On the Micro-Macro Limit in Traffic Flow

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Joint work with Rinaldo M. Colombo
Traffic

Continuum Macroscopic Model

Discrete Microscopic Model
Traffic

Continuum Macroscopic Model

- Density of vehicles.
- Partial Differential Equation (PDE).

Discrete Microscopic Model

- Positions of \( n \) vehicles.
- System of Ordinary Differential Equations (ODE).

Analysis

Limit

Numerics

\[ \tilde{\rho} \]

\[ E_n \]

\[ \tilde{p} \]

\[ \text{ODE solution} \]

\[ p_C \]

\[ \rho_n \]

\[ n \to \infty \]

\[ \rho \]

\[ \text{PDE solution} \]

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Continuum Macroscopic Model

\[ \rho = \rho(t, x) \]
Continuum Macroscopic Model

\[ \rho = \rho(t, x) \]

Conservation Law
Continuum Macroscopic Model

\[ \rho = \rho(t, x) \]

Conservation Law

Continuum Speed Law

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} \left( \rho v(\rho) \right) &= 0 \\
\rho(0, x) &= \tilde{\rho}(x)
\end{align*}
\]
Continuum Macroscopic Model

\[ \rho = \rho(t, x) \]

Conservation Law

\[ \Downarrow \]

Continuum Speed Law

\[ \Downarrow \]

**LWR model** [Lighthill, Whitham, 1955, and Richards, 1956]

\[
\begin{aligned}
\partial_t \rho + \partial_x (\rho v(\rho)) &= 0 \\
\rho(0, x) &= \tilde{\rho}(x)
\end{aligned}
\]
Continuum Macroscopic Model

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Conservation Law \[ \Downarrow \] Continuum Speed Law

**LWR model** [Lighthill, Whitham, 1955, and Richards, 1956]

\[
\begin{align*}
\partial_t \rho + \partial_x (\rho v(\rho)) &= 0 \\
\rho(0, x) &= \tilde{\rho}(x)
\end{align*}
\]

\[ R_m = \left\{ \rho \in L^1(\mathbb{R}, [0, 1]) : \int_{\mathbb{R}} \rho(x) \, dx = m > 0 \text{ and } \text{spt } \rho \text{ is compact} \right\} \]
Discrete Microscopic Model

\[ p_i = p_i(t) \text{ for all } i = 1, \ldots, n + 1 \]
Discrete Microscopic Model

\[ p_i = p_i(t) \text{ for all } i = 1, \ldots, n + 1 \]

\[ \dot{p}_i = w (p_{i+1} - p_i) \]
Discrete Microscopic Model

\[ p_i = p_i(t) \text{ for all } i = 1, \ldots, n + 1 \]

\[ \dot{p}_i = w (p_{i+1} - p_i) \quad \dot{p}_{n+1} = V(t) \]
Discrete Microscopic Model

\[ p_i = p_i(t) \text{ for all } i = 1, \ldots, n + 1 \]

\[ \dot{p}_i = w(p_{i+1} - p_i) \]

\[ \dot{p}_{n+1} = V(t) \]

First order **Follow-the-Leader** model

\[
\begin{cases}
\dot{p}_i = w(p_{i+1} - p_i) & i = 1, \ldots, n \\
\dot{p}_{n+1} = V(t) \\
p_i(0) = \tilde{p}_i & i = 1, \ldots, n + 1
\end{cases}
\]
Discrete Microscopic Model

\[ p_i = p_i(t) \] for all \( i = 1, \ldots, n + 1 \)

\[ \dot{p}_i = w(p_{i+1} - p_i) \]

\[ \dot{p}_n+1 = V(t) \]

First order **Follow-the-Leader** model

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\dot{p}_{n+1} = V(t) \\
p_i(0) = \tilde{p}_i & i = 1, \ldots, n + 1
\end{cases}
\]

\[ P_n = \left\{ (p_1, \ldots, p_{n+1}) \in \mathbb{R}^{n+1} : p_{i+1} - p_i \geq \ell_n \ \forall i = 1, \ldots, n \right\} \]
Well-Posedness

Assume the following conditions on $v$:

- $v \in C^{0,1}([0, 1], \mathbb{R})$
- $v'(\rho) \leq 0$ for a.e. $\rho \in [0, 1]$
- $v(1) = 0$
- $v(0) = v_{\text{max}}$ for a suitable positive $v_{\text{max}}$
Well-Posedness

Assume the following conditions on \( \nu \):

- \( \nu \in C^{0,1}([0, 1], \mathbb{R}) \)
- \( \nu'(\rho) \leq 0 \) for a.e. \( \rho \in [0, 1] \)
- \( \nu(1) = 0 \)
- \( \nu(0) = \nu_{\text{max}} \) for a suitable positive \( \nu_{\text{max}} \)

**Proposition (Continuum)**

For any initial datum \( \tilde{\rho} \in R_m \cap BV(\mathbb{R}; [0, 1]) \), there exists a unique solution \( \rho \in C^{0,1}(\mathbb{R}^+, R_m) \) to

\[
\begin{aligned}
\partial_t \rho + \partial_x (\rho \nu(\rho)) &= 0 \\
\rho(0, x) &= \tilde{\rho}(x)
\end{aligned}
\]
Well-Posedness

Assume the following conditions on $w$:

- $w \in C^{0,1}([\ell_n, +\infty[, \mathbb{R})$
- $w'(\delta) \geq 0$ for a.e. $\delta \geq \ell_n$
- $w(\ell_n) = 0$
- $\lim_{\delta \to +\infty} w(\delta) = w_{\text{max}}$ for a suitable positive $w_{\text{max}}$
Well-Posedness

Assume the following conditions on $w$:

- $w \in C^{0,1}([\ell_n, +\infty[, \mathbb{R})$
- $w'(\delta) \geq 0$ for a.e. $\delta \geq \ell_n$
- $w(\ell_n) = 0$
- $\lim_{\delta \to +\infty} w(\delta) = w_{\max}$ for a suitable positive $w_{\max}$

Proposition (Discrete)

For any initial datum $\tilde{p} \in P_n$, there exists a unique solution $p \in C^{0,1}(\mathbb{R}^+, P_n)$ to

\[
\begin{cases}
\dot{p}_i = w(p_{i+1} - p_i) & i = 1, \ldots, n \\
\dot{p}_{n+1} = V \\
p_i(0) = \tilde{p}_i & i = 1, \ldots, n + 1
\end{cases}
\]
Continuum and discrete descriptions are related through particle paths:

\[
\begin{align*}
\dot{p} &= v(\rho(t, p(t))) \\
p(0) &= \tilde{p}
\end{align*}
\]
Micro–Macro Connection

Continuum and discrete descriptions are related through particle paths:

\[
\begin{align*}
\dot{p} &= \nu (\rho (t, p(t))) \\
p(0) &= \bar{p}
\end{align*}
\]

\[\implies \text{Connection:} \]

each equation in the discrete model is a particle path.
Micro–Macro Connection

$$\left\{ \begin{array}{l} \partial_t \rho + \partial_x (\rho v(\rho)) = 0 \\ \rho(0,x) = \tilde{\rho}(x) \end{array} \right.$$  $$\begin{aligned} \dot{p}_i &= w(p_{i+1} - p_i) & i &= 1, \ldots, n \\ \dot{p}_{n+1} &= V \\
 p_i(0) &= \tilde{p}_i & i &= 1, \ldots, n + 1 \end{aligned}$$
Micro–Macro Connection

\[
\begin{aligned}
\frac{\partial_t \rho}{\partial x}(\rho \nu(\rho)) &= 0 \\
\rho(0, x) &= \tilde{\rho}(x)
\end{aligned}
\]

\[
\begin{aligned}
\dot{p}_i &= w(p_{i+1} - p_i) & i &= 1, \ldots, n \\
\dot{p}_{n+1} &= V \\
p_i(0) &= \tilde{p}_i & i &= 1, \ldots, n + 1
\end{aligned}
\]

\[w(\delta) = v \left( \ell_n / \delta \right)\]
Micro–Macro Connection

\[
\begin{aligned}
\begin{cases}
\partial_t \rho + \partial_x (\rho v(\rho)) = 0 \\
\rho(0, x) = \tilde{\rho}(x)
\end{cases}
\end{aligned}
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\dot{\rho}_i = w(p_{i+1} - p_i) & i = 1, \ldots, n \\
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\rho_i(0) = \tilde{\rho}_i & i = 1, \ldots, n + 1
\end{cases}
\end{aligned}
\]

\[w(\delta) = v\left(\ell_n / \delta\right)\]

