Generic second order traffic flow models (GSOM).

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Outline of the presentation

1. Basics of: LWR and macroscopic intersection modeling
2. The invariance principle
3. Discussion of internal state intersection model
4. GSOM model
5. GSOM intersection modeling
6. GSOM lagrangian HJ and variational interpretation
7. Numerical solution schemes

Scope

- The GSOM model is close to the LWR model
- It is nearly as simple (non trivial explicit solutions fi)
- But it accounts for driver variability (attributes)
- More scope for lagrangian modeling, driver interaction, individual properties
- Admits a variational formulation
- Expected benefits: numerical schemes, data assimilation

The LWR model

The LWR model in a nutshell

- Introduced by Lighthill, Whitham (1955), Richards (1956)
- The equations:

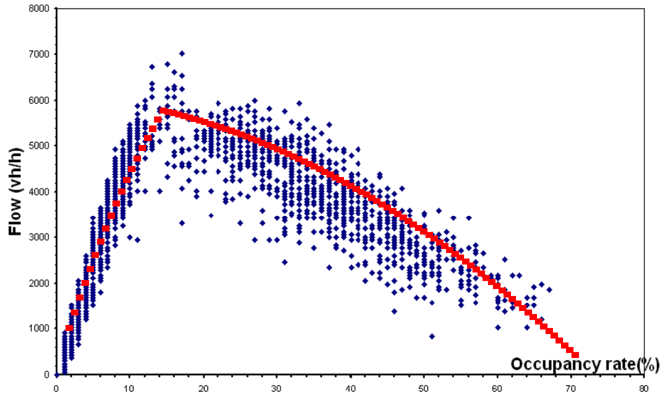
$$\frac{\partial \rho}{\partial t}+\frac{\partial q}{\partial x}=0 \quad \text{Conservation equation}$$
$$q=\rho v \quad \text{definition of } v$$
$$v=V_e(\rho,x) \quad \text{Behavioural equation}$$

- Or:

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial t}Q_e(\rho,x)=0$$

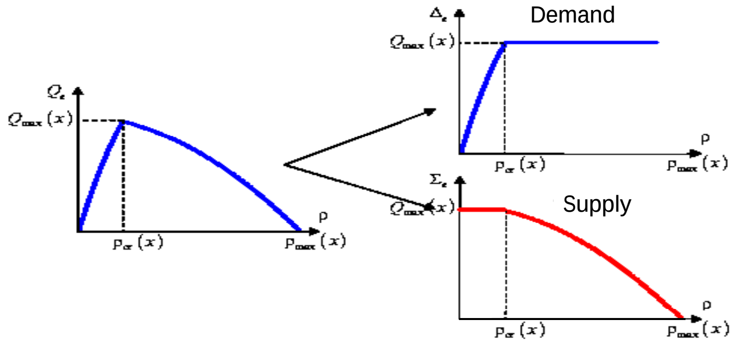
Example of FD (vs field data)

- (PQ model assumed true for the rest of the presentation)
- Field data: traffic from a highway south of Paris



The LWR model: supply / demand

- the *equilibrium supply* Σ_e and *demand* Δ_e functions (Lebacque, 1993-1996)



The LWR model: the min formula

- The local supply and demand:

$$\Sigma(x,t)=\Sigma_e\left(\rho(x+,t),x+\right)$$
$$\Delta(x,t)=\Delta_e\left(\rho(x-,t),x-\right)$$

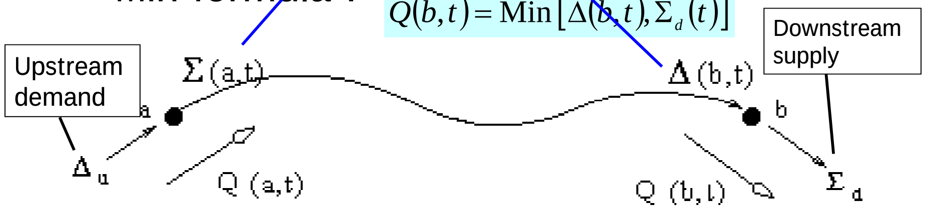
- The min formula

$$Q(x,t)=\mathrm{Min}\left[\Sigma(x,t),\Delta(x,t)\right]$$

- Usage: numerical schemes, boundary conditions → intersection modeling

Supply-demand boundary conditions

- Link supply : $\Sigma(a,t) = \Sigma_e(\rho(a+,t),a)$
- Link demand : $\Delta(b,t) = \Delta_e(\rho(b-,t),b)$
- Min formula : $Q(a,t) = \mathrm{Min}[\Delta_u(t),\Sigma(a,t)]$ $\quad Q(b,t) = \mathrm{Min}[\Delta(b,t),\Sigma_d(t)]$



Intersection modeling: basics

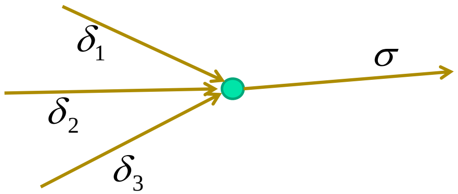
- A node model must combine:
  - Upstream link Boundary Conditions (entry)
  - Downstream link Boundary Conditions (exit)
  - Possibly: internal node dynamics
  - Vehicle conservation relations
  - Flow constraints (can be imposed)
- Some references: Holden-Risebro 1995, Lebacque-Khoshyaran 1998-2002-2005, Coclite-Piccoli 2002, Garavello-Piccoli 2005, Lebacque-Mammar-Haj-Salem 2006, Rascle-Herty 2006

LWR intersection modelling

Some intersection models

- SSMT
- STRADA
- Jin-Zhang
- Daganzo (diverge)
- Rascle (merge)

STRADA merge model (Buisson, Lebacque, Lesort 1993-1996)



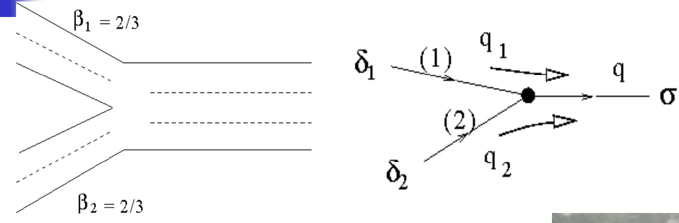
$$q_i = \frac{\min(\delta_i, \beta_i \sigma)}{\max\left[\sigma, \sum_\ell \min(\delta_\ell, \beta_\ell \sigma)\right]} \sigma$$

$$q = \min\left[\sigma, \sum_\ell \min(\delta_\ell, \beta_\ell \sigma)\right]$$

- Link outflow = Minimum (link demand, partial downstream supply)
- Total flow less than supply

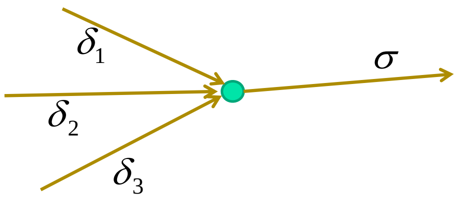
$\Sigma_e(N)$

The STRADA intersection model: the split coefficients



- The downstream link supply is split according to fixed ratios among upstream link demands

