Road junction modelling using a scheme based on Hamilton-Jacobi equation

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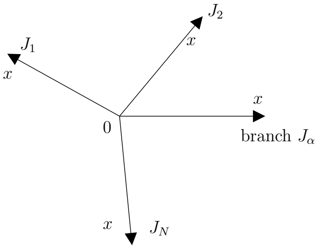


Proposition (Junction model [IMZ '11])

$$\begin{cases} u^\alpha_t + H_\alpha(u^\alpha_x) = 0, & x > 0, \quad \alpha = 1, \dots, N \\ u^\alpha(0,t) := u(0,t), & x = 0, \\ u_t + \max\limits_{\alpha=1,\dots,N} H^-_\alpha(u^\alpha_x) = 0, & x = 0. \end{cases} \tag{1.1}$$

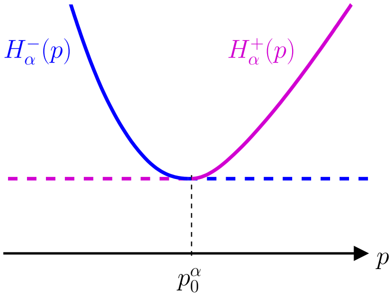
with the initial condition $u^\alpha(0,x) = u^\alpha_0(x)$.

Assumptions

For all $\alpha = 1, \dots, N$,

(A0) The initial condition u_0^α is Lipschitz continuous.

(A1) The Hamiltonians H_α are $C^1(\mathbb{R})$ and convex such that:



Proposition (Numerical Scheme)

Let us consider the discrete space and time derivatives:

$$p_i^{\alpha,n} := \frac{U_{i+1}^{\alpha,n} - U_i^{\alpha,n}}{\Delta x} \quad \textit{and} \quad (D_t U)_i^{\alpha,n} := \frac{U_i^{\alpha,n+1} - U_i^{\alpha,n}}{\Delta t}$$

Then we have the following numerical scheme:

$$\begin{cases} (D_t U)_i^{\alpha,n} + \max\{H_\alpha^+(p_{i-1}^{\alpha,n}), H_\alpha^-(p_i^{\alpha,n})\} = 0, & i \geq 1, \quad \alpha = 1, \ldots, N \\ U_0^n := U_0^{\alpha,n}, & i = 0, \\ (D_t U)_0^n + \max\limits_{\alpha=1,\ldots,N} H_\alpha^-(p_0^{\alpha,n}) = 0, & i = 0 \end{cases} \tag{2.2}$$

With the initial condition $U_i^{\alpha,0} := u_0^\alpha(i\Delta x)$.

Δx and Δt = space and time steps satisfying a CFL condition

CFL condition

The natural CFL condition is given by:

$$\frac{\Delta x}{\Delta t} \geq \sup_{\substack{\alpha=1,\dots,N \\ i\geq 0,\ 0\leq n\leq n_T}} |H'_\alpha(p_i^{\alpha,n})| \tag{2.3}$$

First result

Theorem (Time and Space Gradient estimates)

Assume (A0)-(A1). If the CFL condition (2.3) is satisfied and

(i) *[Time] If* $M^n := \sup_{\alpha,i} (D_t U)_i^{\alpha,n}$ *and* $m^n := \inf_{\alpha,i} (D_t U)_i^{\alpha,n}$, *then*

$$m^0 \le m^n \le m^{n+1} \le M^{n+1} \le M^n \le M^0.$$

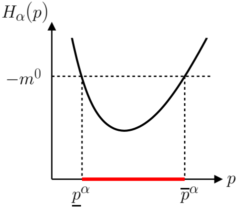
(ii) *[Space] If* $\underline{p}_\alpha := (H_\alpha^-)^{-1}(-m^0)$ *and* $\overline{p}_\alpha := (H_\alpha^+)^{-1}(-m^0)$, *then*

$$\underline{p}_\alpha \le p_i^{\alpha,n} \le \overline{p}_\alpha, \quad \text{for all} \quad i \ge 0, \quad n \ge 0 \quad \text{and} \quad \alpha = 1, \ldots, N.$$

Stronger CFL condition

As for any $\alpha = 1, \dots, N$, we have that:

$$\underline{p}_\alpha \leq p_i^{\alpha,n} \leq \overline{p}_\alpha \qquad \text{for all} \quad i, n \geq 0$$



Then the CFL condition becomes:

$$\frac{\Delta x}{\Delta t} \geq \sup_{\substack{\alpha=1,\dots,N \\ p_\alpha \in [\underline{p}_\alpha, \overline{p}_\alpha]}} |H'_\alpha(p_\alpha)| \tag{2.4}$$

