Kinetic ODEs Modeling Vehicular Traffic in Time and Space

Andrea Tosin*

Istituto per le Applicazioni del Calcolo "M. Picone"
Consiglio Nazionale delle Ricerche
Rome, Italy

a.tosin@iac.cnr.it
http://www.iac.cnr.it/~tosin

Sophia Antipolis, March 20-22, 2013

*Joint with **Luisa Fermo** (University of Cagliari, Italy)

Keywords and Motivations

Continuous flow

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Granularity

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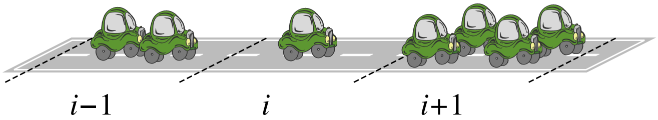
Stochastic games

The model



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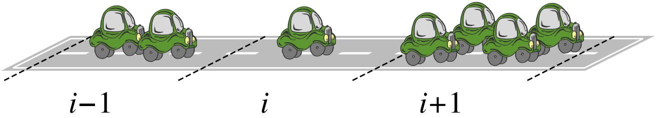

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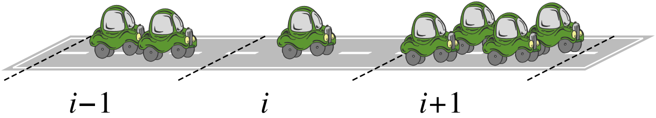

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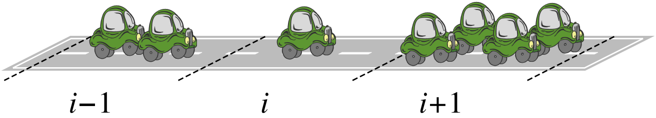

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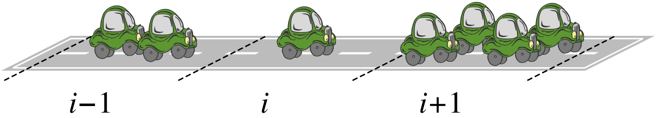

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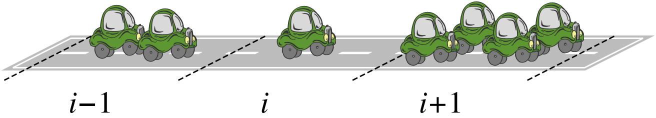

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$$\frac{df_{ij}}{dt} + v_j \left(\Phi_{i,i+1} f_{ij} - \Phi_{i-1,i} f_{i-1,j} \right) =$$

$$\Phi_{i,i+1} = \begin{cases} \dfrac{1-\rho_{i+1}}{\rho_i} & \text{if } \rho_i + \rho_{i+1} > 1 \\ 1 & \text{if } \rho_i + \rho_{i+1} \leq 1, \end{cases}$$

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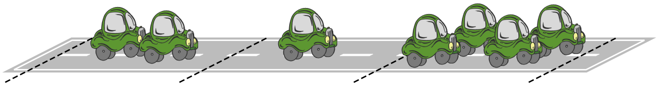

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$$\frac{df_{ij}}{dt} + v_j\left(\Phi_{i,i+1} f_{ij} - \Phi_{i-1,i} f_{i-1,j}\right) = \sum_{h,k=1}^{n} \eta_{hk}(i)\mathcal{A}_{hk}^{j}(i) f_{ih} f_{ik} - f_{ij} \sum_{k=1}^{n} \eta_{jk}(i) f_{ik}$$

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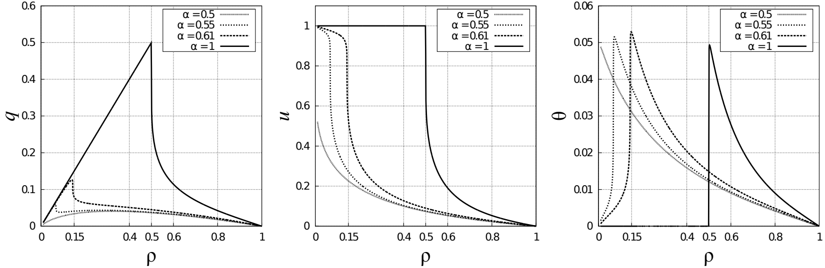
Case studies



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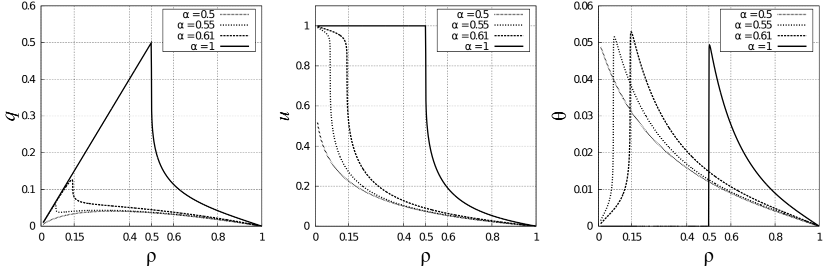