The Jin-Zhang fair merge model (Jin and Zhang 2002-2004)



$$q_i = \frac{\delta_i}{\sum_\ell \delta_\ell} \sigma$$

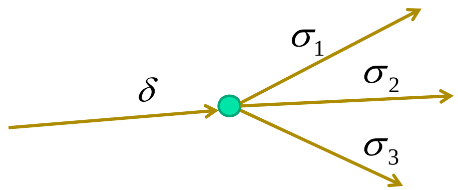
$$q = \min\left[\sigma, \sum_\ell \delta_\ell\right]$$

- Link outflow proportional to demand
- Total flow less than downstream supply

The Daganzo diverge model (Daganzo 1994-1995)

$$q = \min_i \left[\delta, \frac{\sigma_i}{\alpha_i}\right]$$

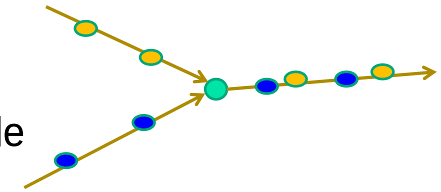
$$q_i = \min\left[\alpha_i \delta, \sigma_i\right]$$



- Partial outflow = min (partial demand, downstream supply)

Rascle model

- ARZ (Aw-Rascle-Zhang 2000-2002)
- Lagrangian discretization
- Fixed proportions of particles cross the intersection.
- Close to GSOM → GSOM compatible

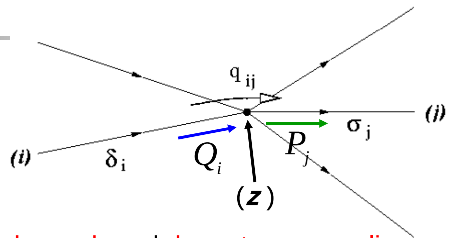


Intersection model building: the invariance principle

- Lebacque Khoshyaran 2005
- Not all models are consistent

LWR intersection modeling: point wise (Lebacque Khoshyaran 1998-2005)

$$
\begin{array}{ll}
0 \le Q_i \le \delta_i \overset{def}{=} \Delta_i(z-,t) & \forall i \\
0 \le P_j \le \sigma_j \overset{def}{=} \Sigma_j(z+,t) & \forall j \\
P_j - \sum_i \gamma_{ij} Q_i = 0 & \forall j
\end{array}
$$



- Flows are constrained by upstream demands and downstream supplies (+ flow conservation)
- Flows must respect assignment coefficients (constant directional fractions, information responsive user path choice etc...)
- More complicated intersection situations: Flows can be constrained by traffic lights, priority conflicts inside the intersection

Invariance principle for node models

- Node models $(Q,P) = f(\delta,\sigma)$ must satisfy the invariance principle (Lebacque-Khoshyaran 2005)
- They must be stable by the following transformation:

$$
\left[
\begin{array}{lll}
\delta_i \to Q_{i,max} & \text{if} & Q_i < \delta_i \\
\sigma_j \to P_{j,max} & \text{if} & P_j < \sigma_j
\end{array}
\right.
$$

- (if the node is congested the upstream links have maximum demand)
- (if the node is not congested the downstream links have maximum supply)

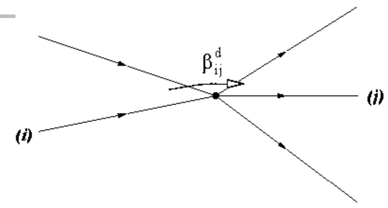
Meaning of these structural constraints

- Generalized Riemann problem for the intersection
  ⇒ supply and demand constraints
- Self-similarity + feasibility of solution ⇒ invariance principle
- The invariance principle guaranties
  - consistent models and convergent numerical schemes
  - that waves travel in the right directions in the vicinity of the intersection

An example of a node model satisfying the invariance principle: the optimization node model (Holden-Risebro 1995, Lebacque-Khoshyaran 2005)

$$Max \sum_i \Phi_i(Q_i) + \sum_j \Psi_j(R_j)$$

$$\left| \begin{array}{ll} Q_i \le \delta_i & \forall i \\ R_j \le \sigma_j & \forall j \\ \sum_i \gamma_{ij} Q_i - R_j = 0 & \forall j \end{array} \right.$$



- γ_{ij} : turning movement coefficients (deduced from the assignment coefficients)
- Constraints:
  - Node inflows less than upstream demands
  - Node outflows less than downstream supplies
  - Conservation of node out-flows

Optimization node model

- The Karush-Kuhn-Tucker optimality conditions yield (***s_j***) coefficient of the outflow (***j***) conservation equation)

$$\left| \begin{array}{l} Q_i = \mathrm{Min}\left[\delta_i, \Phi_i'^{-1}\left(-\sum_l \gamma_{il} s_l\right)\right] \\ R_j = \mathrm{Min}\left[\Psi_j'^{-1}(s_j), \sigma_j\right] \\ \sum_i \gamma_{ij} Q_i - R_j = 0 \quad \forall j \end{array} \right.$$

- The in- and out-flows are given by a Min-formula ⇒ The model satisfies the invariance principle

$$\left| \begin{array}{l} Q_i = \mathrm{Min}\left[\delta_i, \varphi_i\right] \\ R_j = \mathrm{Min}\left[\psi_j, \sigma_j\right] \end{array} \right.$$

Optimization node model (cont'd)

- Interpretation of the criterion:
  - Φ_i : → partial supply of node (for link (*i*))
  - Ψ_j : → partial demand of node (for link (*j*))
- Coefficients ***s_j*** : "node state"

$$\varphi_i \overset{def}{=} \Phi_i'^{-1}\left(-\sum_\ell \gamma_{i\ell} s_\ell\right)$$

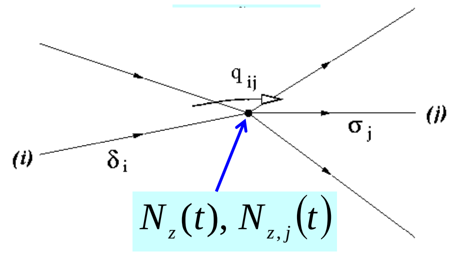
$$\psi_j \overset{def}{=} \Psi_j'^{-1}(s_j)$$

- Other models satisfy the invariance principle (dynamic pointwise, equilibrium)
- Daganzo merge model (1994-1995) is an optimisation model with quadratic node supply generators Φ_i

Internal state node modelling

Internal state node model
(Lebacque Khoshyaran 1998-2002-2005-2009)

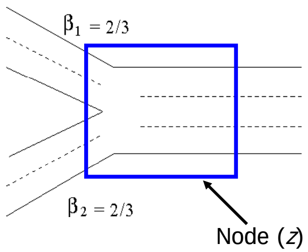
- The extension of the node is neglected (point wise intersection)
- The node is assumed to exhibit a global behavior: global supply and demand
- The dynamics inside of the intersection are modeled approximatively
- Node ⇔ "buffer" between upstream and downstream link Boundary Conditions



