On the Micro-Macro Limit in Traffic Flow

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Joint work with Rinaldo M. Colombo

Traffic

Continuum Macroscopic Model

Discrete Microscopic Model

Traffic

Continuum Macroscopic Model

- Density of vehicles.
- Partial Differential Equation (PDE).

Discrete Microscopic Model

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- Positions of n vehicles.
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Analysis | Limit | Numerics

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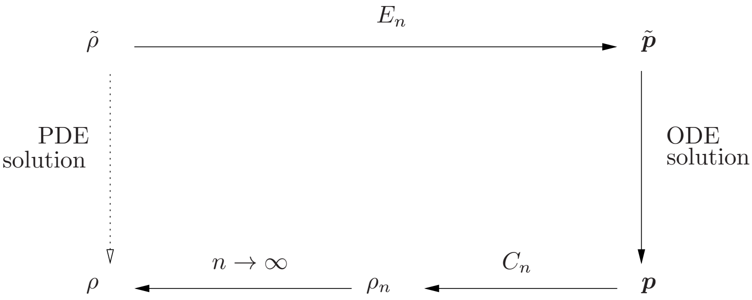

Continuum Macroscopic Model

$$\rho = \rho(t, x)$$

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Conservation Law

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Continuum Speed Law

Continuum Macroscopic Model

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Conservation Law $\Downarrow$

Continuum Speed Law $\Downarrow$

LWR model [Lighthill, Whitham, 1955, and Richards, 1956]

$$\begin{cases} \partial_t \rho + \partial_x(\rho v(\rho)) = 0 \\ \rho(0, x) = \tilde{\rho}(x) \end{cases}$$

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$$\begin{cases} \partial_t \rho + \partial_x (\rho v(\rho)) = 0 \\ \rho(0, x) = \tilde{\rho}(x) \end{cases}$$

$$R_m = \left\{ \rho \in \mathbf{L}^1(\mathbb{R}, [0,1]) : \int_{\mathbb{R}} \rho(x)\, dx = m > 0 \text{ and } \operatorname{spt} \rho \text{ is compact} \right\}$$

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First order **Follow-the-Leader** model

$$\begin{cases} \dot{p}_i = w(p_{i+1} - p_i) & i = 1, \ldots, n \\ \dot{p}_{n+1} = V(t) & \\ p_i(0) = \tilde{p}_i & i = 1, \ldots, n+1 \end{cases}$$

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$$P_n = \left\{(p_1, \ldots, p_{n+1}) \in \mathbb{R}^{n+1} : p_{i+1} - p_i \geq \ell_n \ \forall i = 1, \ldots, n\right\}$$

Well-Posedness

Assume the following conditions on v:

- $v \in \mathbf{C}^{0,1}([0,1],\mathbb{R})$
- $v'(\rho) \leq 0$ for a.e. $\rho \in [0,1]$
- $v(1) = 0$
- $v(0) = v_{\max}$ for a suitable positive $v_{\max}$

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Proposition (Continuum)

For any initial datum $\tilde{\rho} \in R_m \cap \mathbf{BV}(\mathbb{R};[0,1])$, there exists a unique solution $\rho \in \mathbf{C}^{0,1}(\mathbb{R}^+, R_m)$ to

$$\begin{cases} \partial_t \rho + \partial_x(\rho v(\rho)) = 0 \\ \rho(0,x) = \tilde{\rho}(x) \end{cases}$$

Well-Posedness

Assume the following conditions on w:

- $w \in \mathbf{C}^{0,1}([\ell_n, +\infty[, \mathbb{R})$
- $w'(\delta) \geq 0$ for a.e. $\delta \geq \ell_n$
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Proposition (Discrete)

For any initial datum $\tilde{\mathbf{p}} \in P_n$, there exists a unique solution $\mathbf{p} \in \mathbf{C}^{0,1}(\mathbb{R}^+, P_n)$ to

$$\begin{cases} \dot{p}_i = w(p_{i+1} - p_i) & i = 1, \ldots, n \\ \dot{p}_{n+1} = V & \\ p_i(0) = \tilde{p}_i & i = 1, \ldots, n+1 \end{cases}$$

Micro–Macro Connection

Continuum and discrete descriptions are related through **particle paths**:

$$\begin{cases} \dot{p} = v\left(\rho\left(t, p\left(t\right)\right)\right) \\ p\left(0\right) = \tilde{p} \end{cases}$$

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$\Longrightarrow$ Connection:

each equation in the discrete model is a particle path.

