From 1st order to Higher order Sliding modes

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Introduction
A simple stabilization problem: double integrator

How to stabilize?

\[ \ddot{x} = u \]  \hspace{1cm} (1)

\[ \dot{x}_1 = x_2 \]
\[ \dot{x}_2 = u \]  \hspace{1cm} (2)

F1 car.
Introduction
A simple stabilization problem: double integrator

Classical solution: State feedback
State feedback which is $\equiv$ to frequency approach or polynomial approach

Assume that $(x_1, x_2)$ is available.

Stabilization even with a bounded control !!

Sate feedback: stabilization of $\begin{pmatrix} 2 \end{pmatrix}$ with $u = -x_1 - \frac{1}{\sqrt{2}}x_2$
Introduction
A simple stabilization problem: double integrator

A variable structure controller
If the speed is not available: Observer (dynamic extension)
An alternative solution with output feedback?

\[ u = f(x_1) \]

The simplest function being:

\[ u = \alpha x_1 \]

(variable \( \alpha \))

\[ \dot{x}_1 = x_2 \]
\[ \dot{x}_2 = \alpha x_1 \]
Introduction
A simple stabilization problem: double integrator

Strategy 1:
position \( x \) available and the signum of \( \dot{x} \) (in fact of \( x\dot{x} \))
How to play with \( \alpha \)?

\[
\ddot{x} + \alpha x = 0
\]

\[
x_1(t) = x_0 \cos(\sqrt{\alpha}t) + \frac{\dot{x}_0}{\sqrt{\alpha}} \sin(\sqrt{\alpha}t)
\]

\[
x_2(t) = -x_0 \sqrt{\alpha} \sin(\sqrt{\alpha}t) + \dot{x}_0 \cos(\sqrt{\alpha}t)
\]

\[
(x_0 \sqrt{\alpha} x_1 + \frac{\dot{x}_0}{\sqrt{\alpha}} x_2)^2 + (\dot{x}_0 x_1 - x_0 \dot{x}_2)^2 = (x_0^2 \sqrt{\alpha} + \frac{\dot{x}_0^2}{\sqrt{\alpha}})^2
\]

Solutions are ellipsoids
Introduction
A simple stabilization problem: double integrator

Area I : $x_1 x_2 < 0, \alpha_I = 1$

Area II : $x_1 x_2 > 0, \alpha_{II} = 2$

After $2k + 1$ switching:

$$l_{2k+1} = \frac{\alpha_I}{\alpha_{II}} l_0$$
Introduction
A simple stabilization problem: double integrator

Strategy 2:
position $x$ and velocity $\dot{x}$ are available
How to play with $\alpha$?

$\alpha = 1$

$\ddot{x} + x = 0$

Solutions are ellipsoids

Phase portrait
Introduction
A simple stabilization problem: double integrator

\( \alpha = -1 \)

\( \ddot{x} - x = 0 \)

Solutions are hyperbolas

\( x(t) = x_0 \cosh(t) + \dot{x}_0 \sinh(t) \)

Phase portrait

One stable and one unstable manifold
Introduction
A simple stabilization problem: double integrator

Area I: \( \alpha_I = -1 \)

\[ x_1 < 0 \land x_1 + x_2 \geq 0 \]

or

\[ x_1 > 0 \land x_1 + x_2 \leq 0 \]

Area II: \( \alpha_{II} = 1 \)

\[ x_1 \leq 0 \land x_1 + x_2 < 0 \]

or

\[ x_1 \geq 0 \land x_1 + x_2 > 0 \]
Introduction
Some first questions

Problems:

- Notion of solution,
- Discontinuous Control (damaging the actuators),
- How to find the switching logic?

Panis (Canada 97)
General Problem formulation for VSS:

\[ \dot{x} = f_i(t, x, u_i) \]

Find the switching logic and the control?
SMC (1rst order and Higher):

“Slap” principle
Introduction
Sliding Mode Control

Objective
To constrain the trajectories of system $\dot{x} = f(x) + g(x)u$ to reach, in a finite time, and then, to stay onto the sliding surface chosen according to the control objectives

Sliding mode control

$$u = \begin{cases} 
  u^+(s) & \text{if } \text{sign}(s(x)) > 0 \\
  u^-(s) & \text{if } \text{sign}(s(x)) < 0
\end{cases} \quad \text{with } u^+ \neq u^-$$

A simple sliding mode control design

$$u = u_{eq} + u_{disc}$$

W. Perruquetti 1rst to HOSM
**Introduction**

**Sliding Mode Control**

**Objective**

To constrain the trajectories of system $\dot{x} = f(x) + g(x)u$ to reach, in a finite time, and then, to stay onto the sliding surface chosen according to the control objectives.

**Sliding mode control**

$$u = \begin{cases} u^+(s) & \text{if } \text{sign}(s(x)) > 0 \\ u^-(s) & \text{if } \text{sign}(s(x)) < 0 \end{cases}$$

with $u^+ \neq u^-$

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A simple sliding mode control design

$$u = u_{eq} + u_{disc}$$

- given by $s = \dot{s} = 0$, (invariance of the sliding surface)
- $u_{disc} = -k\text{sign}(s)$, (convergence in finite time onto the surface)
**Objective**

To constrain the trajectories of system $\dot{x} = f(x) + g(x)u$ to reach, in a finite time, and then, to stay onto the sliding surface chosen according to the control objectives.

**Sliding mode control**

$$u = \begin{cases} u^+(s) & \text{if } \text{sign}(s(x)) > 0 \\ u^-(s) & \text{if } \text{sign}(s(x)) < 0 \end{cases} \quad \text{with } u^+ \neq u^-$$

**A simple sliding mode control design**

$$u = u_{eq} + u_{disc}$$

- given by $s = \dot{s} = 0$, (invariance of the sliding surface)
- $u_{disc} = -k\text{sign}(s)$, (convergence in finite time onto the surface)
Introduction

Sliding Mode Control: Advantages vs disadvantages

Advantages:
- System order reduction
- Finite time convergence (adjust time response)
- Robustness w.r.t. parametric uncertainties and disturbances

Disadvantages:
- Chattering phenomena (actuator damage)
- Noise sensitivity (??)
- Output feedback (??)
Sliding mode control design:

- hitting phase (or reaching phase), and the
- sliding phase.