Recall

(A2) Technical assumption (Legendre-Fenchel transform)

$$H_\alpha(p) = \sup_{q\in\mathbb{R}} (pq - L_\alpha(q)) \quad \text{with} \quad L''_\alpha \geq \delta > 0, \quad \text{for all index } \alpha$$

Theorem (Existence and uniqueness [IMZ, '11])

Under (A0)-(A1)-(A2), there exists a unique viscosity solution u of (1.1) on the junction, satisfying for some constant $C_T > 0$

$$|u(t,y) - u_0(y)| \leq C_T \quad \textit{for all} \quad (t,y) \in J_T.$$

Moreover the function u is Lipschitz continuous with respect to (t,y).

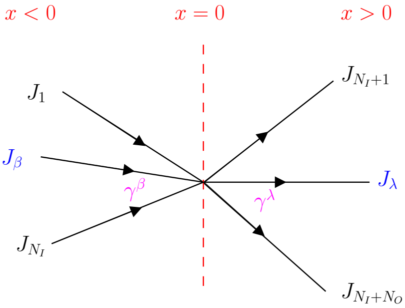
Second result

Theorem (Convergence from discrete to continuous [CLM, '13])

Assume that (A0)-(A1)-(A2) and the CFL condition (2.4) are satisfied. Then the numerical solution converges uniformly to u the unique viscosity solution of (1.1) when $\varepsilon \to 0$, locally uniformly on any compact set $\mathcal{K}$:

$$\limsup_{\varepsilon \to 0} \sup_{(n\Delta t, i\Delta x) \in \mathcal{K}} |u^\alpha(n\Delta t, i\Delta x) - U_i^{\alpha,n}| = 0$$

Setting

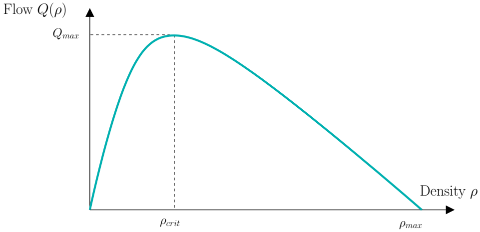


N_I incoming and N_O outgoing roads

Densities

LWR model [Lighthill, Whitham '55; Richards '56] on branch α:

$$\rho_t^\alpha + (Q^\alpha(\rho^\alpha))_x = 0$$



$$Q^\alpha(\rho^\alpha) = \rho^\alpha V^\alpha(\rho^\alpha) \quad \text{with} \quad V^\alpha \quad \text{velocity function}$$

Getting the HJ equation

LWR model on branch α:

$$\rho_t^\alpha + (Q^\alpha(\rho^\alpha))_x = 0$$

By definition

$$\rho^\alpha = \gamma^\alpha \partial_x U^\alpha \quad \text{on branch } \alpha$$

And

$$\begin{cases} u^\alpha(x,t) = -U^\alpha(-x,t), & x > 0, \text{ for incoming roads} \\ u^\alpha(x,t) = -U^\alpha(x,t), & x > 0, \text{ for outgoing roads} \end{cases}$$

where the continuous car label u^α solves the HJ equation on branch α:

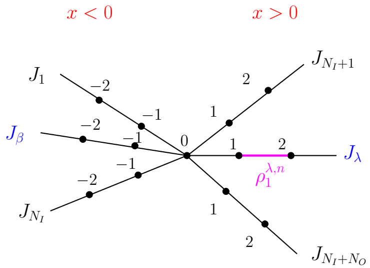
$$u_t^\alpha + H^\alpha(u_x^\alpha) = 0, \quad \text{for } x > 0$$

Discrete car densities

Definition (Discrete car density)

The discrete car density $\rho_i^{\alpha,n}$ *with* $n \geq 0$ *and* $i \in \mathbb{Z}$ *is given by:*

$$\rho_i^{\alpha,n} := \begin{cases} \gamma^\alpha p_{|i|-1}^{\alpha,n} & \text{for} \quad \alpha = 1, ..., N_I, \quad i \leq -1 \\ \\ -\gamma^\alpha p_i^{\alpha,n} & \text{for} \quad \alpha = N_I+1, ..., N_I+N_O, \quad i \geq 0 \end{cases} \tag{3.5}$$



Traffic interpretation

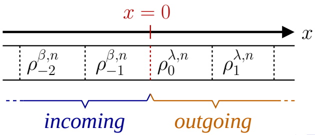
Proposition (Scheme for vehicles densities)

