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Dynamical Phenomena induced by Bottleneck

Ingenuin Gasser
Department of Mathematics
University of Hamburg, Germany

Tilman Seidel, Gabriele Sirito, Bodo Werner

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- Dynamics of the microscopic model (without bottleneck)
- Dynamics of the microscopic model (with bottleneck)
- Macroscopic view (with bottleneck)
- From micro to macro (without bottleneck)
- Fundamental diagrams (with and without bottleneck)
- Future
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Basic concept: Take a very simple microscopic model (Bando), study the full dynamics, take a macroscopic view on the results.

Dynamics of the microscopic model (without bottleneck)

Bando model on a circular road (scaled)

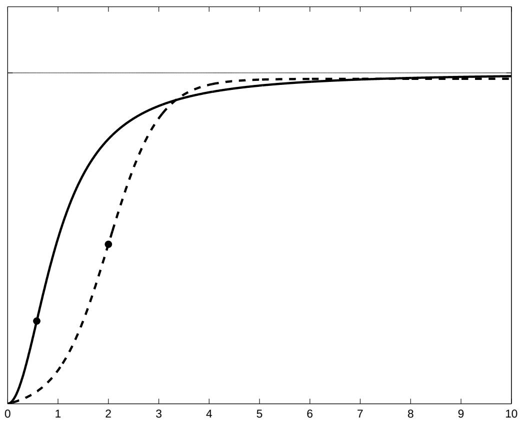
N **cars on a circular road of lenght** L:

Behaviour: x_j position of the j-th car

$$\ddot{x}_j(t) = -\left\{V\left(x_{j+1}(t) - x_j(t)\right) - \dot{x}_j(t)\right\}, \qquad j = 1, ..., N, \quad x_{N+1} = x_1 + L$$

$V = V(x)$ optimal velocity function:

$V(0) = 0$, $\qquad V$ strictly monoton increasing , $\qquad \lim_{x\to\infty} V(x) = V^{max}$



System for the headways: $y_j = x_{j+1} - x_j$

$$\begin{aligned} \dot{y}_j &= z_j \\ \dot{z}_j &= -\left\{V(y_{j+1}) - V(y_j) - \dot{z}_j\right\}, \qquad j = 1, ..., N, \quad y_{N+1} = y_1 \end{aligned}$$

Additional condition: $\sum_{j=1}^{N} y_j = L$

"quasistationary" solutions: $y_{s;j} = \frac{L}{N}, \quad z_{s;j} = 0, \quad j = 1, ..., N.$

Linear stability-analysis around this solution gives for the Eigen-values λ:

$$(\lambda^2 + \lambda + \beta)^N - \beta^N = 0, \qquad \beta = V'(\frac{L}{N})$$

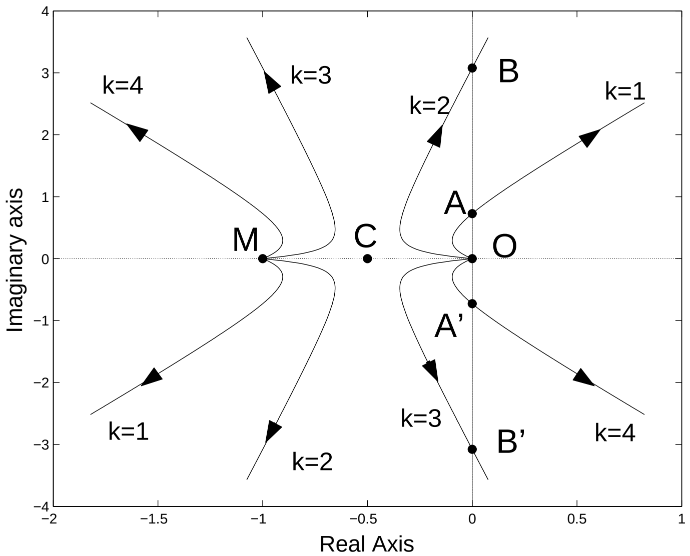
Result (Huijberts ('02)):

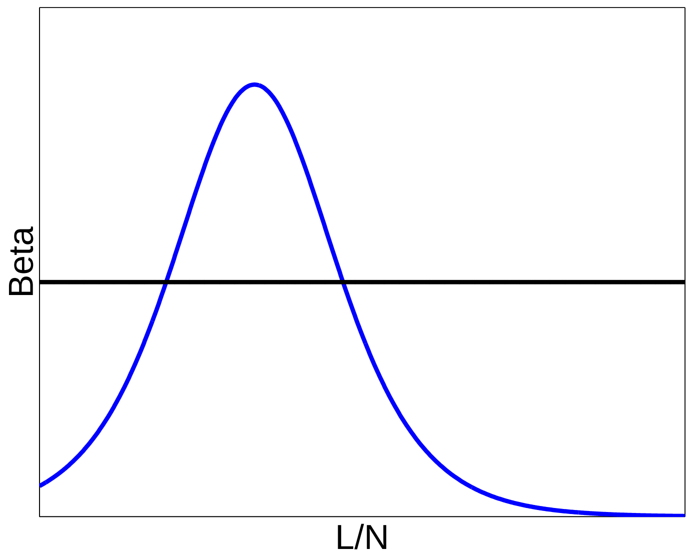
For $\frac{1}{1+\cos\frac{2\pi}{N}} > \beta^{max} = max_x V'(x)$ asymptotic stability

For $\frac{1}{1+\cos\frac{2\pi}{N}} = V'(\frac{L}{N})$ loss of stability

What kind of loss of stability? (I.G., G.Sirito, B. Werner '04):

Eigenvalues as functions of $\beta = V'(\frac{L}{N})$





Bifurcation analysis gives a Hopfbifurcation.

