

Various possibilities for solving Riemann problems at junctions

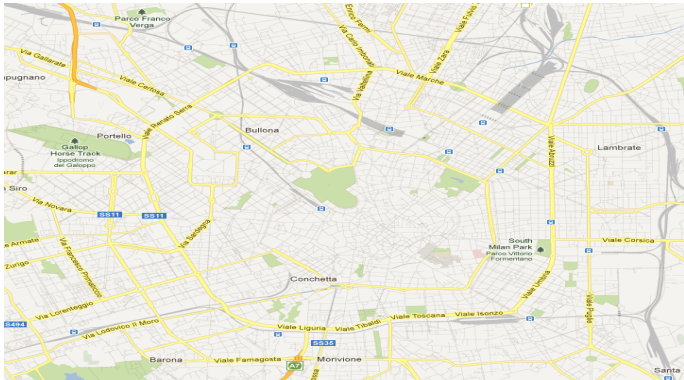
Mauro Garavello

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University of Milano Bicocca
mauro.garavello@unimib.it

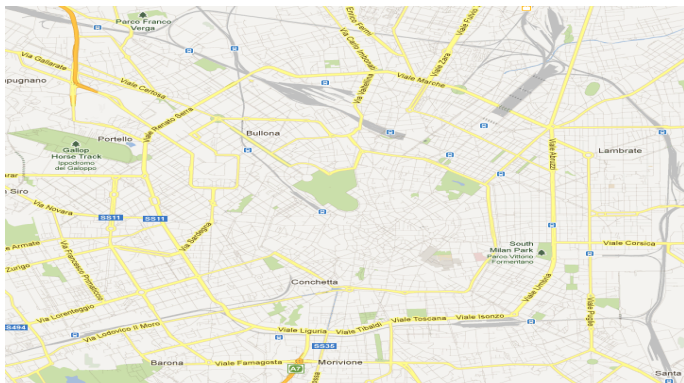
Joint works with: G. M. Coclite, P. Goatin, B. Piccoli



Road networks

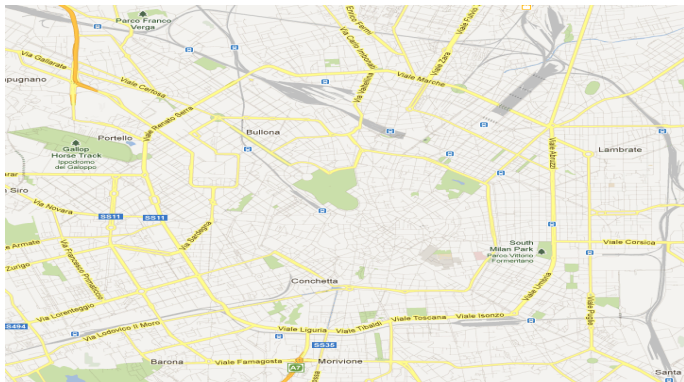


Road networks



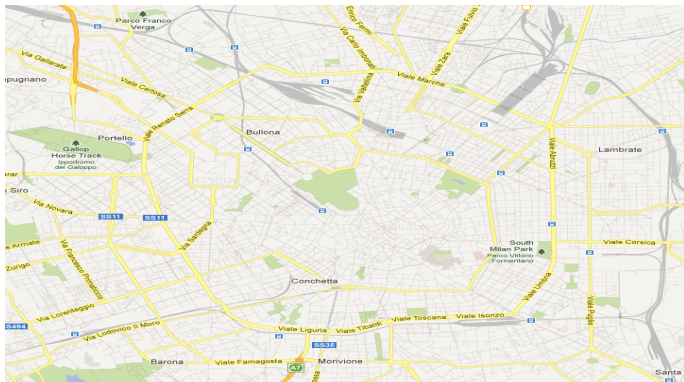
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Road networks



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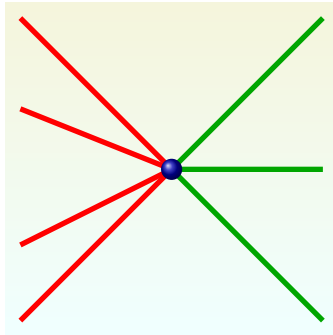
Road networks



- finite number of roads and junctions
- each arc can be modeled by $[a, b]$
- a macroscopic traffic model on each arc