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$$\mathbf{w(\delta) = v\left(\ell_n/\delta\right)}$$

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v satisfies the conditions for the continuum model $\iff$ w satisfies the conditions for the discrete model

Two Useful Operators: C_n and E_n

Discretization $\quad \mathbf{E_n} \quad R_m \to P_n$

Continuum $\to$ Discrete

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Anti-discretization $\mathbf{C_n}$ $\quad P_n \to R_m$

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Discretization $\mathbf{E_n}$ $R_m \to P_n$

Continuum → Discrete

Anti-discretization $\mathbf{C_n}$ $P_n \to R_m$

Discrete → Continuum

$$E_n \circ C_n = \mathrm{Id}$$

$$\lim_{n \to +\infty} C_n \circ E_n = \mathrm{Id}$$

Micro-Macro Limit

As the number of vehicles grows to infinity:

$$\text{Discrete-model} \overset{n\to+\infty}{\longrightarrow} \text{Continuum-model}$$

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Theorem

Fix $T > 0$. Choose $\tilde{\rho} \in R_m \cap \mathbf{BV}(\mathbb{R}, [0,1])$.

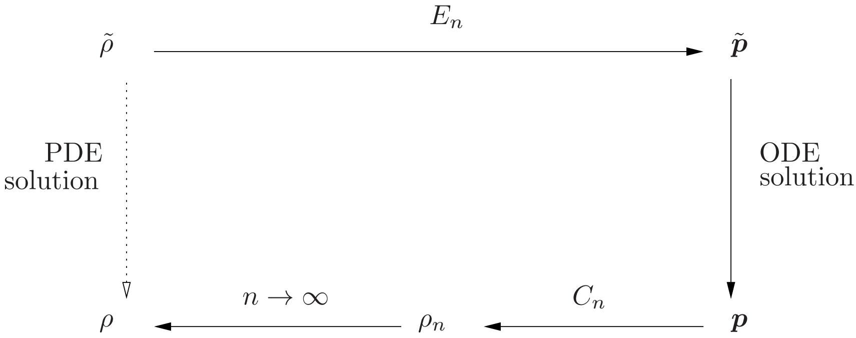
Set $\tilde{\mathbf{p}} = E_n(\tilde{\rho})$ and let $\mathbf{p}(t)$ be the corresponding discrete–solution.

Define $\rho_n(t) = C_n \mathbf{p}(t)$.

If there exists $\rho \in \mathbf{L}^\infty\left([0,T], R_m)\right)$ such that $\lim_{n\to+\infty} \rho_n = \rho$ a.e., then ρ is a continuum–solution with initial datum $\tilde{\rho}$.

Micro-Macro Limit

The Theorem above states that the following diagram commutes:



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$\tilde{\rho}$

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$$\tilde{\rho} \xrightarrow{\quad E_n \quad} \tilde{\boldsymbol{p}}$$

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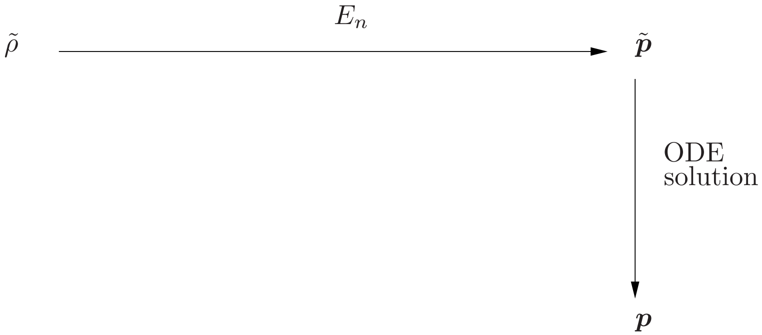

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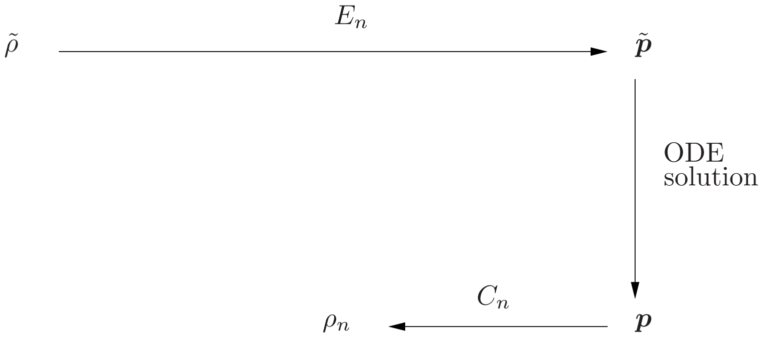