Therefore we have locally periodic solutions.

Are these solutions stable? (i.e. is the bifurcation sub- or super-critical?)

Criterion: Sign of the first Ljapunov-coefficient l

Theorem:

$$l = c^2 \left\{ V''' \left(\frac{L}{N} \right) - \frac{\left(V''(\frac{L}{N}) \right)^2}{V'(\frac{L}{N})} \right\}$$

Conclusion: For the mostly used (Bando et al (95))

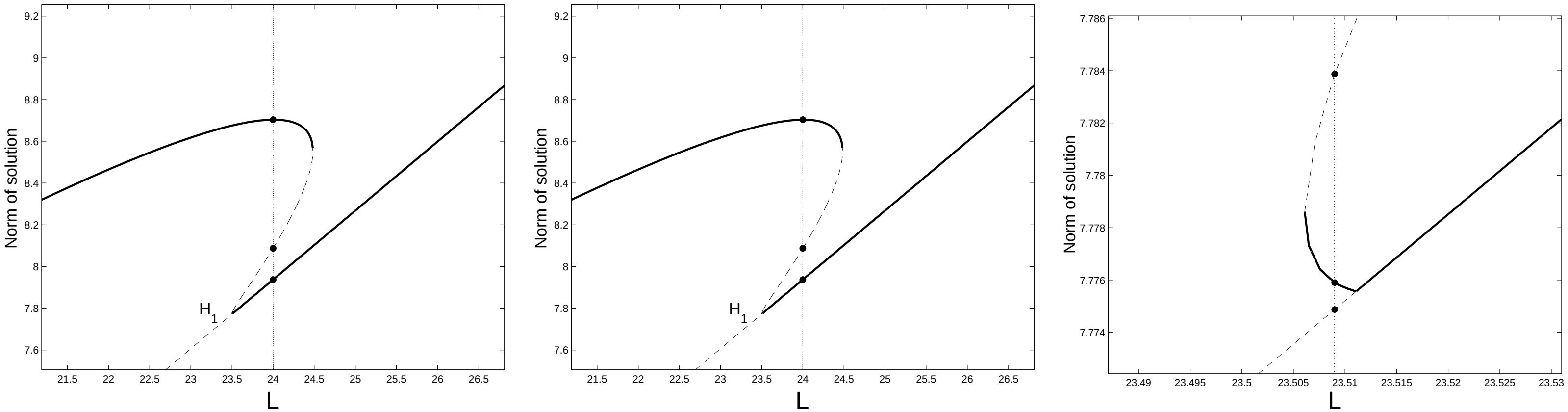
$$V(x) = V^{max} \frac{\tanh(a(x-1)) + \tanh a}{1 + \tanh a}$$

the bifurction is supercritical (i.e. stable periodic orbits).

But: “Similar” functions V give also subcritical bifurcations.

Problem: It seems to be very sensitive with respect to V

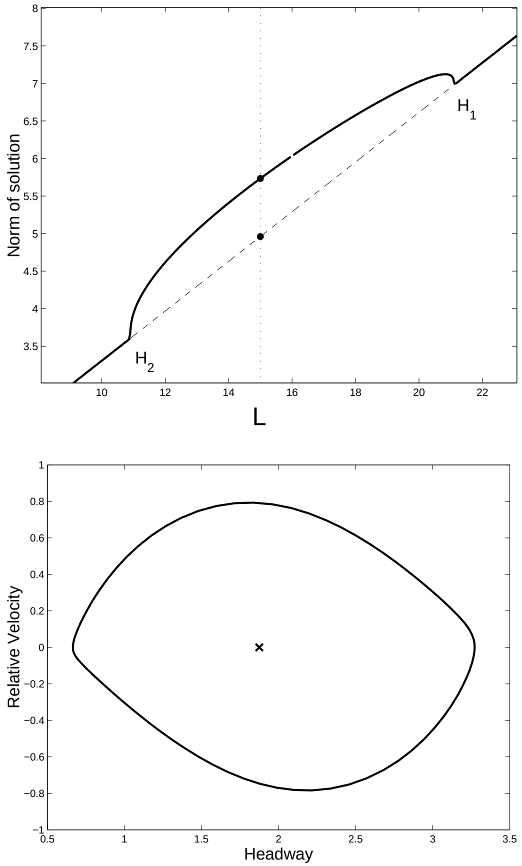
Global bifurcation analysis: numerical tool (AUTO2000)



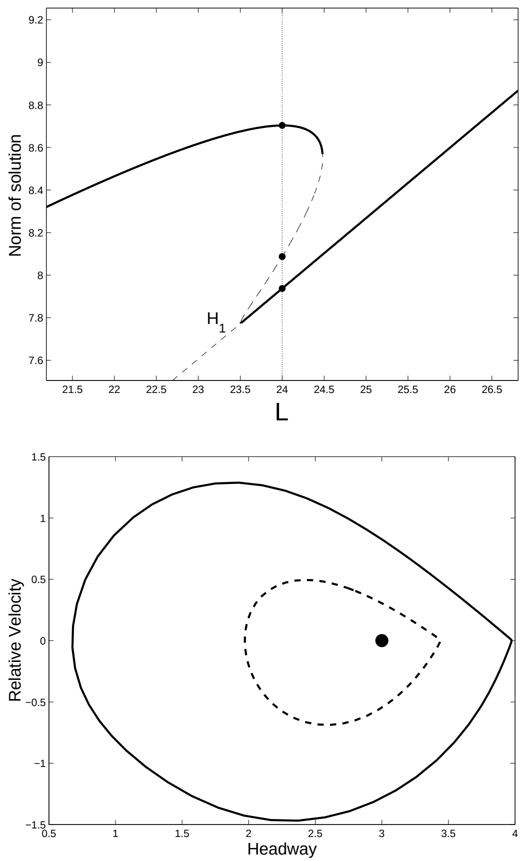
Conclusion: Globally "similar" functions V give similar behaviour. The bifurcation is "macroscopically" subcritical