Stability/attractivity concepts:

- existence of sliding motions is a contraction property (locally),
- shaping procedure: stabilization problem ("tune" the shape of the sliding: in sliding minimum phase system).
This system “seems” complex, however, if we set

\[ z_1 = x_1 (1 + x_2^2) \]
\[ z_2 = x_2 \]

(note that it defines a global diffeomorphism), then one obtains
Introduction

Sliding Mode Control: One more time ...

\[
\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= u
\end{aligned}
\]  

(8)

and it becomes obvious that if in sliding mode \( z_2 = -z_1 \), then \( z_1 \) converges asymptotically to zero (\( \dot{z}_1 = z_2 = -z_1 \)) and thus \( z_2 \) also converges. In this step of design (the “sliding phase”), the shape of the sliding manifold arises naturally.
Now, we need to force the system to evolve on the constraint \( z_2 = -z_1 \). For this, let us define the sliding surface as

\[
S = \{ z \in \mathbb{R}^2 : s(z) = 0 \} \quad (9)
\]

\[
s(z) = z_2 + z_1 \quad (10)
\]

Then, according to the equivalent control method, we need the control to satisfy

\[
u(z) = \begin{cases} 
u^+(z) & \text{if } s(z) > 0 \\ 
u^-(z) & \text{if } s(z) < 0 \end{cases}
\]

\[
\min(u^+(z), u^-(z)) < u_{eq} = -z_2 < \max[u^+(z), u^-(z)]
\]

in order to ensure that a sliding mode exists on \( S \).
This leads to various design controls, for example,

\[ u(z) = \begin{cases} 
-1 & \text{if } s(z) > 0 \\
1 & \text{if } s(z) < 0 
\end{cases} \]

which ensures a finite time convergence to \( S \) as soon as the initial conditions are close enough to the surface and satisfy \(|z_2| < 1\). But, can we provide a better characterization of the initial conditions leading to a sliding mode?
An alternative to this control is

$$u(z) = \begin{cases} -z_2 - 1 & \text{if } s(z) > 0 \\ -z_2 + 1 & \text{if } s(z) < 0 \end{cases}$$

which ensures a finite time convergence to $S$, whatever the initial conditions. But since the chattering problem remains, can we stabilize the system while reducing the chattering?
\[ \dot{x} = f(x, x), \forall x \in \mathcal{X} \setminus S \] (12)

where \( \mathcal{X} \) is the state manifold (locally diffeomorphic to \( \mathbb{R}^n \)).

**Problem**: \( f \) is not defined on a manifold of codimension one (if \( S = \{ x \in \mathbb{R}^n : s(x) = 0 \} \) and \( s \) is a scalar function) thus Cauchy-Lipschitz and Peano Theorem does not apply (existence (and uniqueness) of solutions).

**Notion of solutions on the manifold**: extend the vector field \( f \) on the manifold \( S \). (Aizerman, Filipov, Utkin, . . . .)
Differential Inclusion: Notion of solution

Main points of view:

- Real world (system is not discontinuous), just take into account (delays, hysterisis, saturation) in a small vicinity of the sliding manifold $S_\varepsilon = \{ x \in \mathbb{R}^n : \| s(x) \| \leq \varepsilon \}$ ($\varepsilon$ radius), then use the usual results, then $\varepsilon \to 0$: Sliding mode are then limit of “classical solution”. That is Aizerman’s point of view.

- Embed the discontinuous system into a Differential Inclusion (Filipov).

- Equivalent control theory (Utkin).
Filipov’s points of view: replace the ODE with discontinuous right-hand side

\[ \dot{x} = f(t, x), \forall x \in \mathcal{X} \setminus \mathcal{S} \subset \mathbb{R}^n \]

with the following differential inclusion

\[ \dot{x} \in F(t, x) \]

which capture the behaviors of the original system, where

\[ F(t, x) = \bigcap_{\varepsilon > 0} \bigcap_{\mu(M) = 0} \text{conv}(f(t, B_\varepsilon(x) - M)) \]
\[ \dot{x} = -\text{sign}(x), \quad x \in \mathbb{R} \]

\[
F(x) = \begin{cases} 
-1 & \text{if } x < 0 \\
1 & \text{if } x > 0 \\
[-1, 1] & \text{if } x = 0 
\end{cases}
\]
Differential Inclusion: Notion of solution

\[ \dot{x} = f_0(t, x) \]

\[ x \in S \]

\[ f_0 \text{ should be in } T_x S. \]

\[ f_0(t, x) \in \text{conv}\{f^+(t, x), f^-(t, x)\} \cap T_x S \]
Differential Inclusion: Notion of solution

\[ f_0 = \alpha f^+ + (1 - \alpha) f^-, \quad \alpha \in [-1, 1] \]

\[ f_0(t, x) \in T_x S \iff \langle ds, f_0 \rangle = 0 \]
\[ \iff \alpha \langle ds, f^+ \rangle + (1 - \alpha) \langle ds, f^- \rangle = 0 \]

\[ \alpha = \frac{\langle ds, f^- \rangle}{\langle ds, f^- - f^+ \rangle} \]

\[ \dot{x} = f_0 = \frac{\langle ds, f^- \rangle}{\langle ds, f^- - f^+ \rangle} f^+ - \frac{\langle ds, f^+ \rangle}{\langle ds, f^- - f^+ \rangle} f^- \]
Differential Inclusion: Notion of solution

Utkin’s points of view: On the sliding manifold, replace the dynamics by the following ODE (equivalent dynamics)

\[ \dot{x} = f_{eq}(t, x, u_{eq}) \]

where \( f_{eq}(t, x, u_{eq}) \) ensure invariance of the sliding manifold that is

\[ f_{eq}(t, x, u_{eq}) : s(x(t)) = 0, \forall t > 0 \]

thus \( s \) is identically zero which implies that \( \dot{s} \) is also zero.