Internal state model: basic variables

- Total number of vehicles, number of vehicles per exit link (*j*)
- Total demand, total supply, which are a function of total number of vehicles inside the node
- In all cases (merges, diverges, T intersections) demand and supply are defined as usual

$$N_z(t), N_{z,j}(t)$$
$$\Delta_z(N(t)), \Sigma_z(N(t))$$

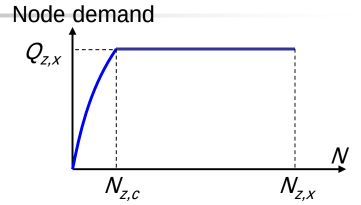


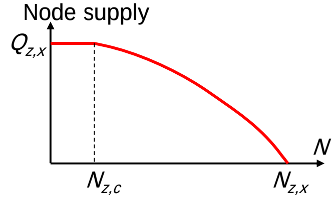
Node supply, demand

- Main parameters

$Q_{z,x}$: through-flow capacity
$N_{z,x}$: storage capacity
$N_{z,c}$: critical number of vhs

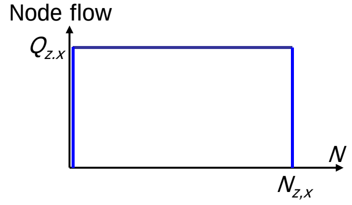
- Demand could be modified to include traffic hysteresis (capacity drop)



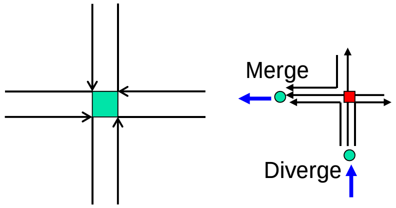


Node modeling

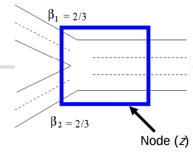
- Node modeling can be simpler
- Example (Herty Lebacque Moutari 2009): only through-flow and storage capacities are retained



- Classical four-branch intersection: 4 merges, four diverges, four conflict zones, can be represented as 8 internal nodes models



Internal state node model



- The node supply is split (constant split coefficient)

$$\Sigma_{z,i}(t) = \beta_i \Sigma_z(N(t))$$

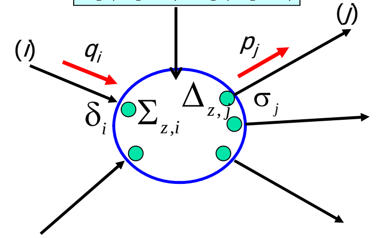
- The node demand is split (proportionally to directional demands)

$$\Delta_{z,j}(t) = \frac{N_{z,j}(t)}{N_z(t)} \Delta_z(N_z(t))$$

- Flows = Min(demand, supply)

$$q_i(t) = \text{Min}\left[\delta_i(t), \Sigma_{z,i}(t)\right]$$
$$p_j(t) = \text{Min}\left[\Delta_{z,j}(t), \sigma_j(t)\right]$$

$$N_z(t),\ N_{z,j}(t)$$
$$\Delta_z(N_z(t)),\ \Sigma_z(N_z(t))$$



Internal state node model: node state dynamics

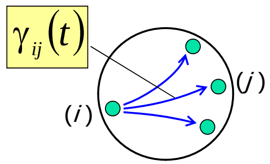
- Min formula for the flows: **the invariance principle is satisfied**
- Conservation equation for the numbers of vehicles stored inside the node
- The assignment coefficients (fraction of users entering from (i), bound to (j)) are one of the following:
  - Constant turning movement coefficients to be estimated (Kalman filter)
  - Dynamic assignment coefficients (user responsive or predictive)
  - Measured directly (GPS/cell phones)

$$q_i(t) = \text{Min}\left[\delta_i(t), \Sigma_{z,i}(t)\right] = \text{Min}\left[\delta_i(t), \beta_i \Sigma_z(N_z(t))\right]$$

$$p_j(t) = \text{Min}\left[\Delta_{z,j}(t), \sigma_j(t)\right] = \text{Min}\left[\frac{N_{z,j}(t)}{N_z(t)} \Delta_z(N_z(t)), \sigma_j(t)\right]$$

$$\frac{d\dot{N}_{z,j}}{dt}(t) = -p_j(t) + \sum_i \gamma_{ij}(t) q_i(t)$$

$$N_z(t) = \sum_j N_{z,j}(t)$$



Introduction of a simpler model: the **equilibrium model**

- The internal state node model requires the **resolution of an ODE** (ordinary differential equation).
- The **time-scale of the node model** is much shorter than the time scale of the link model (due to smaller node dimension)
- Typically the **discretization time-step** of the node equations will be one **order of magnitude less** than the time-step for the discretization of the links (Godunov scheme)
- For better computational times: search for a simpler internal state node model
- Idea: during a Godunov time-step for links **the boundary conditions** (upstream demands, downstream supplies) of the node **stay constant**

Comments

- The internal node model satisfies the invariance principle
- It requires the resolution of an ODE (ordinary differential equation)
- It can be approximated by an **equilibrium model** more easily solvable, which also satisfies the invariance principle
- It can be easily adapted to a variety of constraints and models
  - Traffic lights, conflict induced constraints
  - Capacity drop
  - GSOM (Generic second order modeling = LWR + dynamics of individual driver attributes)
  - Lagrangian setting
  - Stochastic GSOM models

Two other recent approaches related to the invariance principle

- Jin (2008-..): in order to overcome the constraints induced by the invariance principle, introduces a new solution concept: the **filmy states**
- Tampère, Flöterröd, Rohde, Osorio:
  - concentrate on maximizing through-flow + realistic constraints
  - target **complex intersections**

GSOM models

GSOM (Generic second order modelling) models

- (Lebacque, Mammar, Haj-Salem 2005-2007)
- In a nutshell
  - Kinematic waves = LWR
  - Driver attribute dynamics
- Includes many current macroscopic models

GSOM Family: description

- Conservation of vehicles (density) $$\partial_t \rho + \partial_x (\rho v) = 0$$
- Variable fundamental diagram, dependent on a driver attribute (possibly a vecteur) I $$v = \Im(\rho, I)$$
- Equation of evolution for I following vehicle trajectories (example: relaxation) $$\dot{I} = \partial_t I + v \partial_x I = -f(I)$$

GSOM: basic equations

Conservation des véhicules $$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v}{\partial x} = 0$$

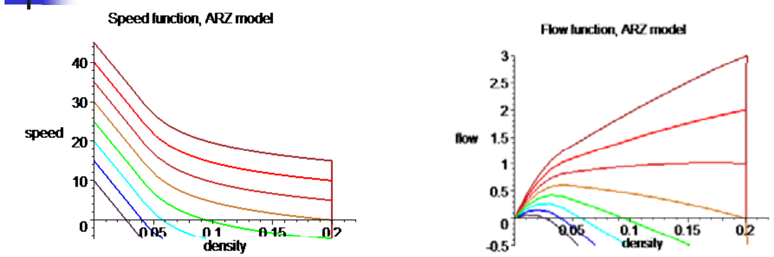
Dynamics of I along trajectories $$\frac{\partial \rho I}{\partial t} + \frac{\partial \rho I v}{\partial x} = -\rho f(I)$$ $$\Leftrightarrow \quad \dot{I} = \partial_t I + v \partial_x I = -f(I)$$