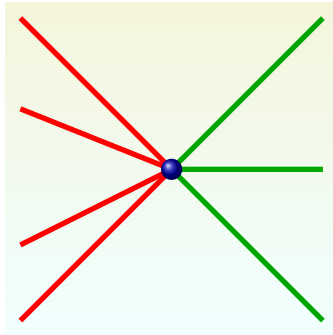
Junctions

- n incoming arcs



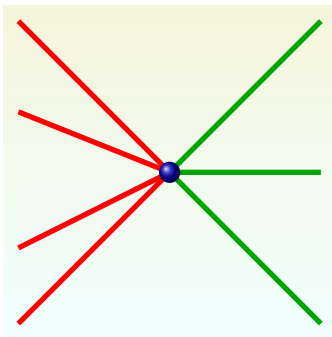
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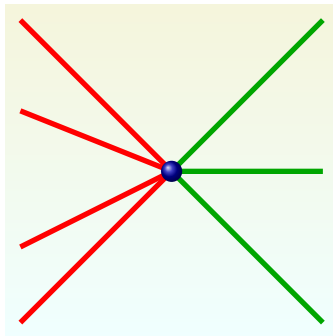
Junctions

- n incoming arcs
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LWR

Aw-Rascle-Zhang

Phase-Transitions models



The LWR model

$$\rho_t + f(\rho)_x = 0$$

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- \mathbf{v} depends only on ρ in a decreasing way

The LWR model

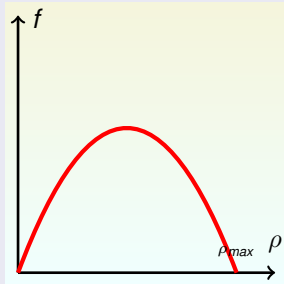
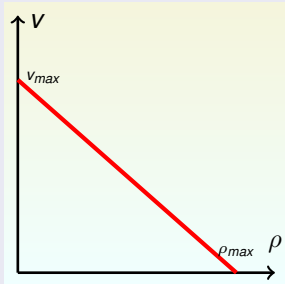
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A Phase-Transition model

Free phase Ω_f :

$$\partial_t \rho + \partial_x (\rho V) = 0$$

Congested phase Ω_c :

$$\begin{cases} \partial_t \rho + \partial_x (\rho v(\rho, q)) = 0 \\ \partial_t q + \partial_x (q v(\rho, q)) = 0 \end{cases}$$

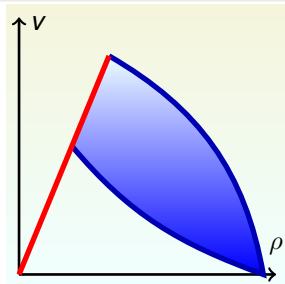
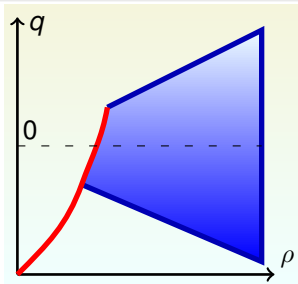
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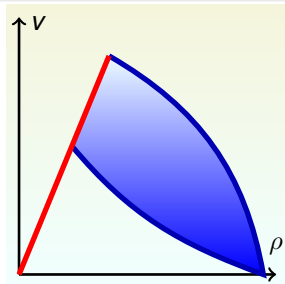
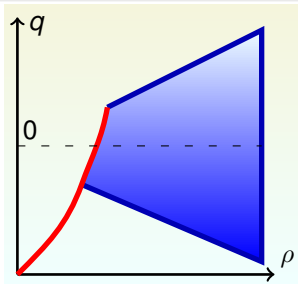
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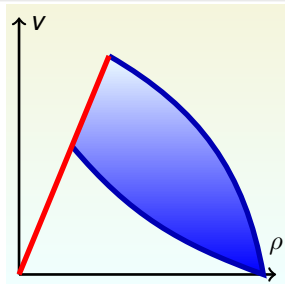
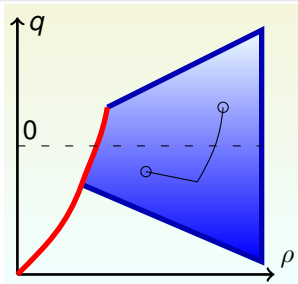
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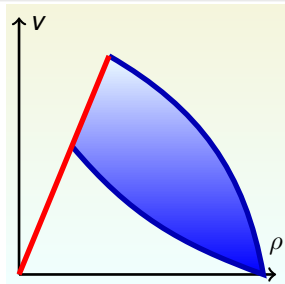
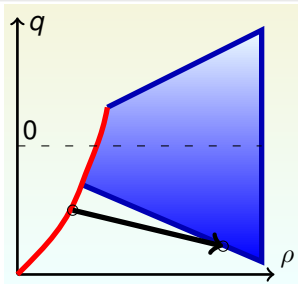
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LWR: Riemann problem at a junction