The scheme deduced from (2.2) for the discrete densities is given by:

$$\frac{\Delta x}{\Delta t}\{\rho_i^{\alpha,n+1}-\rho_i^{\alpha,n}\} = \begin{cases} F^\alpha(\rho_{i-1}^{\alpha,n},\rho_i^{\alpha,n}) - F^\alpha(\rho_i^{\alpha,n},\rho_{i+1}^{\alpha,n}) & \text{for } i\neq 0,-1 \\ F_0^\alpha(\rho_{-1}^{\cdot,n},\rho_0^{\cdot,n}) - F^\alpha(\rho_i^{\alpha,n},\rho_{i+1}^{\alpha,n}) & \text{for } i=0 \\ F^\alpha(\rho_{i-1}^{\alpha,n},\rho_i^{\alpha,n}) - F_0^\alpha(\rho_{-1}^{\cdot,n},\rho_0^{\cdot,n}) & \text{for } i=-1 \end{cases}$$

With

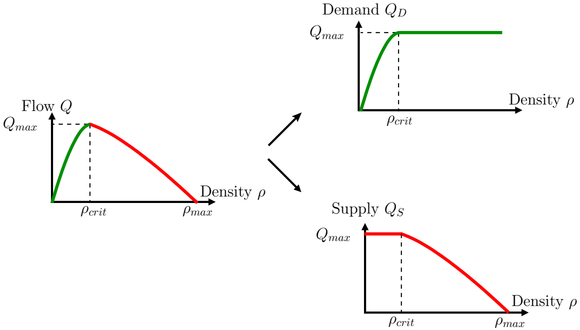
$$\begin{cases} F^\alpha(\rho_{i-1}^{\alpha,n},\rho_i^{\alpha,n}) := \min\left\{Q_D^\alpha(\rho_{i-1}^{\alpha,n}),\; Q_S^\alpha(\rho_i^{\alpha,n})\right\}, \\ F_0^\alpha(\rho_{-1}^{\cdot,n},\rho_0^{\cdot,n}) := \gamma^\alpha \min\left\{\min\limits_{\beta\le N_I}\frac{1}{\gamma^\beta}Q_D^\beta(\rho_{-1}^{\beta,n}),\; \min\limits_{\lambda>N_I}\frac{1}{\gamma^\lambda}Q_S^\lambda(\rho_0^{\lambda,n})\right\} \end{cases}$$



Supply and demand functions

Remark

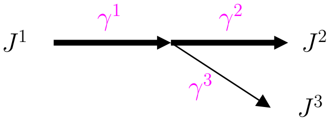
It recovers the classical Godunov scheme with passing flow = minimum between upstream demand Q_D and downstream supply Q_S.



From [Lebacque '93, '96]

Example of a Diverge

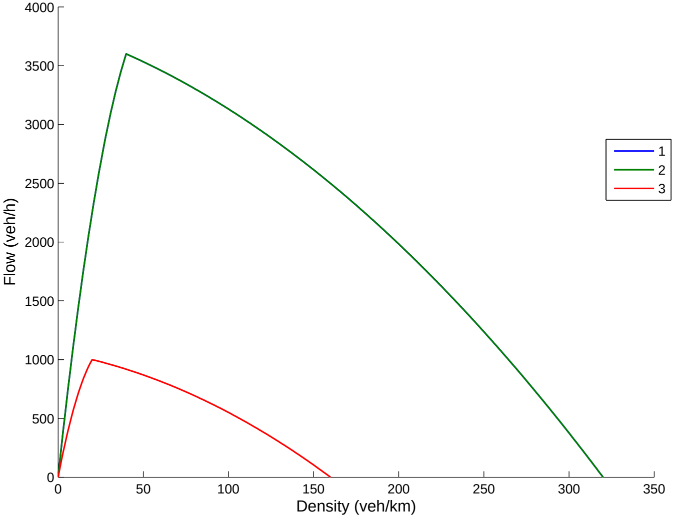
An off-ramp:



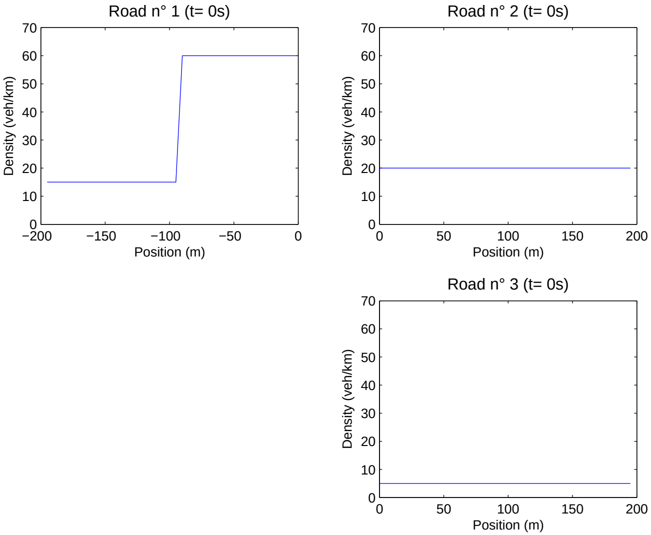
with

$$\begin{cases} \gamma^1 = 1 \\ \gamma^2 = 0.75 \\ \gamma^3 = 0.25 \end{cases}$$

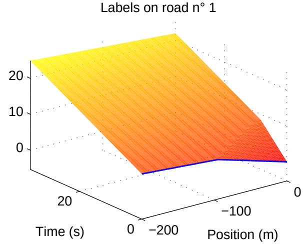
Flow functions Q^α

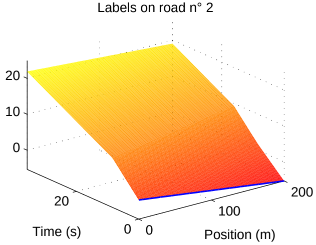


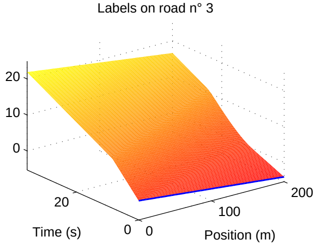
Initial conditions (t=0s)



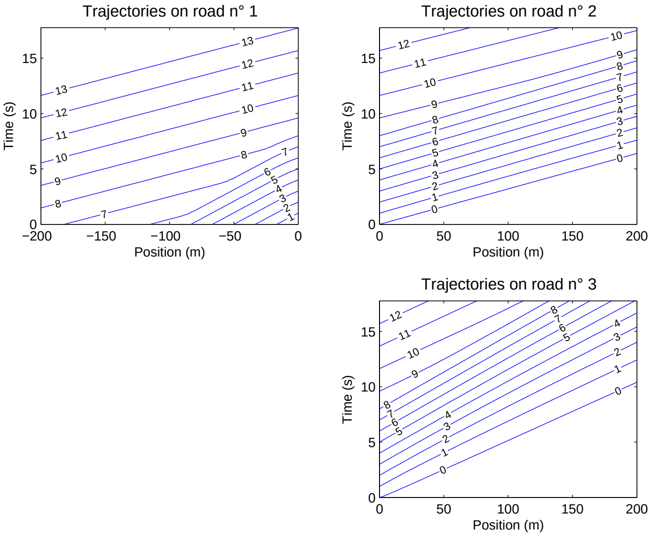
Results for $\Delta x = 5$ m, $\Delta t = 0.16$ s



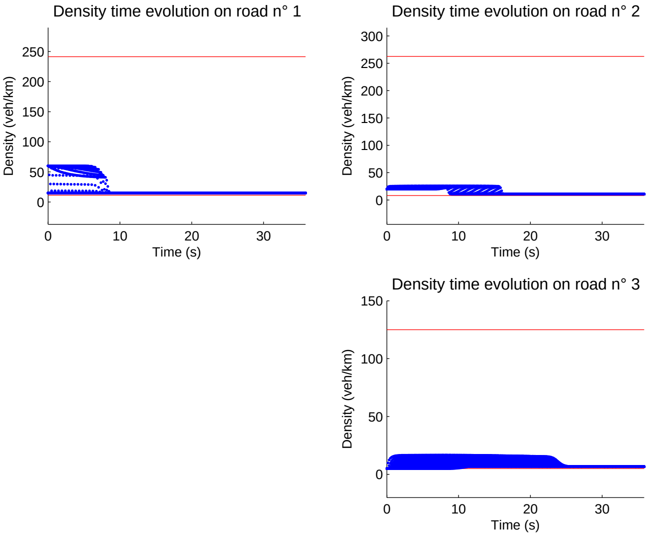




Trajectories



Gradient estimates



Complementary results [CLM '13]:

- Generalization for weaker assumptions on the Hamiltonians
- Numerical simulation for other junction configurations (merge)

Open questions:

- Error estimate
- Non-fixed coefficients γ^α
- Other link models (GSOM)
- Other junction condition

The End

THANKS FOR YOUR ATTENTION

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Some references

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