Variable fundamental Diagram $$v = \Im(\rho, I) \quad \Rightarrow \quad \rho v \overset{def}{=} \Re(\rho, I)$$

Example 0: LWR (Lighthill, Whitham, Richards 1955, 1956)

- No driver attribute $$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v}{\partial x} = 0 \quad \text{Equation de conservation}$$ $$v = V_e(\rho, x) \quad \text{Diagramme fondamental}$$
- One conservation equation $$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial t} Q_e(\rho, x) = 0$$

Example 1: ARZ (Aw,Rascle 2000, Zhang,2002)



- Lagrangien attribute I = difference between actual and mean equilibrium speed

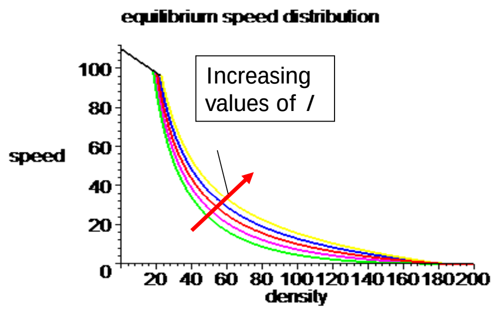
$$I = v - V_e(\rho) \quad \Leftrightarrow \quad v = \Im(\rho, I) = I + V_e(\rho)$$

Example 2: 1-phase Colombo model (Colombo 2002, Lebacque Mammar Haj-Salem 2007)

- **Variable FD** (in the congested domain + critical density)

$$\Im(\rho, I) = \left(I + \frac{q_*}{\rho} \right) v_0(\rho)$$

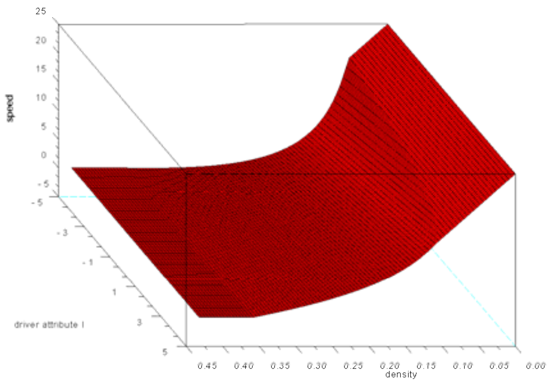
$$v_0(\rho) = 1 - \frac{\rho}{\rho_{max}}$$



- The attribute *I* is the parameter of the family of FDs

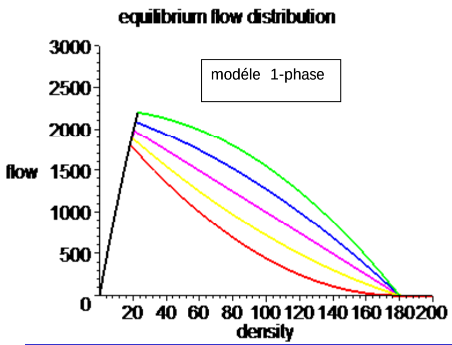
$$v = \Im(\rho, I) \overset{def}{=} \begin{cases} V_{max} - \dfrac{V_{max} - V_{crit}}{\rho_{crit}(I)}\rho & \text{if} \quad \rho \le \rho_{crit}(I) \\ \left(I + \dfrac{q_*}{\rho} \right)\left(1 - \dfrac{\rho}{\rho_{max}} \right) & \text{if} \quad \rho \ge \rho_{crit}(I) \end{cases}$$

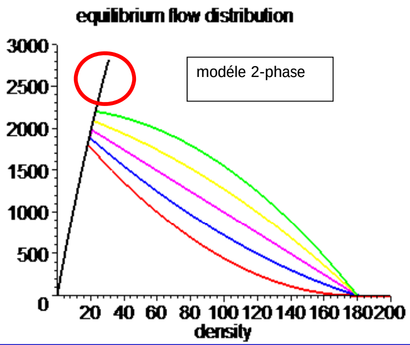
- Fundamental Diagram (speed-density)



Example 2 continued (1-phase Colombo model)

- **1-phase vs 2-phase**: Flow-density FD

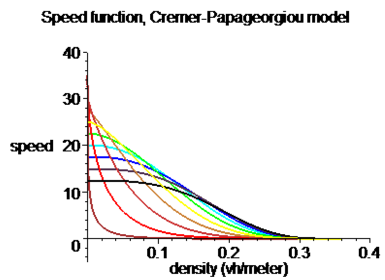


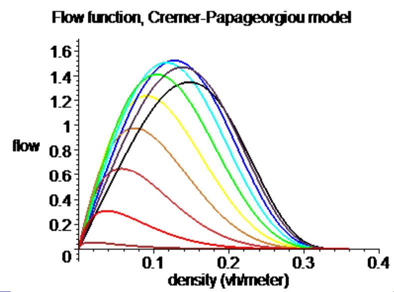


Example 3: Cremer-Papageorgiou

- Based on the Cremer-Papageorgiou FD (Haj-Salem 2007)

$$\Im(\rho, I) = V_f I \left(1 - \left(\frac{\rho}{\rho_{max}} \right)^{I(3-2I)} \right)^m$$

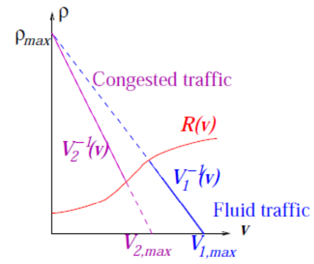




Example 4: « multi-commodity » models

- GSOM Model $$\partial_t I + v\partial_x I = \varphi(I)$$
- + advection (destinations, vehicle type) $$\partial_t \lambda^d + v\partial_x \lambda^d = 0$$
- = multi-commodity GSOM $$\partial_t \mathcal{I} + v\partial_x \mathcal{I} = \begin{pmatrix} \varphi(I) \\ 0 \end{pmatrix}$$

Example 5: multi-lane model



- Impact of multi-lane traffic
- Two states: congestion (strongly correlated lanes) et fluid (weakly correlated lanes)
  - → 2 FDs separated by the phase boundary $R(v)$
- Relaxation towards each regime $$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0$$
- → eulerian source terms

$$\frac{d\alpha}{dt} = \dot{\alpha} \overset{def}{=} \frac{\partial \alpha}{\partial t} + v\frac{\partial \alpha}{\partial x} = \begin{cases} -\alpha/\epsilon & \text{if} \quad \rho \le \mathcal{R}(v) \\ (V_2(\rho) - V_1(\rho) - \alpha)/\epsilon & \text{if} \quad \rho \ge \mathcal{R}(v) \end{cases}$$

Example 6: Stochastic GSOM (Khoshyaran Lebacque 2007-2008)