n incoming and m outgoing arcs

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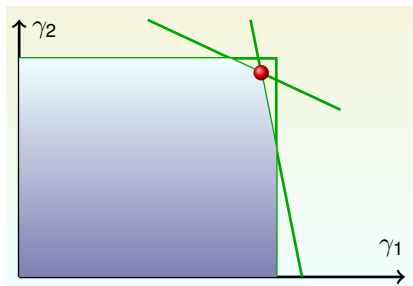
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LWR on a junction: Riemann solver 2

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LWR on a junction: Riemann solver 2

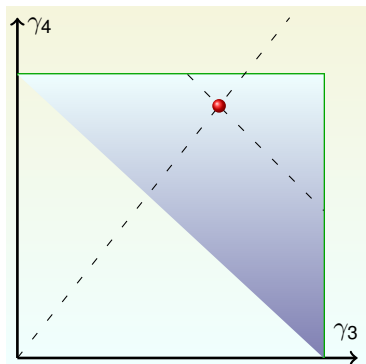
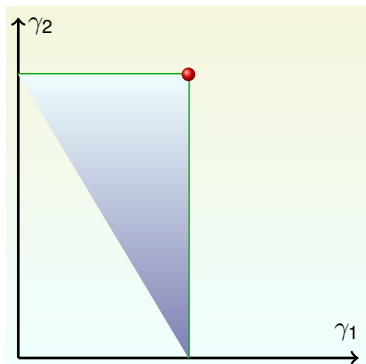
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LWR on a junction: existence result

Theorem [CGP 2005, AIP 2009]

Let \mathcal{RS} be the Riemann solver 1 or 2.

For every $T > 0$, there exists a solution $(\rho_1, \dots, \rho_{n+m})$ for the Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t} \rho_l + \frac{\partial}{\partial x} f(\rho_l) = 0 \\ \rho_l(0, x) = \rho_{l,0}(x) \end{cases} \quad l = 1, \dots, n+m$$

such that

$$\mathcal{RS}(\rho_1(t, 0), \dots, \rho_{n+m}(t, 0)) = (\rho_1(t, 0), \dots, \rho_{n+m}(t, 0))$$

for a.e. $t \in [0, T]$.

LWR on a junction: existence result

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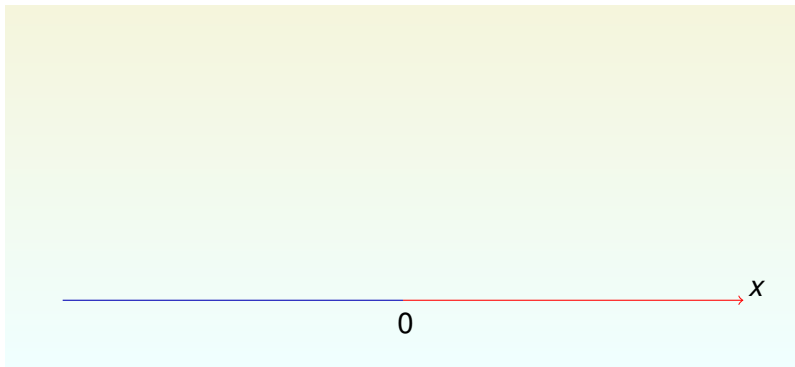
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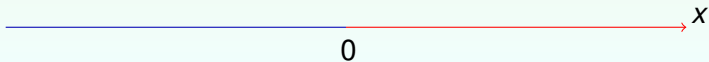
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- $\mathcal{RS2}$: Lipschitz continuous dependence w.r.t. the initial condition