\(v\) satisfies the conditions for the continuum model \iff \(w\) satisfies the conditions for the discrete model
Two Useful Operators: $C_n$ and $E_n$

Discretization $E_n$ \[ R_m \rightarrow P_n \]
Continuum $\rightarrow$ Discrete

Anti-discretization $C_n$ \[ P_n \rightarrow R_m \]
Discrete $\rightarrow$ Continuum

$E_n \circ C_n = \text{Id}$

$\lim_{n \rightarrow +\infty} C_n \circ E_n = \text{Id}$
Two Useful Operators: $C_n$ and $E_n$

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\[ \lim_{n \rightarrow +\infty} C_n \circ E_n = \text{Id} \]
Micro-Macro Limit

As the number of vehicles grows to infinity:

\[
\text{Discrete-model } n \to +\infty \to \text{Continuum-model}
\]
Micro-Macro Limit

As the number of vehicles grows to infinity:

\[
\text{Discrete-model} \xrightarrow{n \to +\infty} \text{Continuum-model}
\]

**Theorem**

*Fix* \( T > 0 \). *Choose* \( \tilde{\rho} \in R_m \cap BV(\mathbb{R}, [0, 1]) \).

*Set* \( \tilde{p} = E_n(\tilde{\rho}) \) *and let* \( p(t) \) *be the corresponding discrete–solution.*

*Define* \( \rho_n(t) = C_n p(t) \).

*If there exists* \( \rho \in L^\infty ([0, T], R_m) \) *such that* \( \lim_{n \to +\infty} \rho_n = \rho \) *a.e., then* \( \rho \) *is a continuum–solution with initial datum* \( \tilde{\rho} \).
Micro-Macro Limit

The Theorem above states that the following diagram commutes:

\[ \tilde{\rho} \quad E_n \quad \tilde{p} \]

\[ \rho \quad n \to \infty \quad \rho_n \quad C_n \quad p \]

PDE solution \quad ODE solution
Micro-Macro Limit

The Theorem above states that the following diagram commutes:

\[ \tilde{\rho} \]
Micro-Macro Limit

The Theorem above states that the following diagram commutes:

\[ \tilde{\rho} \quad \xrightarrow{E_n} \quad \tilde{p} \]
Micro-Macro Limit

The Theorem above states that the following diagram commutes:

\[ \tilde{\rho} \quad E_n \quad \tilde{p} \]

\[ \tilde{\rho} \quad \tilde{p} \quad p \]

ODE solution
Micro-Macro Limit

The Theorem above states that the following diagram commutes:

\[
\begin{array}{c}
\tilde{\rho} \quad \xrightarrow{E_n} \quad \tilde{p} \\
\rho_n \quad \xleftarrow{C_n} \quad p
\end{array}
\]
Micro-Macro Limit

The Theorem above states that the following diagram commutes:

\[ \tilde{\rho} \quad E_n \quad \tilde{p} \]

\[ \rho \quad n \to \infty \quad \rho_n \quad C_n \quad p \]

ODE solution
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The Theorem above states that the following diagram commutes:

\[ \tilde{\rho} \quad \xrightarrow{E_n} \quad \tilde{p} \]

PDE solution

\[ n \to \infty \]

ODE solution

\[ \rho \quad \rho_n \quad p \quad C_n \]
Example
Numerical Integrations

Numerical algorithm to integrate the continuum and discrete models:

- Hyperbolic Conservation Law
- System of ODEs
Numerical Integrations

Numerical algorithm to integrate the continuum and discrete models:

- Hyperbolic Conservation Law
- System of ODEs

ODEs method to integrate PDEs.
Numerical Integrations

Numerical algorithm to integrate the continuum and discrete models:

- Hyperbolic Conservation Law
- System of ODEs

ODEs method to integrate PDEs.