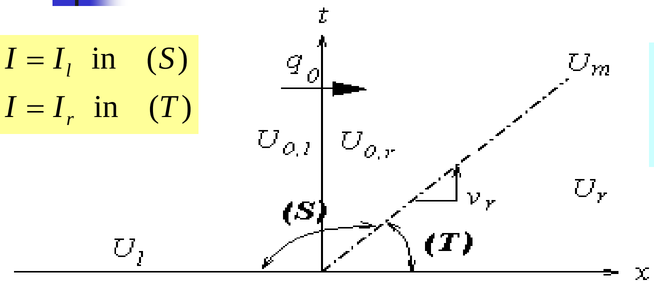
- Idea:
  - Conservation of véhicles $$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v}{\partial x} = 0$$
  - Fundamental Diagram depends on driver attribute *I* $$v = \Im(\rho, I) \quad \Rightarrow \quad \rho v \overset{def}{=} \Re(\rho, I)$$
  - *I* is submitted to stochastic perturbations (other vehicles, traffic conditions, environment) $$\dot{I} = \Phi\left(I, \frac{dB_t}{dt}\right)$$ $$\Rightarrow I = \Xi(N, t; \omega)$$

Two fundamental properties of the GSOM family (homogeneous piecewise constant case)

- **1.** discontinuities of *I* propagate with the speed *v* of traffic flow
- **2.** If the invariant *I* is initially piecewise constant, it stays so for all times $t > 0$
- ⇒ On any domain on which *I* is uniform the GSOM model simplifies to a translated LWR model (piecewise LWR) $$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\Re(\rho, I) = 0$$

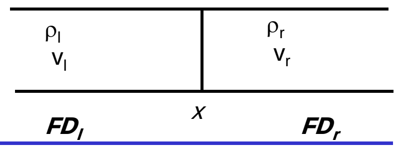
Riemann problem

$I = I_l$ in (S)
$I = I_r$ in (T)



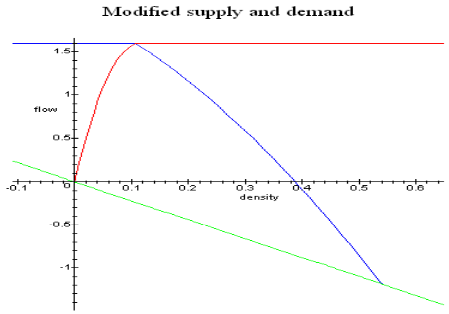
$v_m = \Im^r(\rho_m, I_l) = v_r$
$\Rightarrow \rho_m = \Im_\rho^{r\,-1}(v_r, I_l)$

$U_l \to U_m$: onde LWR
$U_m \to U_r$: discontinuité I



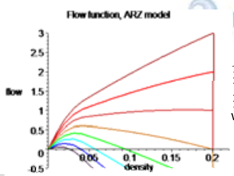
Generalized (translated) supply- Demand (Lebacque Mammar Haj-Salem 2005-2007)

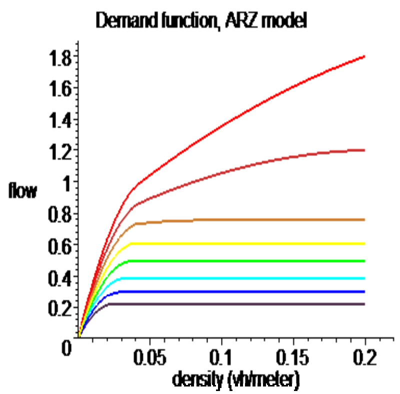
- Translated Supply / Demand = supply resp demand for the « translated » FD (with resp to I)

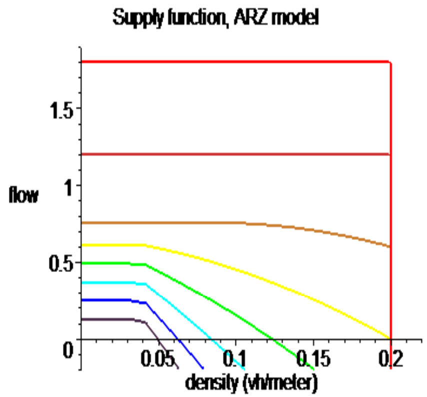


$$\begin{cases} \Delta(\rho, I) \overset{def}{=} \mathrm{Max}_{0 \le \varsigma \le \rho} \Re(\varsigma, I) \\ \Sigma(\rho, I) \overset{def}{=} \mathrm{Max}_{\varsigma \ge \rho} \Re(\varsigma, I) \end{cases}$$

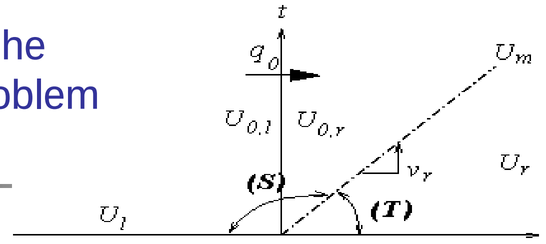
Example of translated supply / demand: the ARZ family





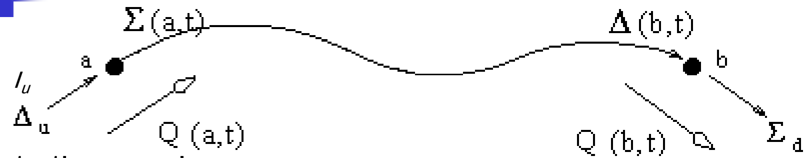


Solution of the Riemann problem (summary)



- Define the upstream demand, the downstream supply offre (which depend on I_l):
  $\delta_l \overset{def}{=} \Delta_{e,l}(\rho_l, I_l), \quad \sigma_r \overset{def}{=} \Sigma_{e,r}(\rho_m, I_l)$
- The intermediate state U_m is given by
  $I_m = I_l$
  $v_m = v_r \quad i.e. \quad \Im_r(\rho_m, I_l) = v_r = \Im_r(\rho_r, I_r)$
- The upstream demand and the downstream supply (as functions of initial conditions) :
  $$\begin{cases} \delta_l \overset{def}{=} \Delta_l(\rho_l, I_l) \\ \sigma_r \overset{def}{=} \Sigma_r(\rho_m, I_l) = \Sigma_r(\Im^{-1}_{r,\rho}(v_r, I_l), I_l) = \Sigma_r(\Im^{-1}_{r,\rho}(\Im_r(\rho_r, I_r), I_l), I_l) \end{cases}$$
- Min Formula:
  $$q_0 = \mathrm{Min}[\delta_l, \sigma_r]$$

Boundary Conditions (revisited for GSOM)



- Note the complex dependancy with respect to driver attributes
- Note: downstream boundary conditions can possibly be expressed in terms of velocity

$$\Sigma(a,t) = \Sigma_e(\rho(a,t), I(a,t), I_u(t); a)$$
$$Q(a,t) = \text{Min}\,[\Delta_u(t), \Sigma(a,t)]$$

$$\Delta(b,t) = \Delta_e(\rho(b,t), I(b,t); b)$$
$$\Sigma_d(t) = \Sigma_e(\rho(b,t), I_d(t), I(b,t); b)$$
$$Q(b,t) = \text{Min}\,[\Delta(b,t), \Sigma(a,t)]$$

GSOM intersection modelling

Intersection modeling for GSOM models

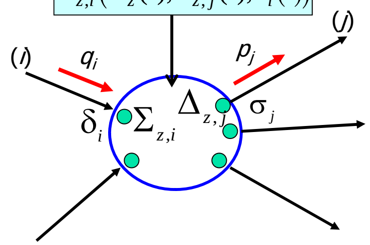
- Supply and demand are well-defined, the min formula appies
- The invariance principle applies
- Optimization models satisfy the invariance principle
- But they are hardly tractable
  - The downstream supply depend on the upstream attribute values
  - Exception: pure assignment problems (FD does not depend on the attributes = destination fi)