Remark

Filipov and Utkin thechnics are equivalent only for system linear in the control that is \( \dot{x} = f(x) + g(x)u \).
Differential Inclusion: Notion of solution

Counter example:

\[
\begin{align*}
\dot{x}_1 &= 0.3x_2 + x_1 u \\
\dot{x}_2 &= -0.7x_1 + 4x_1 u^3
\end{align*}
\] (13) (14)

Sliding manifold defined by:

\[ s(x) = x_2 + x_1 \]

Control: \( u = -\text{sign}(s(x)x_1) \).
Differential Inclusion: Notion of solution

Sliding mode occurs if \( ss' < 0 \) (close to):

\[
\dot{s} = 0.3x_2 + x_1u - 0.7x_1 + 4x_1u^3 \tag{15}
\]

\[
\dot{s} = 0.3x_2 - 0.7x_1 + x_1u(1 + 4u^2) \tag{16}
\]

If \( s(x) \approx 0, x_2 \approx -x_1 \)

\[
\dot{s} = -x_1 + x_1u(1 + 4u^2) \tag{17}
\]

\[
ss' = -sx_1 - 5|x_1| < 0 \tag{18}
\]

Yes sliding will occur.
Equivalent dynamics (Filipov):

\[ f^+(x) = \begin{pmatrix} 0.3x_2 + x_1 \\ 3.3x_1 \end{pmatrix}, \quad f^-(x) = \begin{pmatrix} 0.3x_2 - x_1 \\ -4.7x_1 \end{pmatrix}, \quad (19) \]

\[ \alpha = \frac{1}{1 \ 1} f^-(x) \cdot \begin{pmatrix} -2x_1 \\ -8x_1 \end{pmatrix} = \frac{0.3x_2 - 5.7x_1}{-10x_1} \]

When \( x \) close to the sliding manifold \((x_2 \simeq -x_1)\) we have \( \alpha = \frac{-6x_1}{-10x_1} = \frac{6}{10} \) thus the equivalent dynamics is

\[ \dot{x}_1 = \alpha(0.3x_2 + x_1) + (1 - \alpha)(0.3x_2 - x_1) \quad (20) \]

\[ = \frac{6}{10}(0.3x_2 + x_1) + \frac{4}{10}(0.3x_2 - x_1) \quad (21) \]

\[ = 0.3x_2 + 0.2x_1 = -0.1x_1 \quad (22) \]
Equivalent dynamics (Utkin):

\[ \dot{s} = 0.3x_2 + x_1 u - 0.7x_1 + 4x_1 u^3 \]

When \( x \) close to the sliding manifold \( (x_2 \simeq -x_1) \) we have

\[ \dot{s} = -x_1 + x_1 u(1 + 4u^2) = 0 \]
\[ \Leftrightarrow u(1 + 4u^2) = 1 \lor x_1 = 0 \]  \hspace{1cm} (23)

\[ u(1 + 4u^2) = 1 \Leftrightarrow u = 0.5, u \in \mathbb{R} \]

\[ \dot{x}_1 = 0.3x_2 + 0.5x_1 \]

When \( x \) close to the sliding manifold \( (x_2 \simeq -x_1) \)

\[ \dot{x}_1 = 0.2x_1 \]

Unstable
Non Linear affine systems

\[ \dot{x} = f(x) + g(x)u \]  \hspace{1cm} (25)

Sliding manifold being defined by a $C^1$ function (same dimension as $u$)

\[ S = \{ x \in R^n : s(x) = 0 \} \]  \hspace{1cm} (26)
Differential Inclusion: Notion of solution

Classical first order Sliding Mode
- Attractivity condition and invariance condition of the sliding manifold
- Sliding mode equivalent dynamics
- Robustness with respect to matched disturbance

Higher order sliding mode
Attractivity condition and invariance condition of the sliding manifold

**Sliding Mode Control**: First order sliding mode

Attractivity and invariance condition

\[
\begin{align*}
  u &= \begin{cases} 
    u^+ & \text{if } s(x) > 0 \\
    u^- & \text{if } s(x) < 0 
  \end{cases} \\
  \Rightarrow \quad \min(u^+, u^-) < u_{eq} < \max(u^+, u^-) 
\end{align*}
\]  

(27)

\[s^T \dot{s} < 0 \iff s^T \dot{s} < 0 \iff \min(u^+, u^-) < u_{eq} < \max(u^+, u^-) \]  

(28)

\[\hat{s} = 0 \iff u = u_{eq} \]  

(29)
Be careful, this condition does not imply that the sliding manifold is reached in finite time. Thus, this condition (for the existence of a sliding mode) should be replaced by a more restrictive condition for example (mu-reachability condition)

\[ s^T \dot{s} < -\mu s \]  \hspace{1cm} (30)

Show that \( V(s) = s^T s \) goes to zero in finite time
Sliding Mode Control: First order sliding mode
Attractivity and invariance condition

Let us consider a linear system

\[ \dot{x} = Ax + Bu \]  \hspace{1cm} (31)

with a linear sliding surface

\[ S = \{ x \in \mathbb{R}^n : s(x) = Cx \} \]  \hspace{1cm} (32)

\[ s^T \dot{s} = s^T C(Ax + Bu) < 0 \]  \hspace{1cm} (33)

Equivalent control if \( CB \) invertible

\[ \dot{s} = 0 \Leftrightarrow u_{eq} = -(CB)^{-1}CAx \]  \hspace{1cm} (34)

If \( u = -(CB)^{-1}K \text{sign}(s) + u_{eq} \), \( s^T \dot{s} = -\sum_{i=1}^{m} k_i |s_i| < 0 \)
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3 Higher order sliding mode
Sliding Mode Control: First order sliding mode

Sliding mode equivalent dynamics

\[ \dot{s} = \frac{\partial s}{\partial x} (f(x) + g(x)u) = \mathcal{L}_f s + \mathcal{L}_g s u \]

Equivalent control if \( \mathcal{L}_g s \) invertible

\[ \dot{s} = 0 \iff u_{eq} = - (\mathcal{L}_g s)^{-1} \mathcal{L}_f s \quad (35) \]

Thus the equivalent dynamics are

\[ \dot{x} = f(x) + g(x) \left( - (\mathcal{L}_g s)^{-1} \mathcal{L}_f s \right) \quad (36) \]

\[ = \left( \text{Id} - g(x) \left( - (\mathcal{L}_g s)^{-1} \frac{\partial s}{\partial x} \right) \right) f(x) \quad (37) \]

\( (\text{Id} - g(x) \left( - (\mathcal{L}_g s)^{-1} \frac{\partial s}{\partial x} \right)) \) is a projection operator
Sliding Mode Control: First order sliding mode

Let us consider a linear system

$$\dot{x} = Ax + Bu$$  \hspace{1cm} (38)$$

Equivalent control if $CB$ invertible

$$u_{eq} = -(CB)^{-1}CAx$$  \hspace{1cm} (39)$$

$$\dot{x} = (Id - B(CB)^{-1}C)Ax$$  \hspace{1cm} (40)$$

$$= A_{eq}x$$  \hspace{1cm} (41)$$

$$A_{eq} = (Id - B(CB)^{-1}C)A$$  \hspace{1cm} (42)$$
Sliding Mode Control: First order sliding mode