Conclusion for the application: the critical parameters from the linear analysis are not relevant

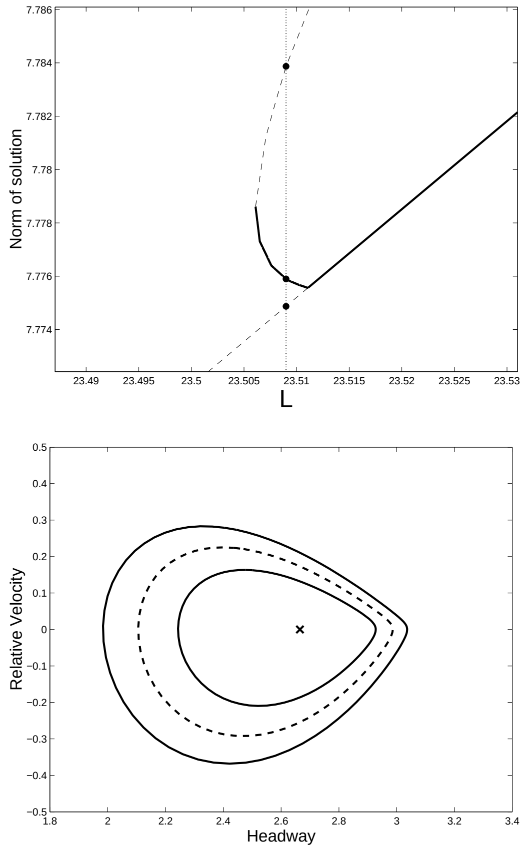
Example 1



Example 2

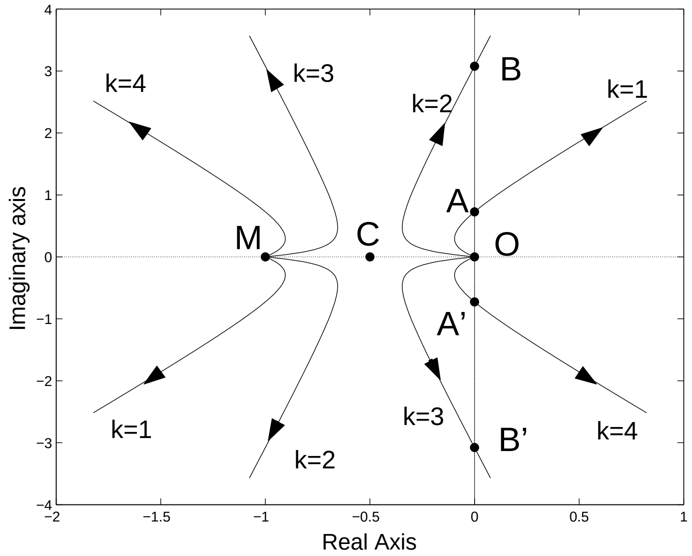


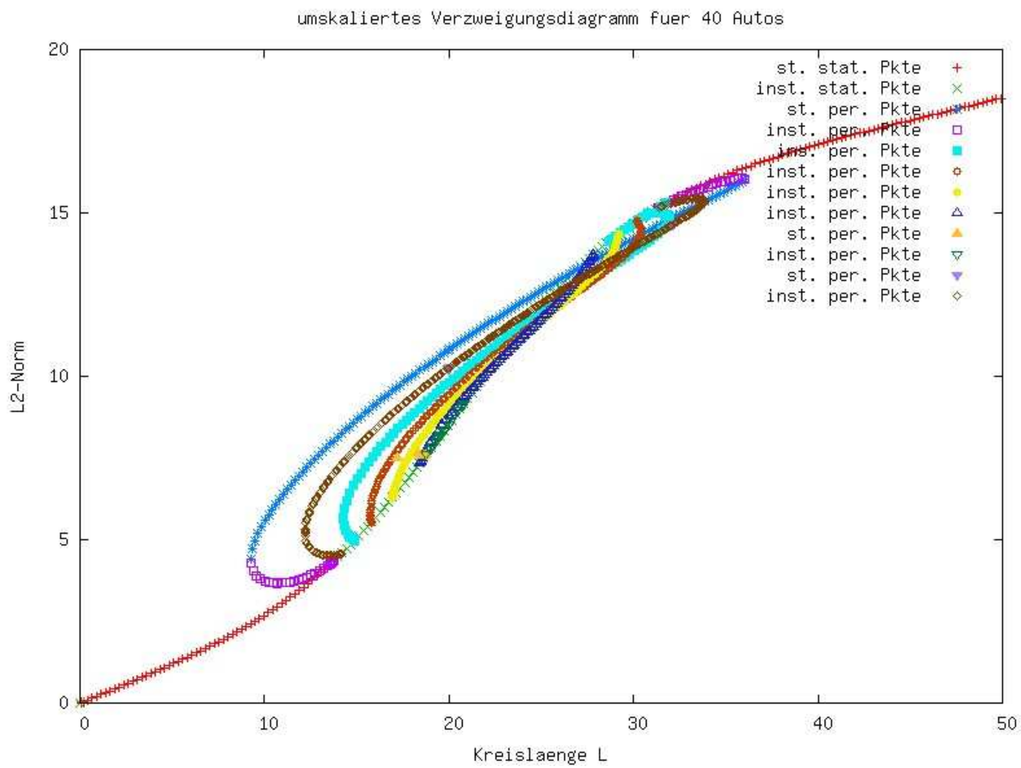
Example 3



More bifurcations:

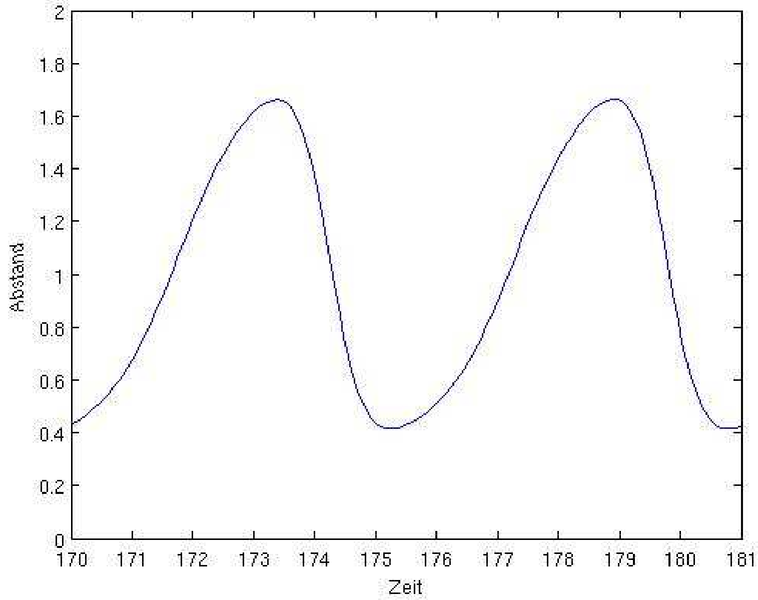
Eigenvalues as functions of $\beta = V'(\frac{L}{N})$

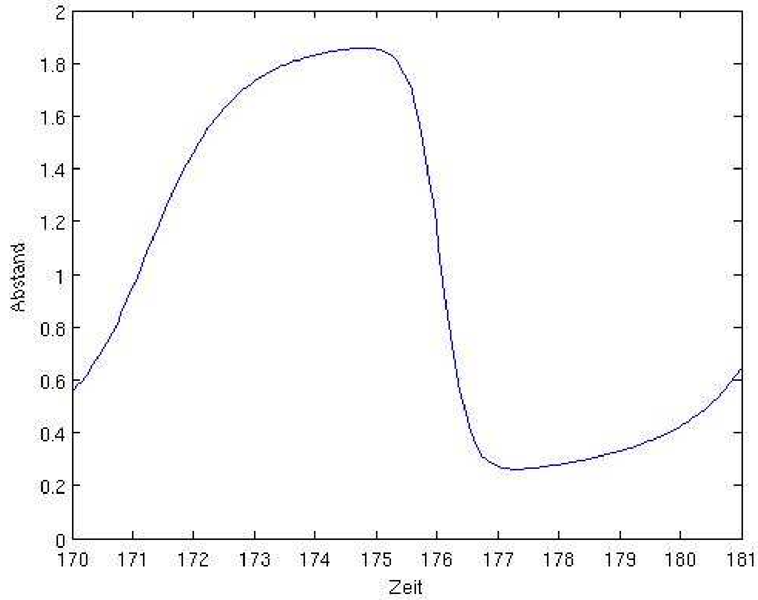




Conclusion: There are many other (weakly unstable) periodic solutions

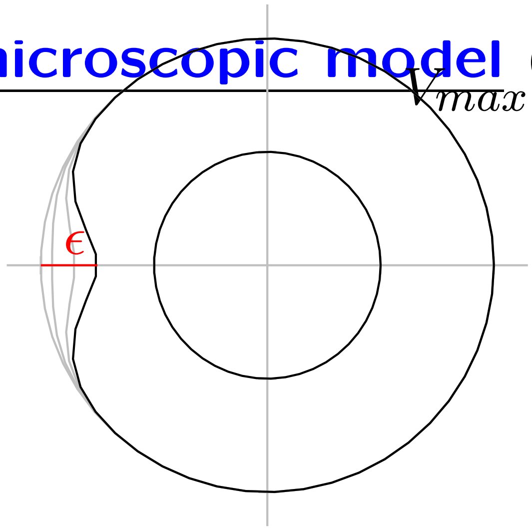
(J.Greenberg '04,'07) Solutions with many oscillations finally tend to a solution with one oscillation





(G. Oroz, R.E. Wilson, B.Krauskopf '04, '05) Qualitatively the same global bifurcation diagram for the model with delay

Dynamics of the microscopic model (with bottleneck)