Internal node models for GSOM

- Conservation equations in the node:

$$N_z = \sum_i q_i I_i - \sum_j p_j L_z$$
$$N_z \dot{L}_z = \sum_i q_i (I_i - L_z)$$
$$N_{z,j} = \sum_i q_i \gamma_{ij} - p_j$$
$$N_{z,j} \dot{L}_{z,j} = \sum_i q_i \gamma_{ij} (I_i - L_{z,j})$$

$$N_z(t), N_{z,j}(t)$$
$$L_z(t), L_{z,j}(t)$$
$$\Delta_{z,j}(N_{z,j}(t), L_{z,j}(t)),$$
$$\Sigma_{z,i}(N_z(t), L_{z,j}(t), I_i(t))$$

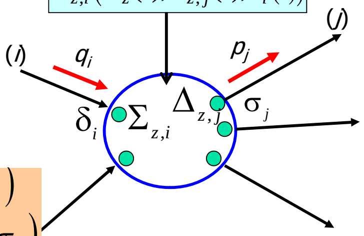


Internal node models for GSOM (continued)

- Supplies, demands:

$$\delta_i = \Delta_i(\rho_i, I_i)$$
$$\sigma_j = \Sigma_j(\rho_j, J_j, L_Z)$$
$$\Delta_{Z,j} = \Delta_{Z,j}(N_{Z,j}, L_{Z,j})$$
$$\Sigma_{Z,i} = \Sigma_{Z,i}(N_Z, L_Z, I_i)$$

$$N_z(t), N_{z,j}(t)$$
$$L_z(t), L_{z,j}(t)$$
$$\Delta_{z,j}(N_{z,j}(t), L_{z,j}(t)),$$
$$\Sigma_{z,i}(N_z(t), L_{z,j}(t), I_i(t))$$



- In / Out flows:

$$q_i = \min(\delta_i, \Sigma_{Z,i})$$
$$p_j = \min(\Delta_{Z,j}, \sigma_j)$$

Comments

- The internal state node acts as a buffer → invariance principle is satisfied
- Behavioral complexity ← assignment (interacts with driver attributes)
- Possibility to construct simpler models based on N_Z instead $N_{Z,j}$ (as LWR)
- Equilibrium models

Lagrangian GSOM; HJ and variational interpretation

Variational formulation of GSOM models

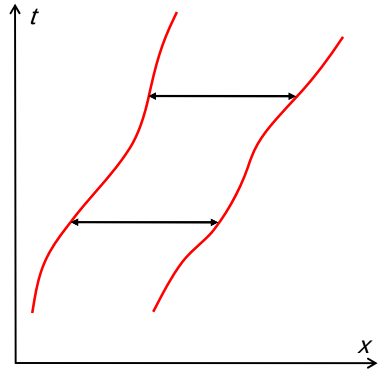
- Motivation
  - Numerical schemes (grid-free cf Mazaré et al 2011)
  - Data assimilation (floating vehicle / mobile data cf Claudel Bayen 2010)
  - Advantages of variational principles

Lagrangian conservation law

- Spacing r
- The rate of variation of spacing r depends on the gradient of speed with respect to vehicle index N

$$\partial_t r + \partial_N v = 0$$

$$v = V_e(1/r) \overset{def}{=} V(r)$$



The spacing is the inverse of density

Lagrangian version of the GSOM model

- Conservation law in lagrangian coordinates
- Driver attribute equation (natural lagrangian expression)

$$\begin{cases} \partial_t r + \partial_N v = 0 & \text{Conservation of vehicles} \\ \partial_t I = \varphi(N, I, t) & \text{Dynamics of the driver attribute } I \\ v = \aleph(r, I) \overset{def}{=} \Im(1/r, I) & \text{Fundamental diagram (FD)} \end{cases}$$

- We introduce position of vehicle N: $X(N,t) \overset{def}{=} \int_{-\infty}^{t} v(N,\tau)d\tau$

- Note that: $v = \partial_t X$, $r = -\partial_N X$

Lagrangian Hamilton-Jacobi formulation of GSOM (Lebacque Khoshyaran 2013)

- Integrate the driver-attribute equation

$$\partial_t I(N,t) = \varphi(N, I, t)$$
$$I(N,0) = i_0(N) \quad \forall N$$

- Solution: $J(N, t, i_0)$

- FD Speed becomes a function of driver, time and spacing $\mathcal{W}(N, r, t) \overset{def}{=} \aleph(r, J(N, t, i_0))$

Lagrangian Hamilton-Jacobi formulation of GSOM

- Expressing the velocity v as a function of the position X

$$\mathcal{W}(N, r, t) \overset{def}{=} \aleph(r, J(N, t, i_0))$$

$$v = \partial_t X$$
$$r = -\partial_N X$$

$$\partial_t X = \mathcal{W}(N, -\partial_N X, t)$$

Associated optimization problem

- Define:

$$\mathcal{M}(N,u,t) = \max_r \left(\mathcal{W}(N,r,t) - u.r\right)$$

$$\mathcal{W}(N,r,t) = \min_u \left(\mathcal{M}(N,u,t) + u.r\right)$$

- Note implication: W concave with respect to r (ie flow density FD concave with respect to density
- Associated optimization problem

$$\mathcal{X}(N_T,T) = \min_{u(.),(N_0,t_0)} \int_{t_0}^{T} \mathcal{M}(N,u,t)\,dt + \xi(N_0,t_0)$$
$$\left|\begin{array}{l} \dot{N} = u \\ N(t_0) = N_0,\ N(T) = N_T \\ (N_0,t_0) \in \mathcal{C} \end{array}\right.$$

Functions M and W

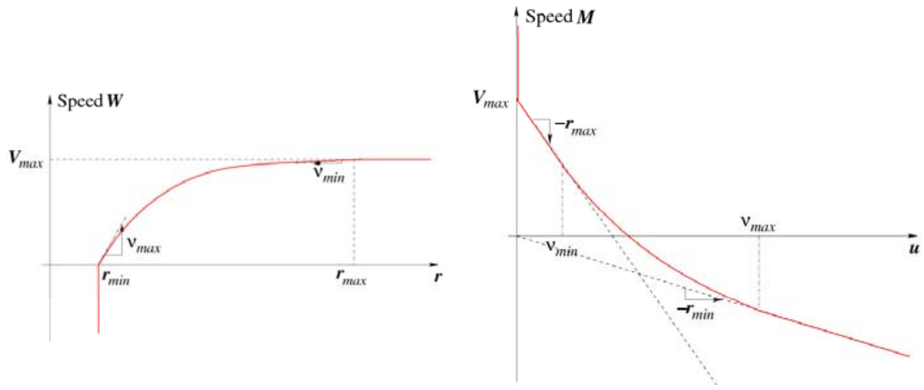
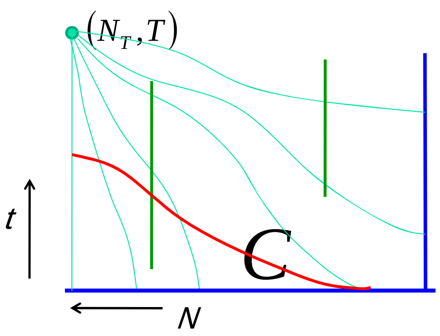