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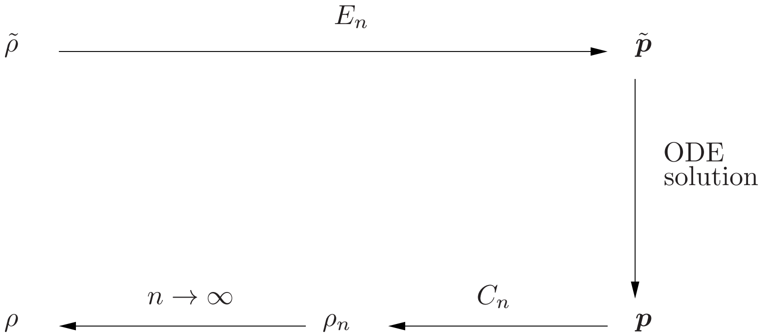

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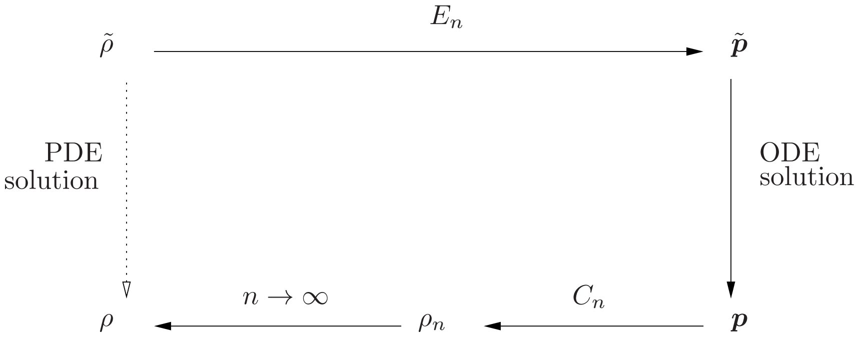
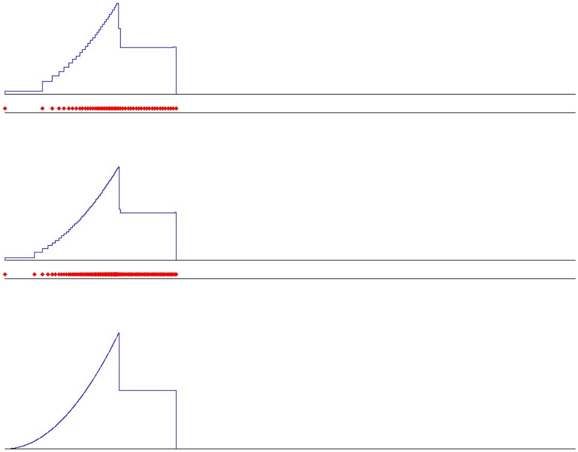


Example



Numerical Integrations

Numerical algorithm to integrate the continuum and discrete models:

- Hyperbolic Conservation Law
- System of ODEs

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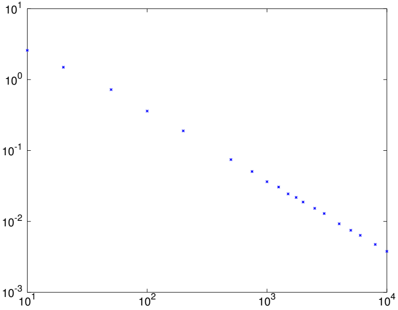
ODEs method to integrate PDEs.

In the numerical integrations we closely follow the diagram seen before.

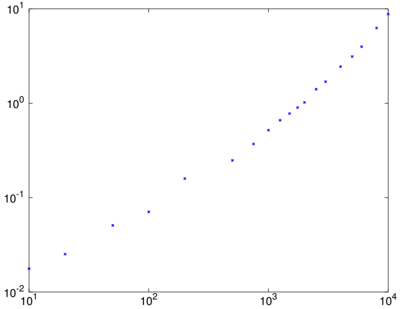
ODEs to integrate PDEs

Shock Wave

Relative error in the $\mathbf{L}^1$-norm vs. n



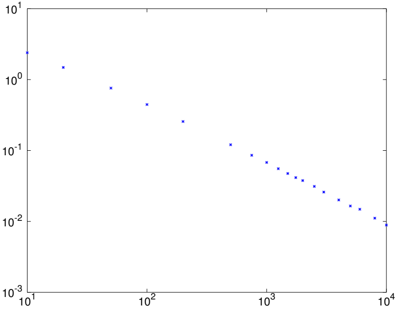
Integration time vs. n



ODEs to integrate PDEs

Rarefaction Wave

Relative error in the $\mathbf{L}^1$-norm vs. n



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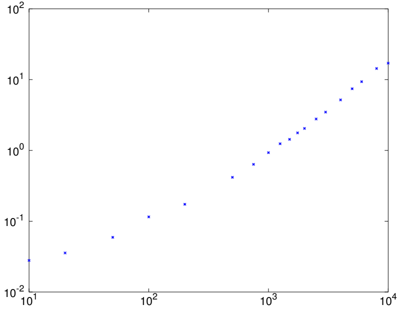