Symmetry breaking, the above theory is not easily applicable

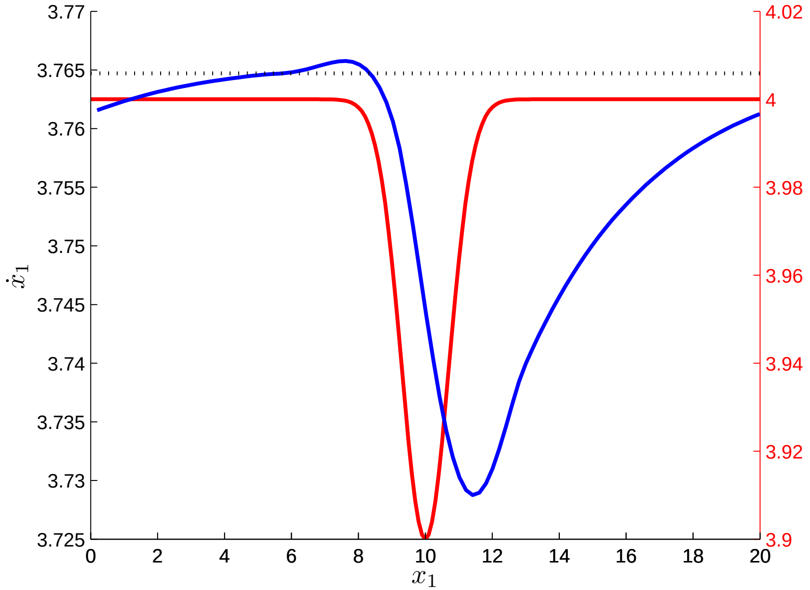
A solution is called ponies on a Merry-Go-Round solution (short POM), if there is a $T \in \mathbb{R}$, such that

(i) $x_i(t+T) = x_i(t) + L \quad (i = 1, \ldots, N)$

(ii) $x_i(t) = x_{i-1}\left(t + \frac{T}{N}\right) \quad (i = 1, \ldots, N)$

hold (Aronson, Golubitsky, Mallet-Paret '91). We call T rotation number and $\frac{T}{N}$ the phase (phase shift).

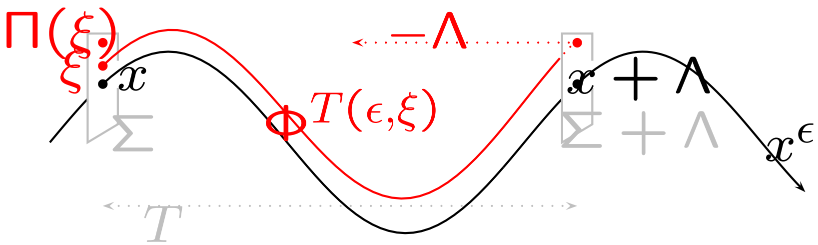
Theorem: The above model has POM solutions for small $\epsilon > 0$.



Velocity of the quasistationary solution (no bottleneck) versus bottleneck solution (The red line indicates maximum velocity).

Technique: Poincare maps

$\Pi(\eta) = \Phi_{T(\eta)}(\eta) - \Lambda$, where Φ is the induced flow and Λ reduces the spacial components by L.



Study fixed points of the corresponding Poincare and reduced Poincare maps. bottleneck are (regular) perturbations.

Four different attractors.:

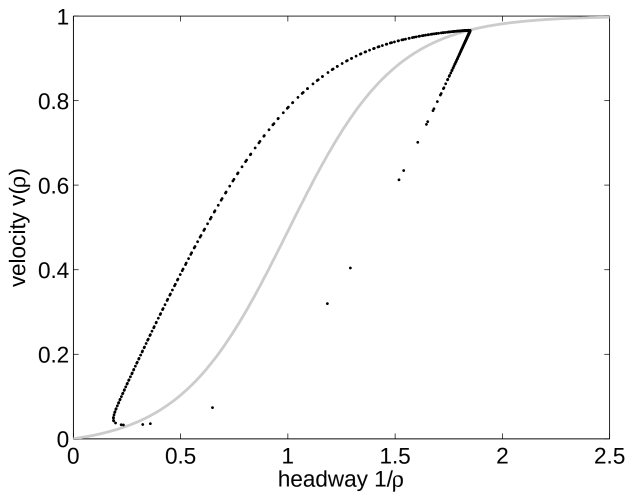
ϵ	I	II
$\epsilon = 0$	trivial POM x^0	Hopf periodic solution
$\epsilon > 0$	POM x^ϵ	quasi-POM