Illustration of the optimization problem:

- Initial/boundary conditions:
  - Blue: IC + trajectory of first vh
  - Green: trajectories of vhs with GPS
  - Red: cumulative flow on fixed detector



Elements of resolution

Characteristics

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- Optimal curves → characteristics (Pontryagin)

$$\begin{cases} \dot N = \partial_r \mathcal{W}(N,r,t) \\ \dot r = -\partial_N \mathcal{W}(N,r,t) \end{cases}$$

- Note: speed of charcteristics: > 0
- Boundary conditions $(\partial_N \xi + r_0, \partial_t \xi - \mathcal{W}_0) \in N_{(x_0,t_0)}C$

Initial conditions for characteristics

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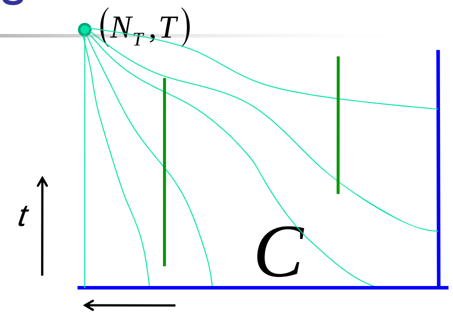
Data $\xi(N, t_0)$

Data $\xi(N_0, t)$



- IC

$$r_0(N) = -\partial_N \xi(N, t_0).$$

- Vh trajectory

$$r_0(t) \overset{def}{=} \mathcal{W}^{-1}(N_0, \xi(N_0, t), t).$$

Initial / boundary condition

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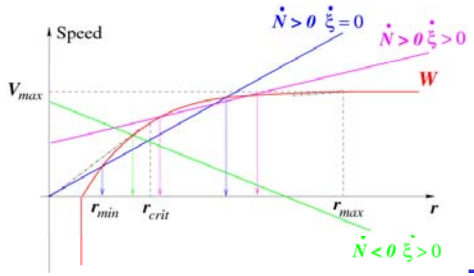
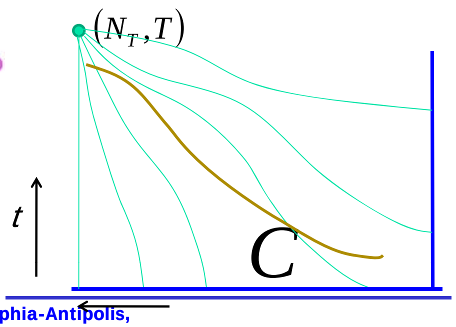
- Usually two solutions $r_0(t)$

Data $\xi(N(t), t)$

$$\frac{d}{dt}\xi(N(t), t) = -\dot N(t) r_0(t) + \mathcal{W}(N(t), r_0(t), t)$$





Example: Interaction of a shockwave with a contact discontinuity Eulerian view

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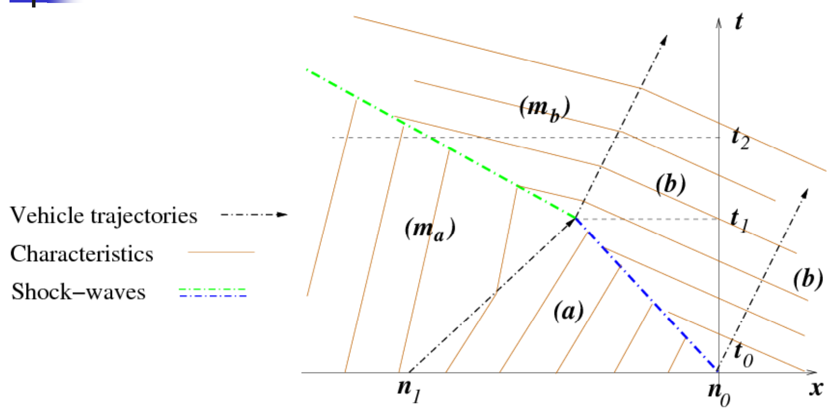
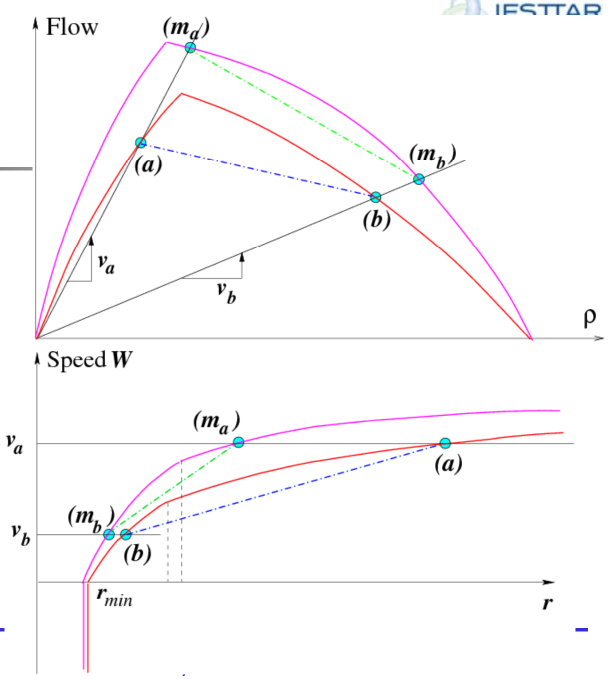
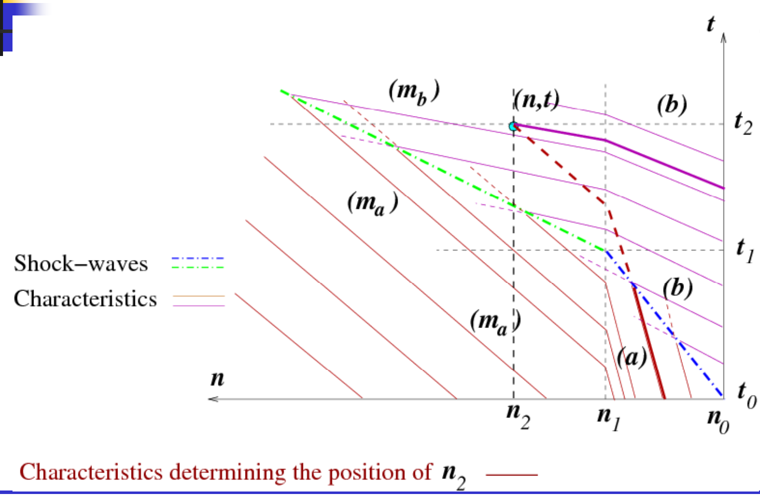


- Initial conditions:
  - Top: eulerian
  - Down: lagrangian



Example: Interaction of a shockwave with a contact discontinuity Lagrangian view



Decomposition property (inf-morphism)