Bottleneck

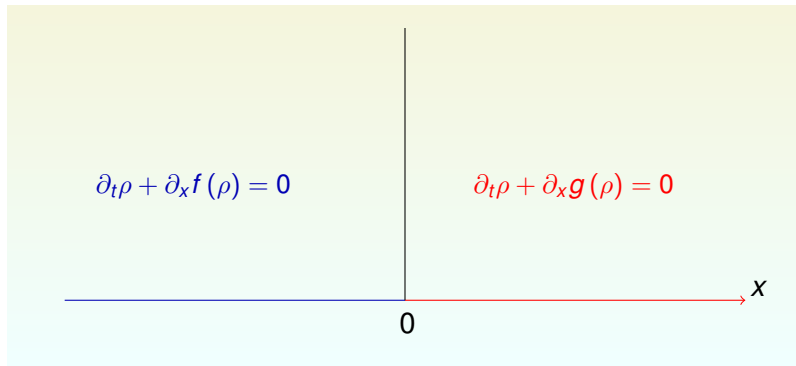


Bottleneck

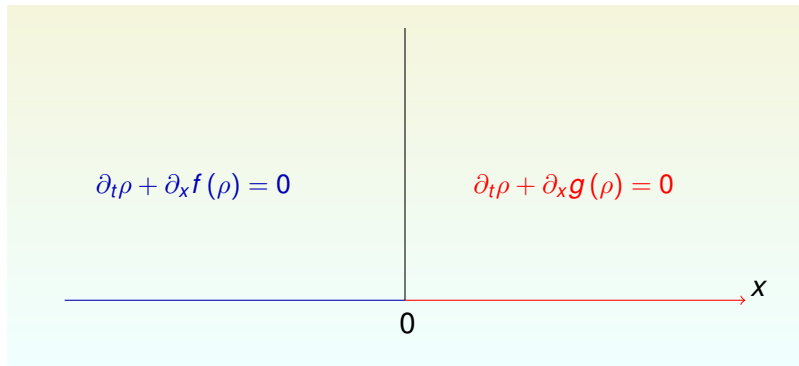
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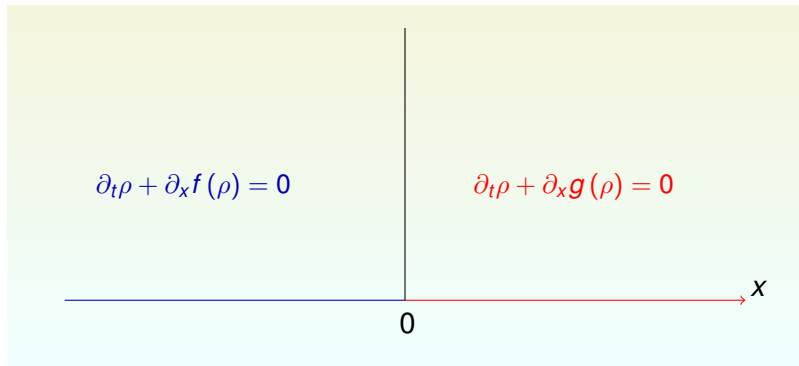


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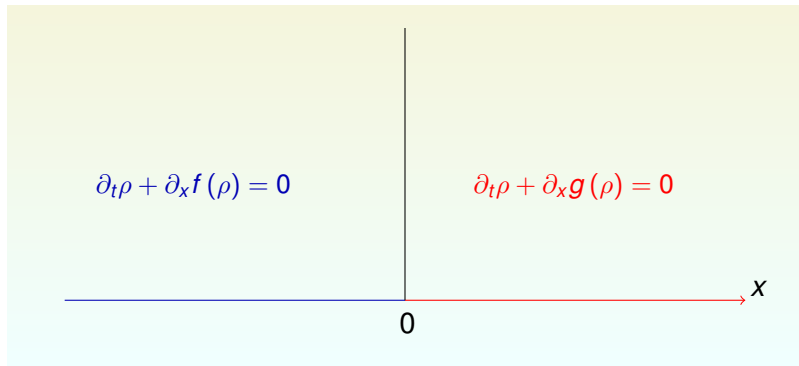
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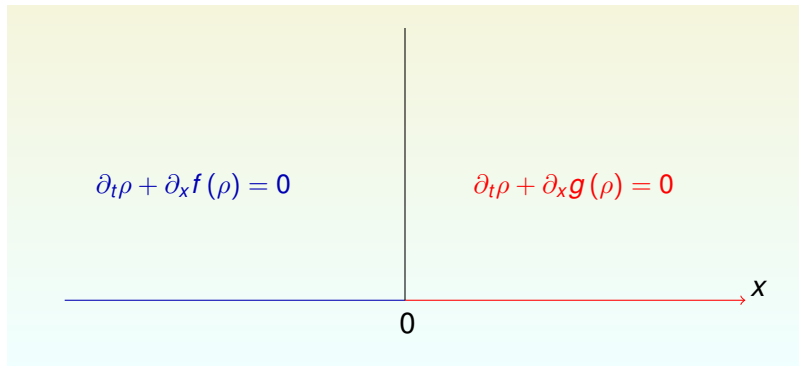
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Bottleneck