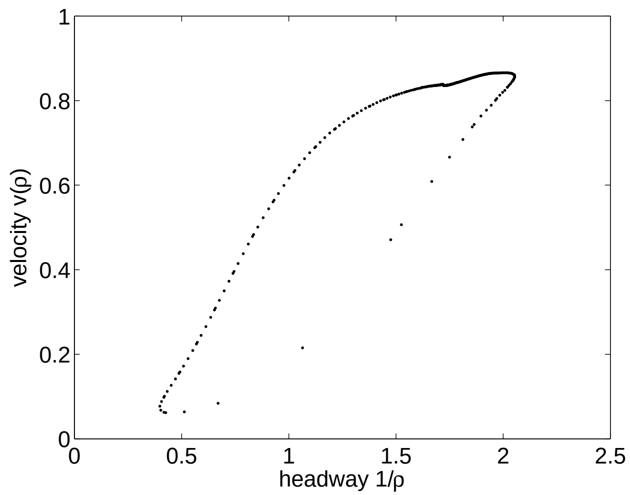
i.e. here

POM's are typically perturbed quasistationary solutions

quasi-POM's are perturbed (Hopf) periodic solutions

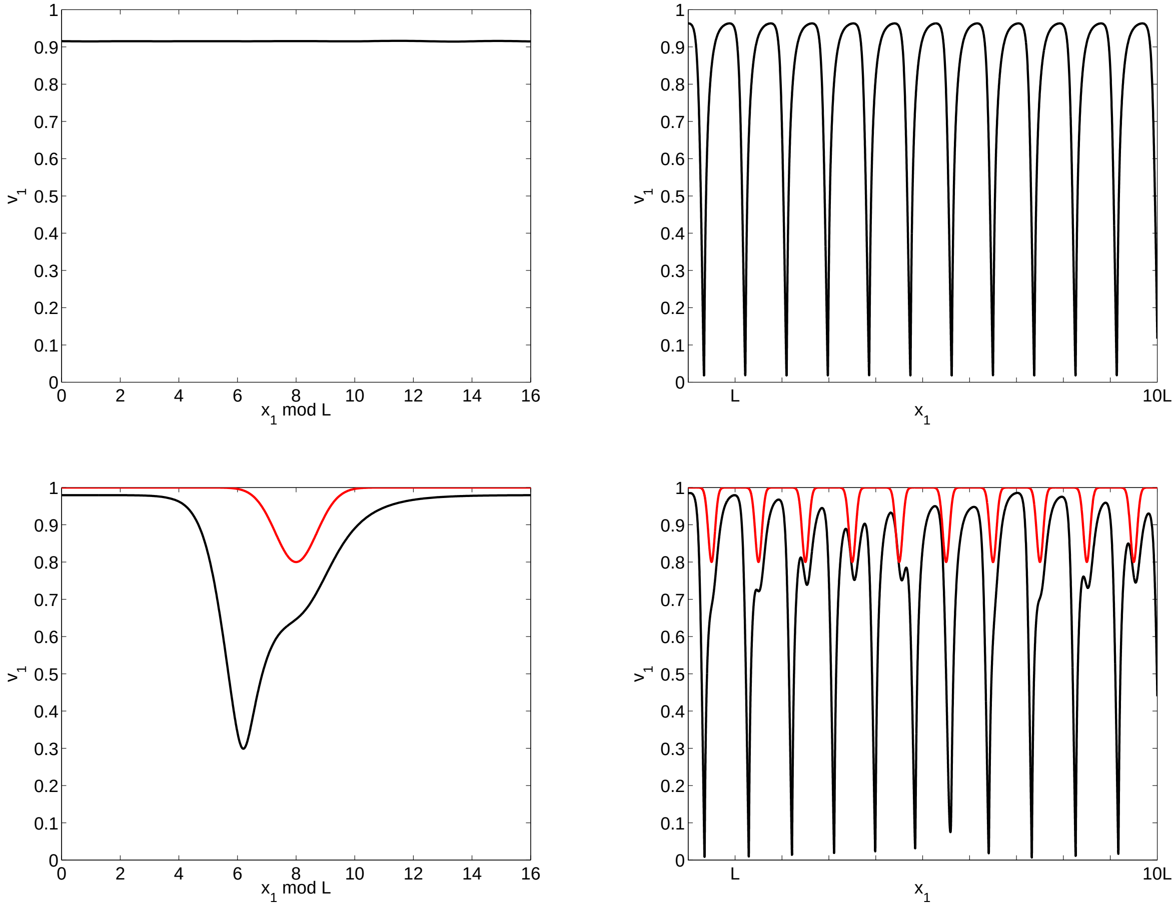
Invariant curves:





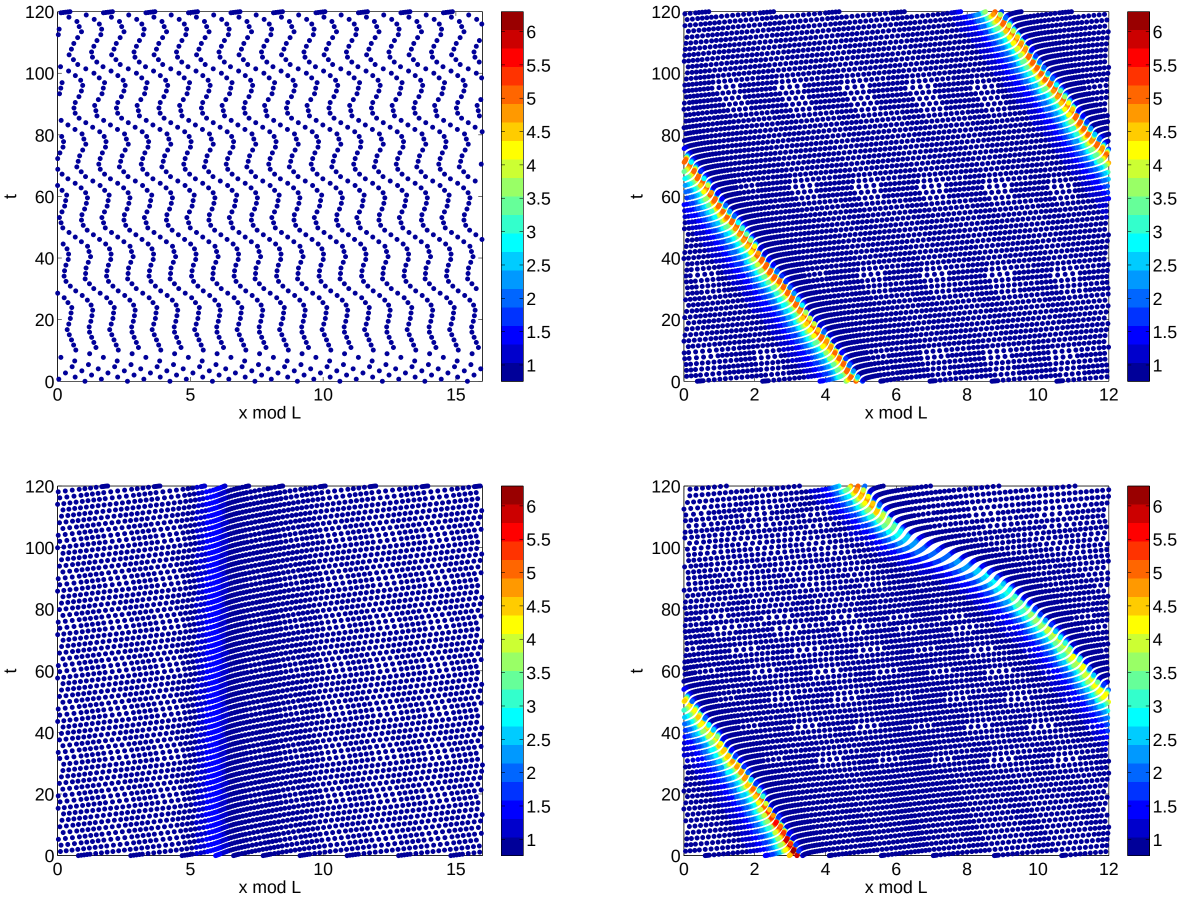
Two closed invariant curves ($\epsilon = 0$ and $\epsilon > 0$) of the reduced Poincaré map π. On the left also the optimal velocity function V_0 is given in gray.