Sliding mode equivalent dynamics

Using a change of coordinates one can obtain \((B_2 \in \mathcal{M}_m(\mathbb{R}))\)

\[
B = \begin{pmatrix} 0 \\ B_2 \end{pmatrix}, \quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}.
\]

\[
A_{eq} = \begin{pmatrix} A_{11} \\ -C_2^{-1}C_1A_{11} \\ -C_2^{-1}C_1A_{12} \end{pmatrix} \quad \text{(43)}
\]

\[
= P^{-1} \begin{pmatrix} A_{11} - A_{12}C_2^{-1}C_1 & A_{12} \\ 0 & 0 \end{pmatrix} P \quad \text{(44)}
\]

\(A_{eq}\) has at least \(m\) zero eigenvalues and at most \(n-m\) non-zero ones.
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Classical first order Sliding Mode
Higher order sliding mode
Attractivity condition and invariance condition of the sliding manifold
Sliding mode equivalent dynamics
Robustness with respect to matched disturbance

**Sliding Mode Control**: First order sliding mode
Robustness with respect to matched disturbance

\[
\dot{x} = Ax + Bu + p \tag{45}
\]

\[
p \in \text{span}(B) \tag{46}
\]

\text{(46)} is called the matching condition, thus we have \( p = Bp^* \).

Put (45) into a controllable canonical form (hereafter \( m = 1 \))

\[
\dot{x}_c = A_c x + B_c (u + p^*) \tag{47}
\]

\[
A_c = \begin{pmatrix}
0 & 1 & \ldots & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & 0 & 0 & 1 \\
-a_{c1} & \cdots & \cdots & \cdots & -a_{cn}
\end{pmatrix},
\]

\[
B_c = \begin{pmatrix}
0 \\
\vdots \\
0 \\
1
\end{pmatrix} \tag{48}
\]
Select a linear sliding manifold $S = \{ x \in \mathbb{R}^n : s(x) = 0 \}$ where

$$s(x) = x_{cn} + \sum_{i=1}^{n-1} a_i x_{ci}$$

$$\dot{s} = - \sum_{i=1}^{n} a_{ci} x_{ci} + u + p^* + \sum_{i=1}^{n-1} a_i x_{ci+1}$$  \hspace{1cm} (49)$$

$$= \sum_{i=1}^{n} a_{ci}^{\diamond} x_{ci} + u + p^*, \quad a_{ci}^{\diamond} = -a_{ci} + a_{ci-1}$$  \hspace{1cm} (50)$$
Sliding Mode Control: First order sliding mode
Robustness with respect to matched disturbance

\textbf{Control}

\begin{align*}
  u &= -k \text{sign}(s) - \sum_{i=1}^{n} a_{ci}^{\bigodot} x_{ci} \\
  s \dot{s} &= -k |s| + |p^{*}| |s|,
\end{align*}

(51)

(52)

If the disturbance is bounded \( \sup |p^{*}| < \infty \), then take

\begin{align*}
  k &= \mu + \sup |p^{*}| \\
  s \dot{s} &< -\mu |s|
\end{align*}

(53)
Sliding Mode Control: First order sliding mode

Robustness with respect to matched disturbance

Equivalent dynamics

\[ \frac{\dot{x}_c}{1} = x_c^2 \quad (53) \]

\[ \vdots = \vdots \quad (54) \]

\[ \frac{\dot{x}_{cn-2}}{1} = x_{cn-1} \quad (55) \]

\[ \frac{\dot{x}_{cn-1}}{1} = x_{cn} = -\sum_{i=1}^{n-1} a_i x_{ci} \quad (56) \]

\( a_i \) (Hurwitz) \( \Leftrightarrow \) No influence of the perturbation once the sliding manifold is reached (only the hitting phase is influenced)
**Sliding Mode Control: First order sliding mode**

Robustness with respect to matched disturbance

**Example: double integrator**

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= u + p, \sup |p| < \infty
\end{align*}
\]  

**Sliding manifold:** \( S = \{ x \in \mathbb{R}^n : s(x) = 0 \} \), \( s(x) = x_2 + a_1 x_1 \)

**Compute Equivalent control** (without disturbance \( p = 0 \))

\[
\dot{s} = 0 = u_{eq} + a_1 x_2 \iff u_{eq} = -a_1 x_2
\]
**Sliding Mode Control : First order sliding mode**
Robustness with respect to matched disturbance

**Example: double integrator**

Control driving the solutions to $S$ in finite time

\[ u = u_{eq} + u_{disc}, \quad u_{disc} = -k \text{sign}(s) \]

\[ s\dot{s} = s(u_{disc} + p) < -\mu|s|, \quad k = \mu + \sup |p|. \]
**Sliding Mode Control: First order sliding mode**

**Advantages**
- Insensibility against perturbations (matching perturbations)
- The choice of surface $s(x, t) = 0$ allows to choose a priori the closed-loop dynamics

**Disadvantages**
- Chattering phenomenon
- $s(x, t)$ must have a relative degree equal to 1 wrt. $u$
- The trajectories are not robust against perturbations during the reaching phase
**Advantages**

- Insensitivity against perturbations (matching perturbations)
- The choice of surface $s(x, t) = 0$ allows to choose *a priori* the closed-loop dynamics

**Disadvantages**

- Chattering phenomenon
- $s(x, t)$ must have a relative degree equal to 1 with respect to $u$
- The trajectories are not robust against perturbations during the reaching phase
Objective
To constrain the system trajectories to evolve onto the sliding surface:

\[ S_r = \left\{ x \in \mathbb{R}^n : s = \dot{s} = \ldots = s^{(r-1)} = 0 \right\} \]
Higher Order Sliding Mode Control

Introduced by A. Levant (Ph. D. supervisor Emel’yanov) in 87

Ideal:

\[ S_r = \left\{ x \in \mathbb{R}^n : s = \dot{s} = \ldots = s^{(r-1)} = 0 \right\} \]

Real:

\[
\begin{align*}
|s| &= O(T_s^r) \quad (59) \\
|\dot{s}| &= O(T_s^{r-1}) \quad (60) \\
\ldots &= \ldots \quad (61) \\
|s^{(r-1)}| &= O(T_s) \quad (62) \\
T_s &= \text{sampling period} \quad (63)
\end{align*}
\]

With respect to a bounded deterministic Lebesgue-measurable noise (bounded by \( \varepsilon \)): 

\[ |s| = O(\varepsilon^{1/2^{r-1}}) \]
Higher Order Sliding Mode Control

Advantages:

- Robustness w.r.t. bounded matching perturbation,
- Reduce the of the sliding dynamics up to at most \((n - r)\) (in fact if counting the added integrators exactly \((n - r)\)).
- Finite Time convergence to \(S_r\),
- Chattering reduction (sometimes see relative degree of \(s\)),
- Higher convergence accuracy.
Higher Order Sliding Mode Control

\[ S_r = \left\{ x \in \mathbb{R}^n : s = \dot{s} = \ldots = s^{(r-1)} = 0 \right\} \]  

Let the set \( S_r \) be non-empty and assume that it consists of Filippov’s trajectories of the discontinuous dynamic system.