- f and g concave functions
- $f(0) = 0$ and $g(0) = 0$
- $f(\rho_{max}) = 0$ and $g(\rho_{max}) = 0$
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- For each $\gamma \in [0, \min \{\max f, \max g\}]$ there exists a Riemann solver \mathcal{RS}_γ , which selects a solution with flux lower than γ at $x = 0$

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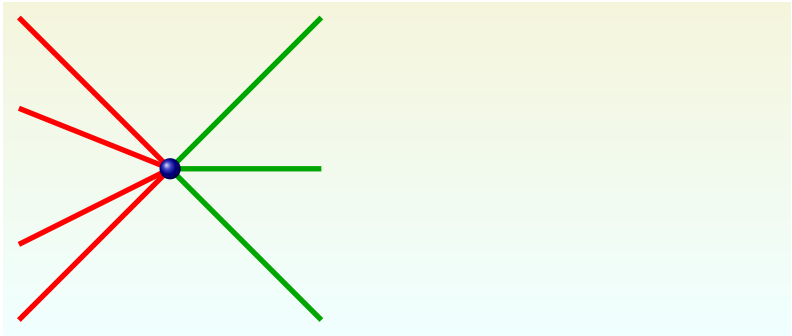
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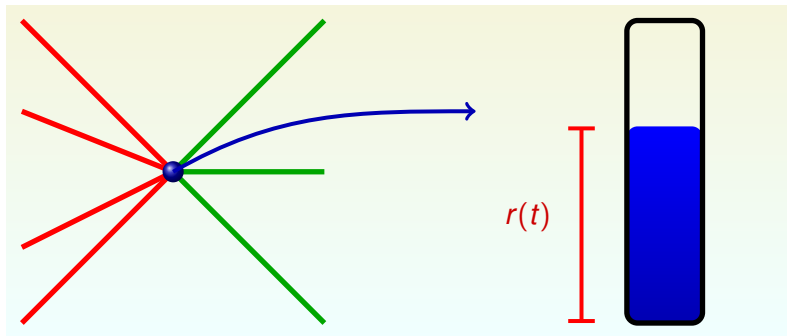
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- [NHM 2007]
- Colombo, Goatin. A well posed conservation law with a variable unilateral constraint. J. Differential Equations 2007.

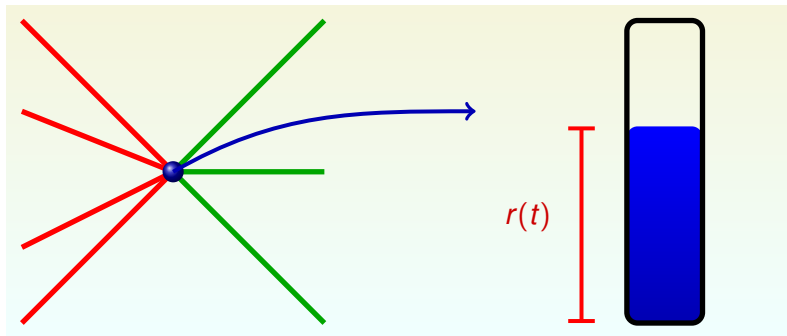
Junction with buffer



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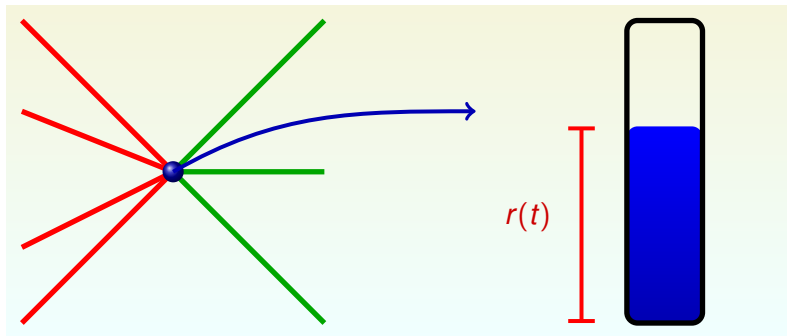