The 4 different scenarios:



above: no bottleneck , below: with bottleneck

Macroscopic view of the 4 different scenarios:



above: no bottleneck , below: with bottleneck

From micro to macro (for simplicity without bottleneck)

We start with Bando: x_j

$$\ddot{x}_j(t) = -\frac{1}{\tau}\left\{V\left(x_{j+1}(t) - x_j(t)\right) - \dot{x}_j(t)\right\}, \quad j = 1, ..., N, \quad x_{N+1} = x_1 + L$$

density: $\rho\left(\frac{x_{j+1}(t)+x_j(t)}{2}, t\right) = \frac{1}{x_{j+1}(t)-x_j(t)}$

equilibrium velocity function: $V_e(\rho(\frac{x_{j+1}+x_j}{2}, t)) = V(x_{j+1} - x_j)$

asymptotics: small parameters (l, L are micro and macro lenght):
$\epsilon = \frac{l}{L}$, τ

position and velocity: $x \approx x_j(t), \qquad v(x,t) \approx v_j(t) = \dot{x}_j(t)$

zeroth order (in ϵ,τ): $\qquad \rho_t + (\rho V_e(\rho))_x = 0$

zeroth order (in ϵ) :

$$\begin{aligned} \rho_t + (\rho v)_x &= 0 \\ (\rho v)_t + \left(\rho v^2\right)_x &= \frac{1}{\tau}\rho(V_e(\rho) - v) \end{aligned}$$

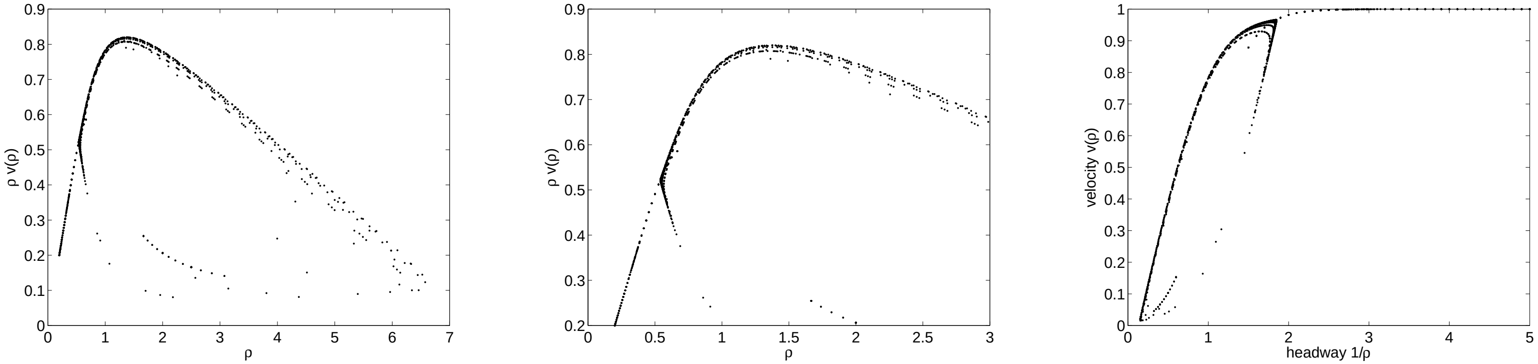
first order (in ϵ) : (see also Aw, Rascle, Klar, Materne 03)

$$\begin{aligned} \rho_t + (\rho v)_x &= 0 \\ (\rho v)_t + \left(\rho v^2 - \frac{\epsilon}{2\tau}V_e(\rho)\right)_x &= \frac{1}{\tau}\rho(V_e(\rho) - v) \end{aligned}$$

Question: are the Hopf periodic solutions in the microscopic world (appearing as traveling waves in the macroscopic presentation) traveling waves of macroscopic equations (scalar or system)?