**Definition**

Any motion (Filipov sense) in the set \( S_r \) is called an \( r \)-sliding mode with respect to the constraint function \( s \).
Higher Order Sliding Mode Control

Sliding mode and relative degree

\[ \dot{x} = f(t, x, u), \quad s = s(t, x) \]

Theorem (H. Sira-Ramirez 89)

A first order sliding mode exists iff the relative degree of \( s \) w.r.t. the above defined system is one.

Equivalent dynamics is stable \( \iff \) system is minimum phase w.r.t. \( s \).

Relative degree \( r \) strictly greater than one: Only an \( r \)-sliding mode algorithm leads to a finite time convergence on the sliding manifold.
Problem: find algorithms ensuring higher order sliding modes. There exist for $r = 1, 2$ and $3$ for any $r > 3$ there no satisfactory constructive algorithm (only the structure is proposed and existence is proved for large enough parameters).

Ideal Algorithms:
- The necessary information increase with the order
- Twisting and Super-twisting [Levant]
- Sub-optimal [Bartolini]
- Nested HOSM [Levant]
- Quasi-continuous HOSM [Levant]

Real Algorithms:
- Good approximation for 2nd order
- Drift algorithm [Emel’yanov]
- Discretized version of ideal ones
Higher Order Sliding Mode Control
2-order sliding mode algorithms

\[ \ddot{s} = a(t, x) + b(t, x, u)u \]

Hypothesis:

1. For any continuous \( u(t) \) s.t. \(|u| \leq U_M, U_M > 1\) the solution of the system is well defined for all \( t \).
2. \( \exists u_1 \in (0, 1) \) s.t. for any continuous function \( u(t) \) with \(|u(t)| > u_1\), \( \exists t_1, \text{s.t } s(t)u(t) > 0\) for each \( t > t_1 \). \( (u(t) = -\text{sign}(s(t_0)), \text{enforces } s = 0 \text{ in finite time}) \)
3. \( \exists s_0 > 0, u_0 < 1, \Gamma_m > 0, \Gamma_M > 0 \) such that if \(|s(t, x)| < s_0\) then

\[ 0 < \Gamma_m \leq |b(t, x, u)| \leq \Gamma_M, \forall |u| \leq U_M, x \in \mathcal{X} \]  \hfill (65)

and the inequality \(|u| > u_0\) entails \( \dot{s}u > 0 \).
4. \( \exists A > 0 \) s.t within \(|s(t, x)| < s_0\) the following inequality holds \( \forall t, x \in \mathcal{X}, |u| \leq U_M \)

\[ |a(t, x)| \leq A \]  \hfill (66)
Higher Order Sliding Mode Control

2-order sliding mode algorithms: Twisting Algorithm (TA) [Levant]

\[
\frac{\partial r(s)}{s} = 1
\]

\[y_1 = s, y_2 = \dot{s}, \text{after some transient}\]

\[|a(t, x)| \leq A, 0 < \Gamma_m \leq b(t, x, u) \leq \Gamma_M, A > 0.\]

\[
\begin{cases}
\dot{y}_1 = y_2 \\
\dot{y}_2 = a(t, x) + b(t, x, u)\dot{u}
\end{cases}
\]

(67)

with \(y_2(t)\) unmeasured but with a possibly known sign.
Higher Order Sliding Mode Control

2-order sliding mode algorithms: Twisting Algorithm (TA) [Levant]

\[
\dot{u}(t) = \begin{cases} 
-u & \text{if } |u| > 1, \\
-\lambda_m \text{sign}(y_1) & \text{if } y_1 y_2 \leq 0; |u| \leq 1, \\
-\lambda_M \text{sign}(y_1) & \text{if } y_1 y_2 > 0; |u| \leq 1. 
\end{cases}
\]  

(68)

Sufficient conditions:

\[
\begin{align*}
\lambda_M &> \lambda_m \\
\lambda_m &> \frac{4\Gamma_M}{s_0} \\
\lambda_m &> \frac{A}{\Gamma_m} \\
\Gamma_m \lambda_M - A &> \Gamma_M \lambda_m + A.
\end{align*}
\]  

(69)
Higher Order Sliding Mode Control

2-order sliding mode algorithms: Twisting Algorithm (TA) [Levant]
Higher Order Sliding Mode Control

2-order sliding mode algorithms: Twisting Algorithm (TA) [Levant]

\[ \partial r(s) = 2 \]

\[ u(t) = \begin{cases} 
-\lambda_m \text{sign}(y_1) & \text{if } y_1y_2 \leq 0 \\
-\lambda_M \text{sign}(y_1) & \text{if } y_1y_2 > 0
\end{cases} \]
Higher Order Sliding Mode Control

2-order sliding mode algorithms: Twisting Algorithm (TA) [Levant]

Example

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= x_1 x_2 + u + p(t) \\
\sup_{t \in \mathbb{R}} |p(t)| &= \pi
\end{align*}
\]
Higher Order Sliding Mode Control

2-order sliding mode algorithms: Twisting Algorithm (TA) [Levant]

Example

Using (70): for

\[ s_1(x) = x_2 + ax_1 \]  \hspace{1cm} (74)

we have

\[ \dot{s}_1 = x_3 + ax_2, \]  \hspace{1cm} (75)
\[ \ddot{s}_1 = x_1x_2 + u + p(t) + ax_3 \]  \hspace{1cm} (76)
\[ \partial r(s_1) = 2, \]  \hspace{1cm} (77)

thus if \( s_1(x) = \dot{s}_1(x) = 0 \) in finite time then the equiv. dynamic is \( \dot{x}_1 = -ax_1 \) thus \( x_1(t) \to 0, x_2(t) \to 0, x_3(t) \to 0. \)
Higher Order Sliding Mode Control

2-order sliding mode algorithms: Twisting Algorithm (TA) [Levant]

Example

Using (70) + (71) \( \dot{x}_1 = x_2, \dot{x}_2 = x_3 \): for

\[
s_2(x) = x_3 + (\omega_n^2 x_1 + 2\zeta \omega_n x_2)
\]  