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- The set of initial/boundary conditions is union of several sets

$$C = \bigcup_{\ell \in \Lambda} C_\ell$$

- Calculate a partial solution (corresponding to a partial set of IBC)

$$X_\ell(N_T,T) = \min_{u_\ell(\cdot),(N_0,t_0)} \int_{t_0}^{T} M(N_\ell,u_\ell,t)\,dt + \xi(N_0,t_0) \quad \left| \begin{array}{l} \dot N_\ell(t) = u_\ell(t) \\ N_\ell(t_0) = N_0,\ N_\ell(T) = N_T \\ (N_0,t_0) \in C_\ell \end{array} \right.$$

- The solution is the min of partial solutions

$$X = \min_{\ell \in \Lambda} X_\ell$$

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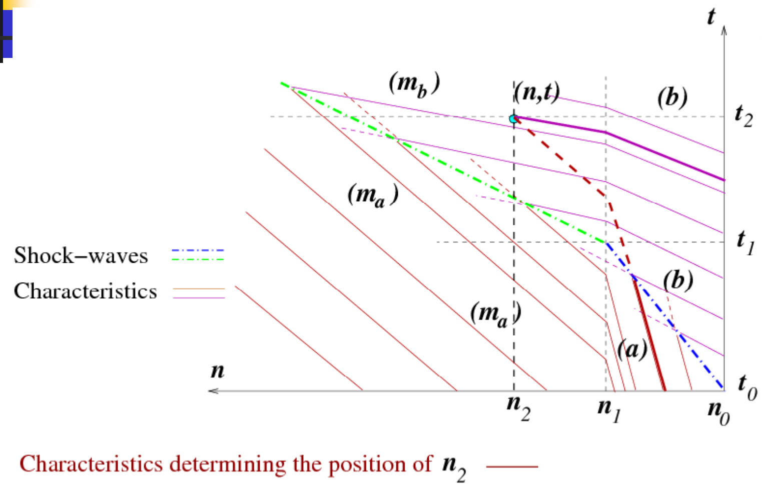

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The optimality pb can be solved on characteristics only

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$$\begin{aligned} X(N_T, T) \quad = \quad & \min_{(N_0, r_0, t_0)} \int_{t_0}^{T} \mathcal{M}\left(N, \partial_r \mathcal{W}(N, r, t), t\right) dt + \xi(N_0, t_0) \\ & \left| \begin{array}{l} N(t) = \partial_r \mathcal{W}(N, r, t) \\ \dot{r}(t) = -\partial_N \mathcal{W}(N, r, t) \\ N(t_0) = N_0, r(t_0) = r_0, N(T) = N_T \\ (N_0, r_0, t_0) \in \mathcal{K} \end{array} \right. \end{aligned}$$

- This is a Lax-Hopf like formula
- Application: numerical schemes based on
  - Piecewise constant data (including the system yielding *I*)
  - Decomposition of solutions based on decomposition of IBC
  - Use characteristics to calculate partial solutions

Numerical scheme based on characteristics (continued)

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- If the initial condition on I is piecewise constant →
- the spacing along characteristics is piecewise constant
- Principle illustrated by the example: interaction between shockwave and contact discontinuity

Alternate scheme: particle discretization

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- Particle discretization of HJ
- Use charateristics
- Yields a Godunov-like scheme (in lagrangian coordinates)
- BC: Upstream demand and downstream supply conditions

Numerical example

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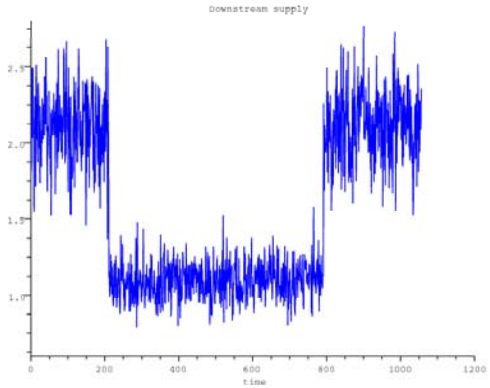
- **Model**: Colombo 1-phase, stochastic
- **Process for *I***: Ornstein-Uhlenbeck, two levels (high at the beginning and end, low otherwise) → refraction of charateristics and waves

- **Demand**: Poisson, constant level
- **Supply**: high at the beginning and end, low otherwise → induces backwards propagation of congestion

- **Particles**: 5 vehicles
- **Duration**: 20 mn
- **Length**: 3500 m

Downstream supply



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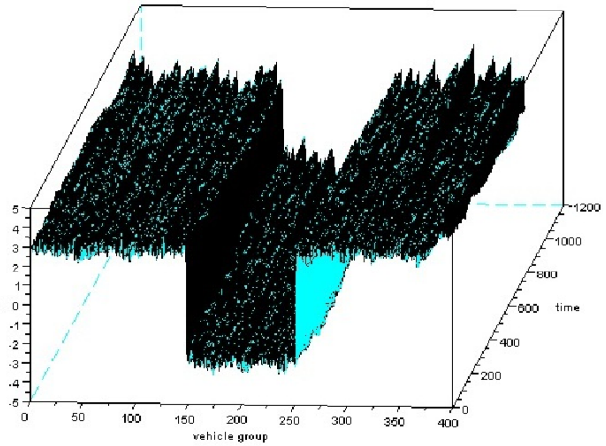

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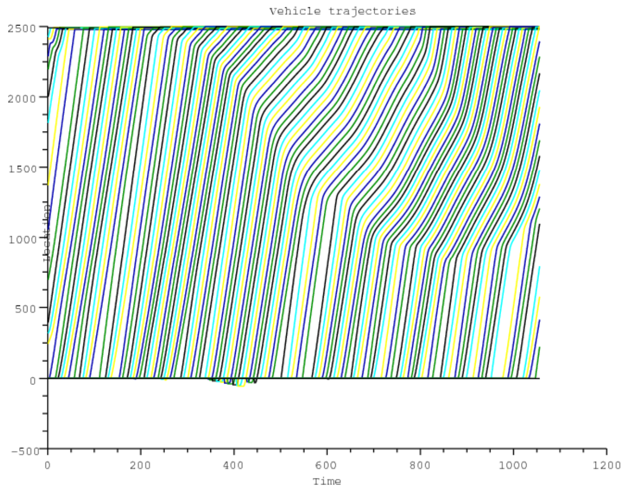
Particle trajectories



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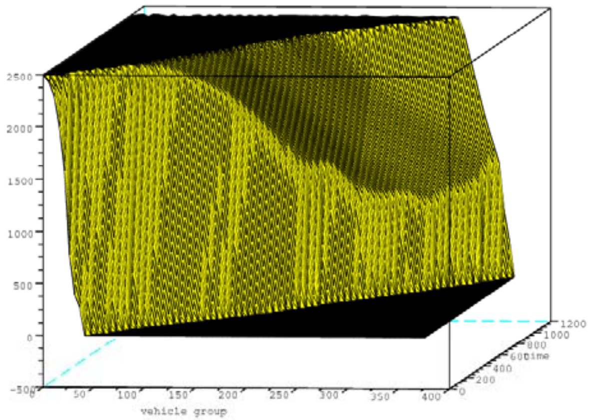
Position of particles



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Conclusion

- Directions for future work:
  - Problem of concavity of FD
  - Eulerian source terms
  - Data assimilation pbs
  - Efficient numerical schemes