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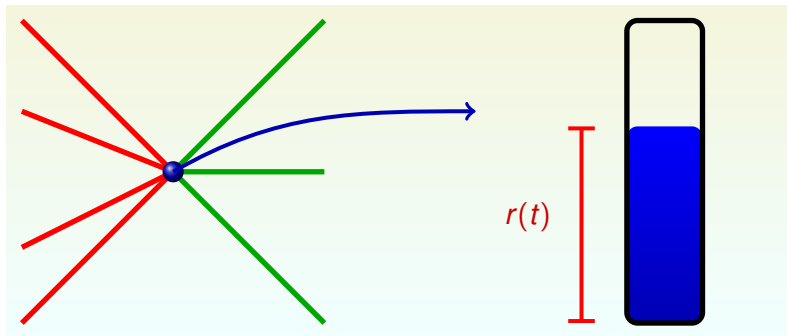
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- Herty, Lebacque, Moutari. A novel model for intersections of vehicular traffic flow. NHM, 2009.

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Case $0 < r_0 < r_{max}$

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Theorem [DCDS-A, 2012]

For every $T > 0$, the Cauchy problem admits a weak solution.

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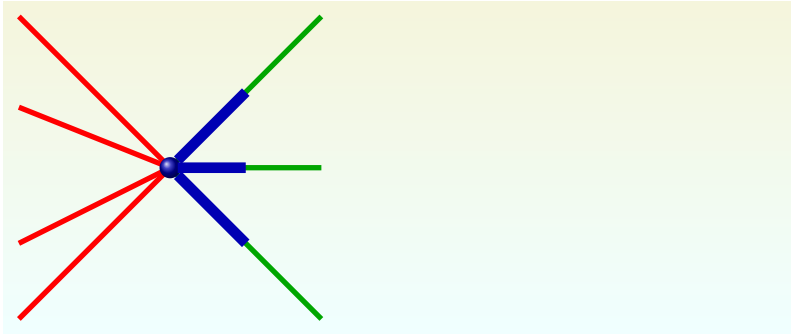
Theorem [DCDS-A, 2012]

For every $T > 0$, the Cauchy problem admits a weak solution. Moreover the solution depends in a Lipschitz continuous way w.r.t the initial conditions.

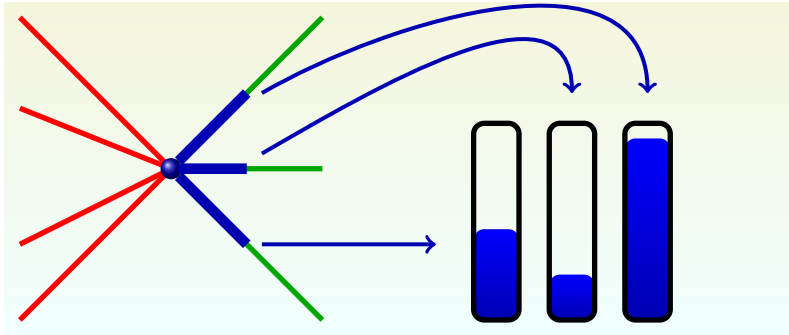
Multi-buffers



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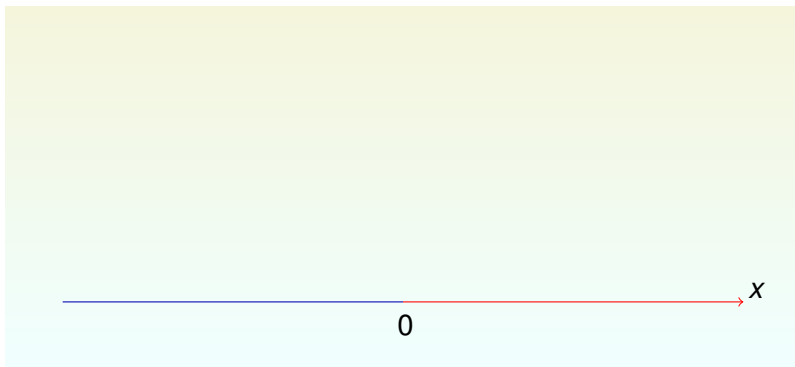
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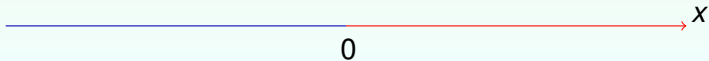
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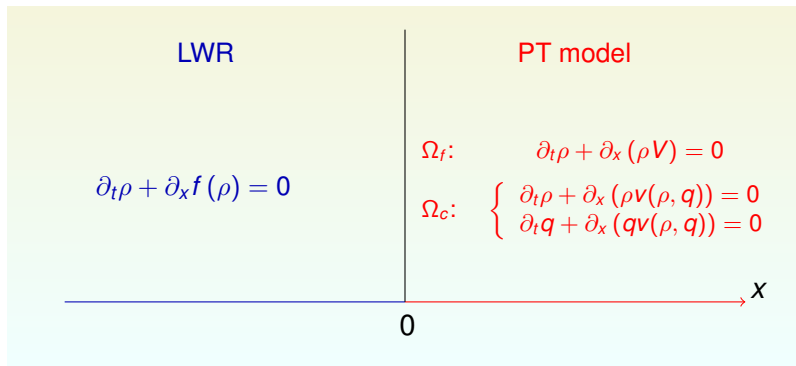
- Maximization