(78)

we have

\[
\dot{s}_2 = x_1 x_2 + u + p(t) + (\omega_n^2 x_2 + 2\zeta \omega_n x_3),
\]

(79)

\[
\partial r(s_2) = 1,
\]

(80)

thus if \( s_2(x) = 0 \) in finite time then the equiv. dynamics is

\( \dot{x}_1 = x_2, \dot{x}_2 = -(\omega_n^2 x_1 + 2\zeta \omega_n x_2) \) thus

\( x_1(t) \rightarrow 0, x_2(t) \rightarrow 0, x_3(t) \rightarrow 0. \)
**Higher Order Sliding Mode Control**

2-order sliding mode algorithms: **Twisting Algorithm (TA)** [Levant]

---

**Example**

**Case 1: 1rst order SM using** $s_2$

\[ s_2(x) = x_3 + (\omega_n^2 x_1 + 2\zeta \omega_n x_2), \]
\[ \dot{s}_2 = x_1 x_2 + u + p(t) + (\omega_n^2 x_2 + 2\zeta \omega_n x_3), \]

**Compute equiv. control (without** $p$**):**

\[ u_{eq} = -x_1 x_2 - (\omega_n^2 x_2 + 2\zeta \omega_n x_3) \]

\[ u = u_{eq} + u_{disc}, \]
\[ u_{disc} = -k \text{sign}(s), k > \pi + \mu \]
\[ s \dot{s} = -k|s| + sp < -\mu|s| \]
Higher Order Sliding Mode Control

2-order sliding mode algorithms: Twisting Algorithm (TA) [Levant]

Example

Case 2: 2nd order SM using $s_2$. Since $\partial r(s_2) = 1$ we add $\int$:

Chattering removal

\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= x_1x_2 + u + p(t) \\
\dot{u} &= v
\end{align*}

(81) \hspace{1cm} (82) \hspace{1cm} (83) \hspace{1cm} (84)
Higher Order Sliding Mode Control

2-order sliding mode algorithms: Twisting Algorithm (TA) [Levant]

Thus we have

\[ s_2(x) = x_3 + (\omega_n^2 x_1 + 2\zeta \omega_n x_2), \]
\[ \dot{s}_2 = x_1 x_2 + u + p(t) + (\omega_n^2 x_2 + 2\zeta \omega_n x_3), \]
\[ \ddot{s}_2 = x_1 x_3 + x_2^2 + v + \dot{p} + (\omega_n^2 x_3 + 2\zeta \omega_n (x_1 x_2 + u + p(t))) \]

\[ = a(x) + v + (\dot{p} + 2\zeta \omega_np) \]
Higher Order Sliding Mode Control

2-order sliding mode algorithms: Twisting Algorithm (TA) [Levant]

Example

Compute equiv. control (without $p$): $v_{eq} = -a(x)$

$$
v = v_{eq} + v_{disc},
$$
$$
v_{disc} = TA(s_2) = \begin{cases} 
-\lambda_m \text{sign}(s_2) & \text{if } s_2 \dot{s}_2 \leq 0 \\
-\lambda_M \text{sign}(s_2) & \text{if } s_2 \dot{s}_2 > 0
\end{cases}
$$

$$
u = \int v \in C^0
$$
Higher Order Sliding Mode Control

2-order sliding mode algorithms: Twisting Algorithm (TA) [Levant]

Example

Case 3: 2nd order SM using $s_1$ Since $\partial r(s_1) = 2$ we can directly use TA (Chattering!!)

$$\dot{s}_1 = x_3 + ax_2,$$
$$\ddot{s}_1 = x_1 x_2 + u + p(t) + ax_3$$

Compute equiv. control (without $p$): $v_{eq} = -x_1 x_2 - ax_3$

$$u = u_{eq} + u_{disc},$$

$$u_{disc} = TA(s_1) = \begin{cases} -\lambda_m \text{sign}(s_1) & \text{if } s_1 \dot{s}_1 \leq 0 \\ -\lambda_M \text{sign}(s_1) & \text{if } s_1 \dot{s}_1 > 0 \end{cases}$$
Higher Order Sliding Mode Control

2-order sliding mode algorithms: **Twisting Algorithm (TA) [Levant]**

Convergence acceleration TA + pole placement (at the same location !!)

\[
u = -\alpha^2 s - 2\alpha \dot{s} + \begin{cases} 
-\lambda_m \text{sign}(s) & \text{if } s\dot{s} \leq 0 \\
-\lambda_M \text{sign}(s) & \text{if } s\dot{s} > 0 
\end{cases}
\]
Higher Order Sliding Mode Control
2-order sliding mode algorithms: sub-optimal [Bartolini et al.]

\[ y_1 = s, y_2 = \dot{s}, \text{ after some transient} \]
\[ |a(t, x)| \leq A, 0 < \Gamma_m \leq b(t, x, u) \leq \Gamma_M, A > 0. \]
\[
\begin{cases}
  \dot{y}_1 = y_2 \\
  \dot{y}_2 = a(t, x) + b(t, x, u)u.
\end{cases} \tag{87}
\]

\( (y_1, y_2) \) Trajectories are confined within limit parabolic arcs.
Control:

\[ v(t) = -\alpha(t)\lambda_M \text{sign}(y_1(t) - \frac{1}{2}y_{1M}), \]
\[ \alpha(t) = \begin{cases} 
  \alpha^* & \text{if} \ [y_1(t) - \frac{1}{2}y_{1M}][y_{1M} - y_1(t)] > 0 \\
  1 & \text{if} \ [y_1(t) - \frac{1}{2}y_{1M}][y_{1M} - y_1(t)] \leq 0
\end{cases}, \tag{88} \]

where \( y_{1M} \) is the last maximum of \( y_1(t) \), i.e. the last value of \( y_1 \)
for \( t \) s.t. \( y_2 = \dot{y}_1 = 0 \).
Higher Order Sliding Mode Control
2-order sliding mode algorithms: sub-optimal [Bartolini et al.]