In the numerical integrations we closely follow the diagram seen before.
ODEs to integrate PDEs

Shock Wave

Relative error in the $L^1$-norm vs. $n$

Integration time vs. $n$
ODEs to integrate PDEs

Rarefaction Wave

Relative error in the $L^1$-norm vs. $n$

Integration time vs. $n$
<table>
<thead>
<tr>
<th>Space mesh size $\Delta x$</th>
<th>PDE</th>
<th>ODE</th>
<th>Number of vehicles $n$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Error</td>
<td>Execution Time</td>
<td>Error</td>
</tr>
<tr>
<td>0.02</td>
<td>1.32E-2</td>
<td>0.06864</td>
<td>0.06262</td>
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<tr>
<td>0.01</td>
<td>6.73E-3</td>
<td>0.1585</td>
<td>0.1910</td>
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<td>0.0064</td>
<td>4.22E-3</td>
<td>0.2737</td>
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<tr>
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<td>3.23E-3</td>
<td>0.3963</td>
<td>0.4242</td>
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<tr>
<td>0.0045</td>
<td>2.76E-3</td>
<td>0.4558</td>
<td>0.4877</td>
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<tr>
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<td>2.51E-3</td>
<td>0.5365</td>
<td>0.5423</td>
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<tr>
<td>0.0025</td>
<td>1.57E-3</td>
<td>1.080</td>
<td>1.216</td>
</tr>
<tr>
<td>0.002</td>
<td>1.20E-3</td>
<td>1.527</td>
<td>1.545</td>
</tr>
<tr>
<td>0.0016</td>
<td>9.93E-4</td>
<td>2.190</td>
<td>2.085</td>
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<tr>
<td>0.001</td>
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<td>4.867</td>
<td>4.551</td>
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</table>
**Same Error**

<table>
<thead>
<tr>
<th>Space mesh size Δx</th>
<th>PDE</th>
<th>ODE</th>
<th>Number of vehicles n</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Time</td>
<td>Error</td>
<td>Time</td>
</tr>
<tr>
<td>0,05</td>
<td>0,02611</td>
<td>3,08E-2</td>
<td>2,87E-2</td>
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<td>2,17E-2</td>
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<tr>
<td>0,002</td>
<td>0,06864</td>
<td>1,32E-2</td>
<td>1,27E-2</td>
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<td>0,016</td>
<td>0,08786</td>
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<td>0,008</td>
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<td>1,080</td>
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<td>0,00125</td>
<td>3,303</td>
<td>7,56E-4</td>
<td>6,94E-4</td>
</tr>
</tbody>
</table>
Two Populations

Continuum model


\[
\begin{cases}
\partial_t \rho_\alpha + \partial_x \left( V_\alpha \rho_\alpha \psi (\rho_1 + \rho_2) \right) = 0 \\
\rho_\alpha (0, x) = \tilde{\rho}_\alpha (x) \\
\end{cases}
\]

\[\alpha = 1, 2\]

\(V_1, V_2 > 0\)

\(\psi \in C^{0,1}([0, 1], [0, 1]), \psi'(r) < 0\) a.e., \(\psi(0) = 1\) and \(\psi(1) = 0\)
Two Populations

Discrete model

\[
\begin{cases}
\dot{p}_{\alpha,i} = V_\alpha \psi \left( (C_{n_1} p_1)(p_{\alpha,i}) + (C_{n_2} p_2)(p_{\alpha,i}) \right) & \alpha = 1, 2 \\
p_{\alpha,i}(0) = \tilde{p}_{\alpha,i} & i = 1, \ldots, n_\alpha + 1
\end{cases}
\]

\(V_1, V_2 > 0\)

\(\psi \in C^{0,1}([0,1], [0,1]), \psi'(r) < 0 \text{ a.e.}, \psi(0) = 1 \text{ and } \psi(1) = 0\)

\[\Psi(r) = \begin{cases} 
\psi(r) & \text{if } r \in [0,1] \\
0 & \text{if } r > 1
\end{cases}\]
Two Populations
Two Populations