
- Reformulate the traffic flow network model step by step: density evolution along roads and linearization of coupling flow
- Define a grid in space and time: discrete timegrid $\mathcal{T} = \{t : t = 0, \dots, n_t\}$ and discrete spatial $\mathcal{K} = \{k : k = 0, \dots, n_i\}$
- The discrete formulation of the objective function is straightforward

$$\max \sum_{t} \sum_{i \in E} \sum_{k=0}^{n_i} f(\rho_{k,i}^t) \Delta t \Delta x + \max \sum_{t} \sum_{i \in E} \hat{\gamma}_i^t \Delta t$$

$$\begin{array}{lll}
\text{control variables:} & A_i^t & \in \mathbb{B} \\
\text{state variables:} & 0 \leq f_{k,i}^t, \hat{\gamma}_i^t, \bar{\gamma}_i^t, \hat{F}_i^t, \bar{F}_i^t \leq f_i^{max}, & \in \mathbb{R} \\
 & 0 \leq \rho_{k,i}^t, \hat{\rho}_i^t, \bar{\rho}_i^t \leq \rho_i^{max}, & \in \mathbb{R} \\
 & \forall i \in E, k \in \mathcal{K}, t \in \mathcal{T} &
\end{array}$$

Density evolution along arcs

- Use staggered Lax-Friedrichs scheme and appropriate CFL condition:



$$\text{left:} \qquad \rho_{0,i}^{t+1} = \frac{1}{4}(3\rho_{0,i}^t + \rho_{1,i}^t) - \frac{\Delta t}{2\Delta x}(f(\rho_{1,i}^t) + f(\rho_{0,i}^t) - 2\bar{\gamma}_i^t)$$

$$\text{central:} \quad \rho_{k,i}^{t+1} = \frac{1}{4}(\rho_{k-1,i}^t + 2\rho_{k,i}^t + \rho_{k+1,i}^t) - \frac{\Delta t}{2\Delta x}(f(\rho_{k+1,i}^t) - f(\rho_{k-1,i}^t))$$

$$\text{right:} \qquad \rho_{n,i}^{t+1} = \frac{1}{4}(\rho_{n-1,i}^t + 3\rho_{n,i}^t) - \frac{\Delta t}{2\Delta x}(2\hat{\gamma}_i^t - f(\rho_{n,i})^t - f(\rho_{n-1,i}^t))$$

Linearizing the coupling flow

For every time step t we need:

$$\max \sum_{i \in \delta^{in}} \hat{\gamma}_i^t$$

such that

$$\bar{\gamma}_j^t = \sum_{j \in \delta^{in_v}} d_{ij} \hat{\gamma}_i^t$$

$$0 \leq \hat{\gamma}_i^t \leq A_i^t \cdot \hat{F}_i^t$$

$$0 \leq \bar{\gamma}_j^t \leq \bar{F}_j^t$$

This can be transformed into

$$\hat{\gamma}_i = \min\left\{ A_i \hat{F}_i, \frac{1}{d_{ij}} \bar{F}_j \right\}$$

including information about permitted traffic light settings.

Linearizing techniques, e.g. min-terms

Given an expression of the form $c = \min\{a, b\}$.

- Introduce a binary variable $\kappa \in \{0, 1\}$ and use additional inequality constraints:

$$\kappa \cdot a \leq c \leq a$$
$$b - M \cdot \kappa \leq c \leq b$$

with M is sufficiently large, such that $M \geq b$ holds.

Linear MIP

The traffic light optimization problem consists of

- Control variables A_i^t,
- State variables $\rho_{i,k}^t$, $f_{i,k}^t, \ldots$,
- Linearization (binary) variables $\kappa_i^t, \ldots$
- $\Rightarrow$ The discrete traffic flow network results in a complex optimization problem where the number of constraints and variables is in $\mathcal{O}(|E| \cdot |n_t| \cdot |n_k|)$!!
- The problem is solved using the commercial solver CPLEX.

Additional restrictions on switching times

- **Idea:** Avoid highly oscillating solutions.
- Add a lower bound on the green phase L:

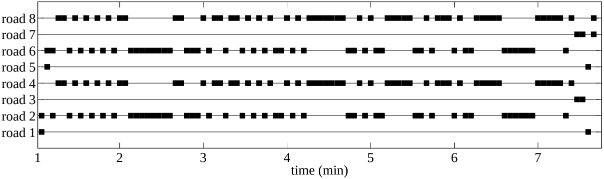
$$\sum_{l=t+1}^{t+L} A_i^l \geq L, \quad \forall t \leq n_t - L$$

- Add an upper bound on the red phase U:

$$\sum_{l=t+1}^{t+U+1} A_i^l \geq 1, \quad \forall t \leq n_t - U - 1$$

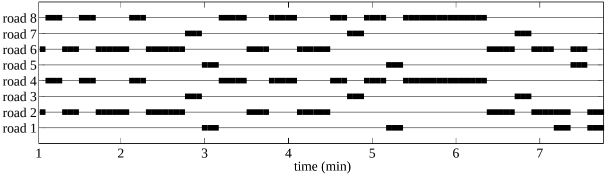
- **Drawback:** Restrictions on the control variables enter the problem.