Same Execution Time

Space mesh size Δx	PDE		ODE		Number of vehicles n
	Error	Execution Time		Error	
0,02	1,32E-2	0,06864	0,06262	1,51E-1	20
0,01	6,73E-3	0,1585	0,1910	4,23E-2	100
0,0064	4,22E-3	0,2737	0,2930	2,87E-2	150
0,005	3,23E-3	0,3963	0,4242	2,17E-2	200
0,0045	2,76E-3	0,4558	0,4877	1,66E-2	225
0,004	2,51E-3	0,5365	0,5423	1,61E-2	250
0,0025	1,57E-3	1,080	1,216	8,95E-3	500
0,002	1,20E-3	1,527	1,545	7,30E-3	600
0,0016	9,93E-4	2,190	2,085	5,76E-3	800
0,001	5,56E-4	4,867	4,551	3,41E-3	1500

Same Error

Space mesh size Δx	PDE		ODE		Number of vehicles n
	Time	Error		Time	
0,05	0,02611	3,08E-2	2,87E-2	0,2930	150
0,032	0,04097	2,04E-2	2,17E-2	0,4242	200
0,025	0,05409	1,60E-2	1,61E-2	0,5423	250
0,002	0,06864	1,32E-2	1,27E-2	0,9298	400
0,016	0,08786	1,07E-2	1,06E-2	0,8723	375
0,001	0,1585	6,73E-3	6,23E-3	1,919	750
0,008	0,2097	5,25E-3	4,99E-3	2,818	1000
0,005	0,3963	3,23E-3	3,41E-3	4,551	1500
0,0045	0,4558	2,76E-3	2,77E-3	6,5346	2000
0,0025	1,080	1,57E-3	1,39E-3	21,80	5000
0,00125	3,303	7,56E-4	6,94E-4	63,94	10000

Two Populations

Continuum model

(Benzoni-Gavage, Colombo, European J. Appl. Math., 2003)

$$\begin{cases} \partial_t \rho_\alpha + \partial_x \left(V_\alpha \rho_\alpha \psi(\rho_1 + \rho_2) \right) = 0 \\ \rho_\alpha(0, x) = \tilde{\rho}_\alpha(x) \end{cases} \qquad \alpha = 1, 2$$

$V_1 \, , V_2 > 0$

$\psi \in \mathbf{C}^{0,1}([0,1],[0,1])$, $\psi'(r) < 0$ a.e., $\psi(0) = 1$ and $\psi(1) = 0$

Two Populations

Discrete model

$$\begin{cases} \dot{p}_{\alpha,i} = V_\alpha \Psi\left(\left(C_{n_1}\mathbf{p}_1\right)\left(p_{\alpha,i}\right) + \left(C_{n_2}\mathbf{p}_2\right)\left(p_{\alpha,i}\right)\right) \\ p_{\alpha,i}(0) = \tilde{p}_{\alpha,i} \end{cases} \qquad \begin{array}{l} \alpha = 1,2 \\ i = 1,\ldots,n_\alpha + 1 \end{array}$$

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$\psi \in \mathbf{C}^{0,1}([0,1],[0,1])$, $\psi'(r) < 0$ a.e., $\psi(0) = 1$ and $\psi(1) = 0$

$$\Psi(r) = \begin{cases} \psi(r) \text{ if } r \in [0,1] \\ 0 \text{ if } r > 1 \end{cases}$$

Two Populations

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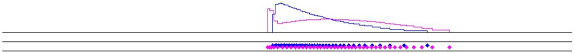

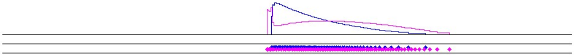

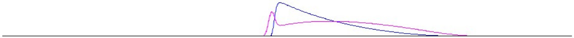

- E. Rossi, *On the Micro–Macro Limit in Traffic Flow*, Master's Thesis, Università Cattolica del Sacro Cuore, Brescia, 2012.

- R. M. Colombo, E. Rossi, *On the Micro–Macro Limit in Traffic Flow*, preprint.