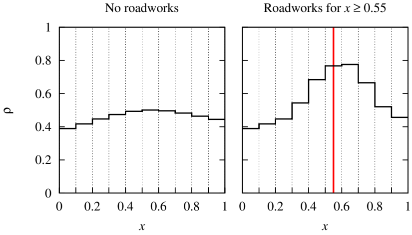

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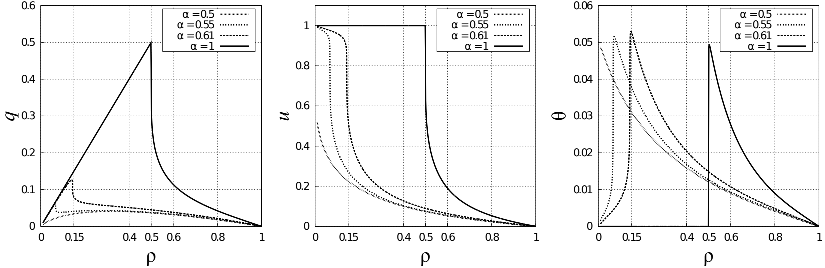

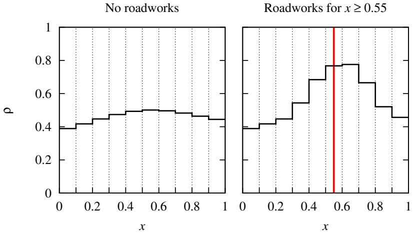

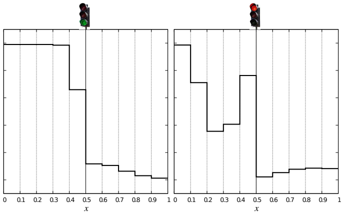

Theory: initial/boundary-value problem

$$\begin{cases} \dfrac{df_{ij}}{dt} + v_j\left(\Phi_{i,i+1}[\rho]f_{ij} - \Phi_{i-1,i}[\rho]f_{i-1,j}\right) = \eta(i)[\rho]\left(\displaystyle\sum_{h,k=1}^{n}\mathcal{A}^j_{hk}(i)[\rho]f_{ih}f_{ik} - f_{ij}\rho_i\right) \\ f_{ij}(0) = f^0_{ij} \in [0,\,1] \quad \forall\, i,\, j \\ f_{0j}(t) = \bar{f}_j(t) \in [0,\,1] \quad \forall\, j,\, \forall\, t \\ \Phi_{0,1} = \bar{\Phi}(t) \in [0,\,1] \quad \forall\, t \end{cases}$$

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Under suitable regularity assumptions on $\Phi_{i,i+1}$, $\eta(i)$, $\mathcal{A}^j_{hk}(i)$:

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$$\|\mathbf{f} - \mathbf{g}\|_\infty \leq \mathcal{C} \left[\|\mathbf{f}^0 - \mathbf{g}^0\|_1 + \int_0^{T_{\max}} \left(|\bar{\Phi}^f(t) - \bar{\Phi}^g(t)| + \|\bar{\mathbf{f}}(t) - \bar{\mathbf{g}}(t)\|_1 \right) dt \right]$$

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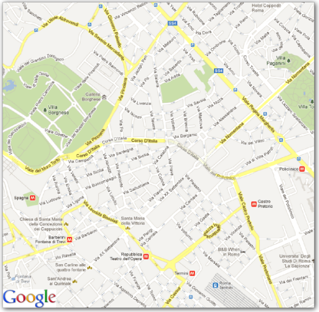
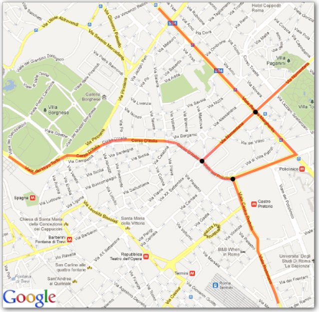
- Boundedness of the solution:

$$0 \leq f_{ij}(t) \leq 1, \quad \forall\, i,\, j,\, \forall\, t$$



- Road networks

Perspectives



- Road networks
- Main and minor roads with different refinements

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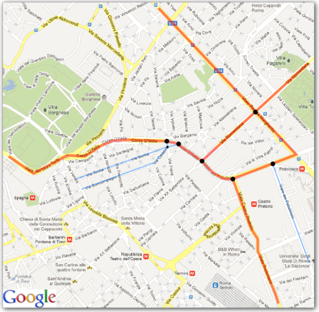

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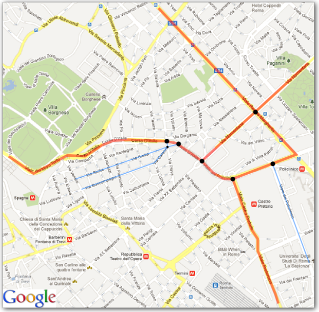

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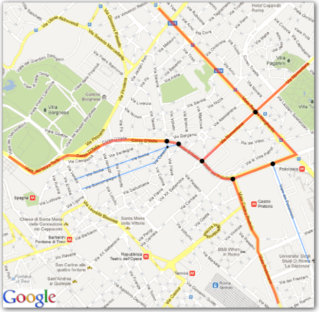

- Road networks
- Main and minor roads with different refinements
- Junctions handled just as easily as cell interfaces
- (Hopefully) easy to extend the theory of IBV problems to the whole network

References

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