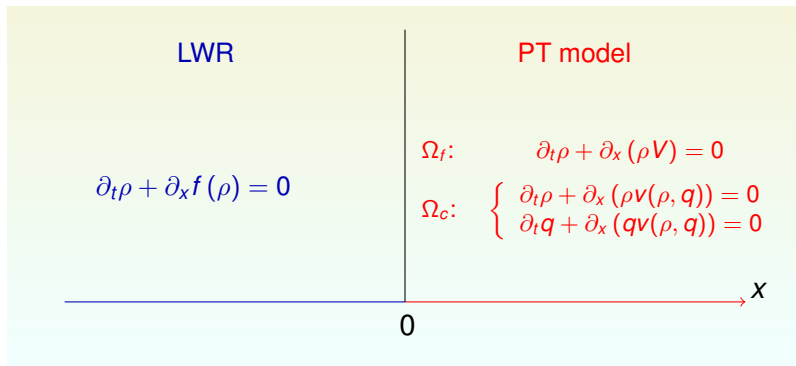


LWR

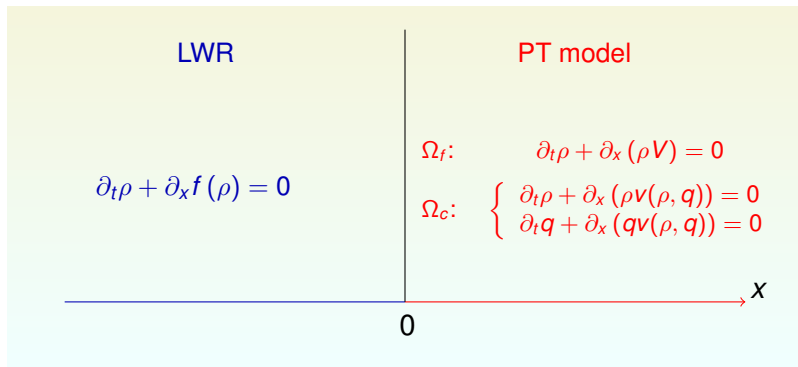
$$\partial_t \rho + \partial_x f(\rho) = 0$$





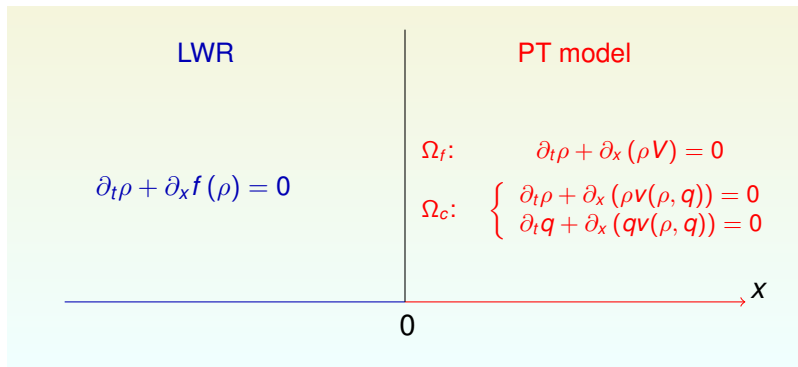


Solution at $x = 0$



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- Conservation of the number of cars: equality of the fluxes



Solution at $x = 0$

- Conservation of the number of cars: equality of the fluxes
- Maximization of the flux

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