Sufficient conditions:

$$\alpha^* \in (0, 1] \cap (0, \frac{3\Gamma_m}{\Gamma_M}),$$

$$\lambda_M > \max \left( \frac{\Phi}{\alpha^* \Gamma_m}, \frac{4\Phi}{3\Gamma_m - \alpha^* \Gamma_M} \right).$$  (89)
Higher Order Sliding Mode Control
2-order sliding mode algorithms: Super twisting Algorithm (STA) [Levant]

The control is given by:

\[ u(t) = u_1(t) + u_2(t) \]

\[ \dot{u}_1(t) = \begin{cases} -u & \text{if } |u| > 1 \\ -W \text{sign}(y_1) & \text{if } |u| \leq 1 \end{cases} \quad (90) \]

\[ u_2(t) = \begin{cases} -\lambda s_0 \rho \text{sign}(y_1) & \text{if } |y_1| > s_0 \\ -\lambda |y_1| \rho \text{sign}(y_1) & \text{if } |y_1| \leq s_0 \end{cases} \]
Higher Order Sliding Mode Control

2-order sliding mode algorithms: Super twisting Algorithm (STA) [Levant]

Sufficient conditions:

\[
\begin{align*}
W & > \frac{\Phi}{\Gamma_m} \\
\lambda^2 & \geq \frac{4A}{\Gamma_m^2} \frac{\Gamma_M(W+A)}{\Gamma_m(W-A)} \\
0 & < \rho \leq \frac{1}{2}
\end{align*}
\] (91)
Higher Order Sliding Mode Control

2-order sliding mode algorithms: Super twisting Algorithm (STA) [Levant]

Simplified version if $b$ does not depend on control, $u$ does not need to be bounded and $s_0 = \infty$:

$$u = -\lambda |s|^\rho \text{sign}(y_1) + u_1,$$

$$\dot{u}_1 = -W \text{sign}(y_1).$$
Higher Order Sliding Mode Control

2-order sliding mode algorithms: Drift Algorithm (DA) [Emelyanov]

Control ($s$ relative degree is 1):

\[
\dot{u} = \begin{cases} 
-u & \text{if } |u| > 1 \\
-\lambda_m \text{sign}(\Delta y_{1i}) & \text{if } y_1 \Delta y_{1i} \leq 0; \ |u| \leq 1 \\
-\lambda_M \text{sign}(\Delta y_{1i}) & \text{if } y_1 \Delta y_{1i} > 0; \ |u| \leq 1 
\end{cases}
\]  

(92)

where $\lambda_m > 0$, $\lambda_M > 0$ are proper positive constants such that $\lambda_m < \lambda_M$ and $\lambda_M^\lambda_m$ is sufficiently large, and

$\Delta y_{1i} = y_1(t_i) - y_1(t_i - \tau), \ t \in [t_i, t_{i+1}).$
Higher Order Sliding Mode Control

2-order sliding mode algorithms: Drift Algorithm (DA) [Emelyanov]

Similar controller (when $s$ is relative degree 2):

$$
\dot{u} = \begin{cases} 
-\lambda_m \text{sign}(\Delta y_{1i}) & \text{if } y_1 \Delta y_{1i} \leq 0 \\
-\lambda_M \text{sign}(\Delta y_{1i}) & \text{if } y_1 \Delta y_{1i} > 0 
\end{cases}
$$
Higher Order Sliding Mode Control

r-order sliding mode algorithms: Homogeneous SM [Levant]

Let $p$ least common multiple of $1, 2, \ldots, r$

\begin{align*}
\varphi_{0,r} &= s \\
N_{1,r} &= |s|^{\frac{r-1}{r}} \\
\varphi_{i,r} &= s^{(i)} + \beta_i N_{i,r} \text{sign}(\varphi_{i-1,r}) \\
N_{i,r} &= (|s|^{\frac{p}{r}} + \ldots + |s^{(i-1)}|^{\frac{p}{r-i+1}}) \frac{r-i}{p} \\
u &= -\lambda \text{sign}(\varphi_{r-1,r}(s, \dot{s}, \ldots, s^{(r)}))
\end{align*}

\(\beta_i\) hard to find but can be set in advance and \(\lambda\) should be large enough!
Higher Order Sliding Mode Control

r-order sliding mode algorithms: Quasi continuous Homogeneous SM [Levant]

\[ s^{(r)} \in [-C, C] + [K_m, K_M]u \]

\[ \varphi_{0,r} = s \quad N_{0,r} = |s| \quad \Psi_{0,r} = \frac{\varphi_{0,r}}{N_{0,r}} = \text{sign}(s) \]

\[ \varphi_{i,r} = s^{(i)} + \beta_{i} N_{i-1,r}^{\frac{r-i}{r-i+1}} \Psi_{i-1,r} \quad N_{i,r} = |s^{(i)}| + \beta_{i} N_{i-1,r}^{\frac{r-i}{r-i+1}} \Psi_{i-1,r} \quad \Psi_{i,r} = \frac{\varphi_{i,r}}{N_{i,r}} \]

\[ u = -\lambda \text{sign}(\Psi_{r-1,r}(s, s', \ldots, s^{(r)})) \]

\[ \beta_i \text{ hard to find but can be set in advance and } \lambda \text{ should be large enough!} \]

1rst to HOSM
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Higher Order Sliding Mode Control

\[ \dot{x} = f(x) \]  \hspace{1cm} (99)

where \( f \) is a continuous vector field or differential inclusion

\[ \dot{x} \in F(x) \]  \hspace{1cm} (100)

where \( F \) is set valued map.
Sufficient condition for ODE (or DI) to be finite time stable:

**Lemma**

*Suppose there exists a Lyapunov function $V(x)$ defined on a neighborhood $\mathcal{U} \subset \mathbb{R}^n$ of the origin of system (99) and some constants $\tau, \gamma > 0$ and $0 < \beta < 1$ such that*

$$\frac{d}{dt}V(x)|_{(99)} \leq -\tau V(x)^\beta + \gamma V(x), \quad \forall x \in \mathcal{U}\setminus\{0\}.$$

*Then the origin of system (99) is FTS. The set $\Omega = \left\{ x \in \mathcal{U} : V(x)^{1-\beta} < \frac{\tau}{\gamma} \right\}$ is contained in the domain of attraction of the origin. The settling time satisfies*

$$T(x) \leq \frac{\ln\left(1-\frac{\gamma}{\tau}V(x)^{1-\beta}\right)}{\gamma(\beta-1)}, \quad x \in \Omega.$$
Let $\lambda > 0$, $r_i > 0$, $i \in \{1, \ldots, n\}$ called weights one can define:

- the vector of weights $r = (r_1, \ldots, r_n)^T$,
- the dilation matrix

$$\Lambda_r = \text{diag}\{\lambda^{r_i}\}_{i=1}^n,$$  \hspace{1cm} (101)

note that $\Lambda_r x = (\lambda^{r_1} x_1, \ldots, \lambda^{r_i} x_i, \ldots, \lambda^{r_n} x_n)^T$.