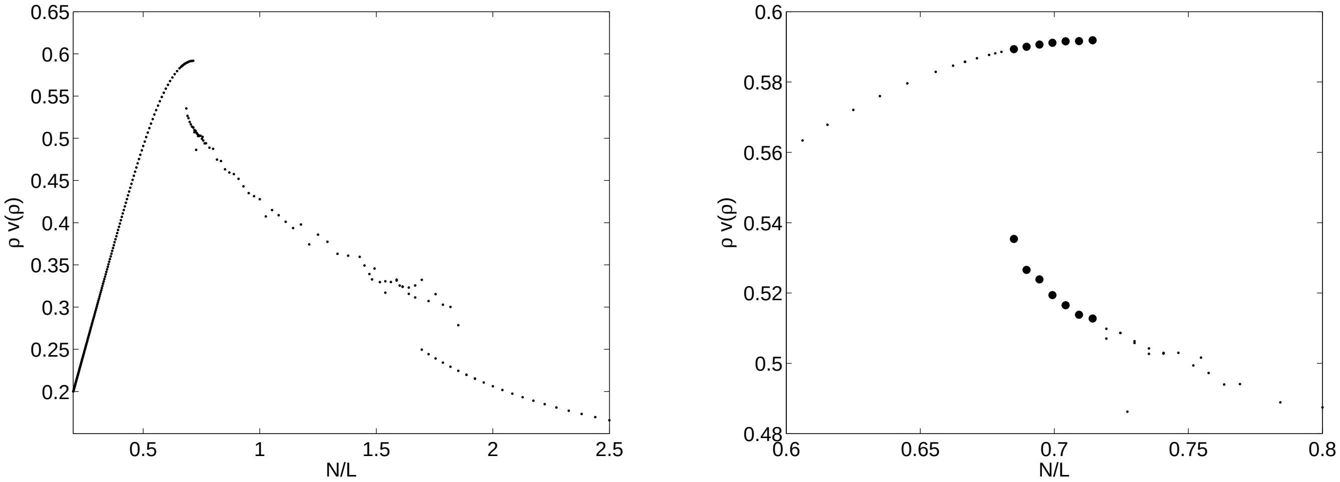
(see Greenberg 01,04,
see Aw, Rascle, Klar, Materne 03,
see Flynn, Kasimov, Nave, Rosales, Seibold 09)

Fundamental diagrams I:



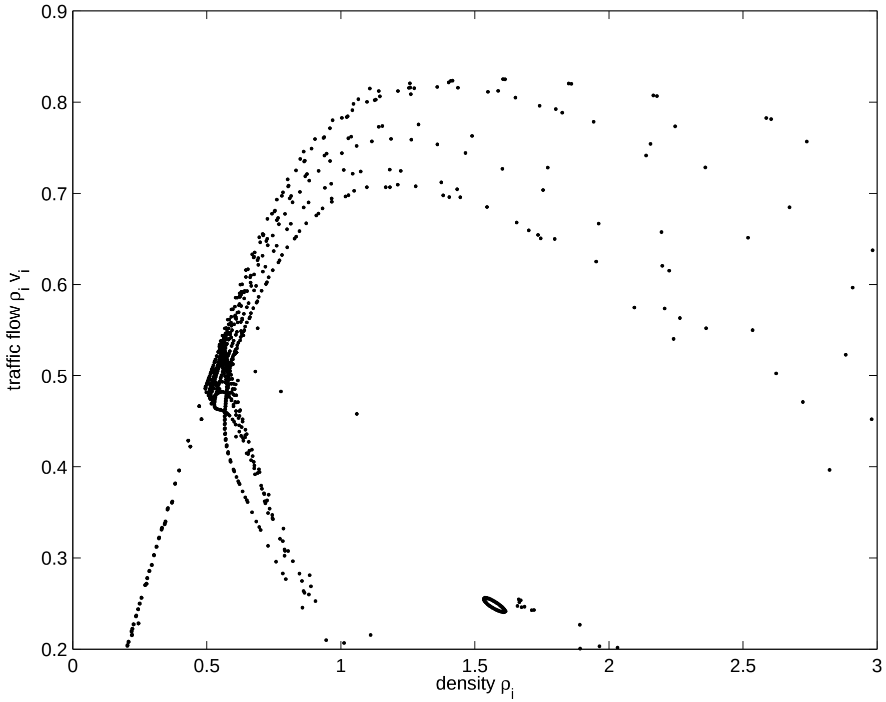
Overlapped fundamental diagrams for $N = 10, L = 50, ..., 4$ measuring at a fixed point.

Fundamental diagrams II:



Fundamental diagram of time-averaged flow versus average density for $N = 10, L = 50 \ldots 4$.

Fundamental diagrams III (with bottleneck):



Overlapped fundamental diagrams for $N = 10, L = 50, ..., 4, \quad \epsilon = 0.1$ measuring at a fixed point.

How to "open" the ring road (B. Werner) (for simplicity without bottleneck)

We start with Bando: x_j

$$\ddot{x}_j(t) = -\frac{1}{\tau}\left\{V\left(x_{j+1}(t) - x_j(t)\right) - \dot{x}_j(t)\right\}, \quad j = 1, ..., N, \quad x_1 = x_1(t) \text{ is given}$$

the parameter L is somehow substituted by $\dot{x}_1$

one can show easily: $\dot{x}_1 = const$ is an asymptotically stable solution of that system

BUT: if $N \to \infty$, the notion of stability has to be clarified!

In fact: we encounter "moving instabilities", i.e. the behaviour of every single car is "stable", but there are perturbations which move backwards and increase!

And there seems to be a bifurcation (similar to the ring road) "separating the moving waves into stable and unstable ones".

Current and future work

- How to open the cicular road in the microscopic world?
- Is this dynamics contained in a macroscopic models?
- There is a time periodicity in the macroscopic pictures!

References

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