Optimal traffic light settings



- Optimized traffic light setting **without** restrictions on the switching times leads to an objective functional value of 96.63.
- **Problem:** High fluctuating traffic light switching and long red phases (cf. road 1, 3, 5 and 7).

Optimal traffic light settings



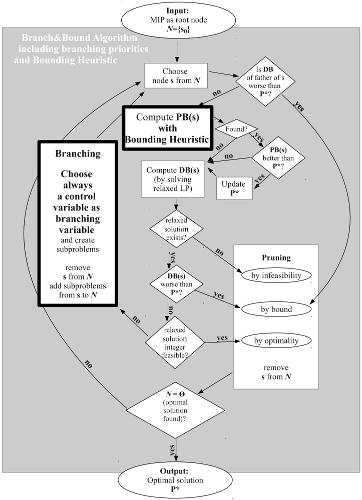
- Optimized traffic light setting **including** restrictions on the switching times leads to an objective functional value of 88.61.

Outline

Challenges and improvements in solving complex linear MIPs

- Complexity of the MIP depends on number of discretization points
- Extremely long computations times
- Fine timegrids $\rightarrow$ accumulation of rounding errors $\rightarrow$ CPLEX unable to find feasible solutions
- **Idea:** Exploiting knowledge from the world of dynamics to easily compute feasible solutions:
  - Utilization of Start-Heuristics
  - Further strategies in using heuristics during branch and bound process

Branch and Bound Tuning



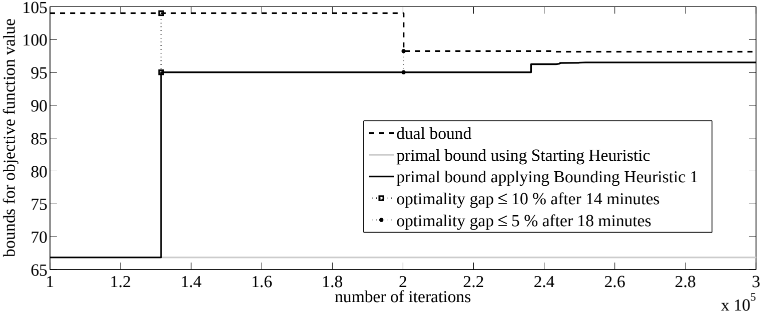
Heuristics

- Bounding Heuristic:
  - Find "good" feasible traffic light setting $\tilde{A}_i$ closest to 1, e.g.

    $$\begin{aligned} &\text{argmax } \tilde{A}_i \\ &\text{s.t. constraints for traffic flow variables} \end{aligned}$$

    where $\tilde{A}_i$ is the solution of the relaxed problem (MIP without integrality constraints).
  - Compute numerical solution.

Evolution of primal and dual bounds



- Comparison of the evolution of primal and dual bounds during the optimization procedure using Starting and Bounding Heuristics.

		Starting Heuristic	Bounding Heuristic
after	primal bound	73.7952	96.6212
18000 s	dual bound	97.1187	98.1543
(5 hours)	optimality gap	31.62%	1.59%
after	primal bound	75.0262	96.6331
259200 s	dual bound	97.0759	98.1538
(3 days)	optimality gap	29.39%	1.57%
optimality	# nodes	–	0 (root node)
gap	# iterations	(not obtained)	131290
$\leq$ 20%	elapsed time	–	813 s ($\approx$ 14 m)
optimality	# nodes	–	0 (root node)
gap	# iterations	(not obtained)	131290
$\leq$ 10%	elapsed time	–	813 s ($\approx$ 14 m)
optimality	# nodes	–	0 (root node)
gap	# iterations	(not obtained)	200173
$\leq$ 5%	elapsed time	–	1108 s ($\approx$ 18 m)
improvement of optimized traffic light setting			44.57%

Final Remarks

- The same strategy can be applied to other dynamic transportation models on networks, e.g. optimal dynamic worker scheduling for production networks.
- **Further interesting questions (long time projects):**
  - Is there any chance to do the optimization in a clever way? Maybe an automatic tool to detect the secure sets...
  - The connection between of Branch and Bound and Adjoint approaches. Recent work [Göttlich/Kolb/Kühn, 2013]. Think about some hybrid model ...
  - Include different kind of data. Intelligent routing?

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Thank you for your attention!