- let $r$ denotes the finite product $r_1 r_2 \ldots r_n$ then the $r$–homogeneous norm of $x \in \mathbb{R}^n$ is defined by:

$$n_r(x) = (|x_1|^{r_1} + \ldots + |x_i|^{r_i} + \ldots + |x_n|^{r_n})^{\frac{1}{r}}. \hspace{1cm} (102)$$
Higher Order Sliding Mode Control

Definition

A function \( h : \mathbb{R}^n \to \mathbb{R} \) is \( r \)-homogeneous with degree \( d_{r,h} \in \mathbb{R} \) if for all \( x \in \mathbb{R}^n \) we have (Hermes 90):

\[
\lambda^{-d_{r,h}} h(\Lambda_r x) = h(x). \tag{103}
\]

When such a property holds, we write \( \deg_r(h) = d_{r,h} \).
Higher Order Sliding Mode Control

Let us note that for any positive real number $\lambda$:

$$\lambda^{-1} n_r(\Lambda_r x) = n_r(x),$$  \hspace{1cm} (104)

this is $\text{deg}_r(n_r) = 1$. Let us introduce the following compact set

$$S_r = \{ x \in \mathbb{R}^n : n_r(x) = 1 \},$$  \hspace{1cm} (105)
Remark

In fact in stead of dealing with $S_r$, one can take any closed curve properly chosen diffeomorphic to $\mathbb{S}^{n-1}$. 
Such homogeneity notion can be also defined for vector fields, ordinary differential system (99)

Definition

A vector field \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is \( r \)-homogeneous with degree \( d_{r,f} \in \mathbb{R} \), with \( d_{r,f} > - \min_{i \in \{1, \ldots, n\}} (r_i) \) if for all \( x \in \mathbb{R}^n \) we have (see Hermes 90):

\[
\lambda^{-d_{r,f}} \Lambda_r^{-1} f(\Lambda_r x) = f(x),
\]  

(106)

which is equivalent to all \( i \)-th component \( f_i \) being \( r \)-homogeneous function of degree \( r_i + d_{r,f} \). When such a property holds, we write \( \text{deg}_r(f) = d_{r,f} \). The system (99) is \( r \)-homogeneous of degree \( d_{r,f} \) if the vector field \( f \) is homogeneous of degree \( d_{r,f} \).
Theorem

If the system (99) is locally AS and \( r \)-homogeneous with negative degree then it is FTS.
\[ \dot{x} = f(x) + g(x)u, s(t, x) \]

with \( \partial r(s) = \text{cte} = \rho \in \mathbb{N}^+ \). HOSM \iff FTS for the following system:

\[
\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= z_3 \\
&\vdots \\
\dot{z}_{\rho-1} &= z_\rho \\
\dot{z}_\rho &= a(x, t) + b(x, t)u
\end{aligned}
\]  

(107)

\[
\begin{aligned}
z &= [z_1, z_2, \ldots, z_{\rho-1}, z_\rho]^T \\
a(x, t) &= L^\rho_f s(x, t) \\
b(x, t) &= L_g L^\rho_f s(x, t)
\end{aligned}
\]
Higher Order Sliding Mode Control

\[ a, b \text{ have known nominal part denoted by } \bar{a}, \bar{b} \text{ their unknown part being described by } \delta_a, \delta_b, \text{ this is:} \]

\[
\begin{align*}
    a &= \bar{a} + \delta_a \\
    b &= \bar{b} + \delta_b
\end{align*}
\]
Higher Order Sliding Mode Control

Assumptions: Assume that the nominal part $\bar{b}$ is invertible.

Using:

$$u = \bar{b}^{-1} (w - \bar{a})$$  \hspace{1cm} (108)

where $w \in \mathbb{R}$ is the knew input (107) leads to:

$$\begin{cases}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= z_3 \\
& \vdots \\
\dot{z}_{\rho-1} &= z_{\rho} \\
\dot{z}_{\rho} &= \vartheta(x, t) + (1 + \zeta(x, t)) w
\end{cases}$$

(109)

where $\vartheta, \zeta$ are given by:

$$\begin{cases}
\vartheta &= \delta_a - \delta_b \bar{b}^{-1} \bar{a} \\
\zeta &= \delta_b \bar{b}^{-1}
\end{cases}$$
Assumption: $\vartheta(x, t), \zeta(x, t)$ bounded: $\exists a(x) > 0$ and $\exists 0 < b \leq 1$

s.t.:

$$\begin{cases} 
|\vartheta(x, t)| & \leq a(x) \\
|\zeta(x, t)| & \leq 1 - b 
\end{cases} \quad (110)$$

Idea: FTS the unperturbed chain of integrator using homogeneity
Higher Order Sliding Mode Control

\[
\begin{align*}
\dot{z}_1 &= z_2 \\
&\vdots \\
\dot{z}_\rho &= w \\
\end{align*}
\]
Higher Order Sliding Mode Control

Theorem (Bhat 2005)

Let \( k_1, \ldots, k_\rho \) positives ctes s.t. \( p^\rho + k_\rho p^{\rho-1} + \ldots + k_2 p + k_1 \) is Hurwitz. Then \( \exists \epsilon \in (0, 1) \) s.t \( \forall \nu \in (1 - \epsilon, 1) \), (111) is FTS by:

\[
    w(z) = -k_1 \text{sign}(z_1)|z_1|^{\nu_1} - \ldots - k_\rho \text{sign}(z_\rho)|z_\rho|^{\nu_\rho}
\]  

(112)

where \( \nu_1, \ldots, \nu_\rho \) are given by:

\[
    \nu_{i-1} = \frac{\nu_i \nu_{i+1}}{2
\nu_{i+1} - \nu_i}, \quad i = 2, \ldots, \rho,
\]  

(113)

with \( \nu_\rho = \nu \) et \( \nu_{\rho+1} = 1 \).
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Objective

To remove the reaching phase

To guarantee the robustness properties against perturbations in the model from the initial time instance

Philosophy

To choose the sliding variable such that the system trajectories are already on the sliding surface at the initial time instance
Integral Sliding Mode Control

**Objective**

To remove the reaching phase

To guarantee the robustness properties against perturbations in the model from the initial time instance

**Philosophy**

To choose the sliding variable such that the system trajectories are already on the sliding surface at the initial time instance
Integral Sliding Mode Control

\( w_{\text{nom}}(z) \) FTS the unperturbed system \((111)\).

\( w_{\text{disc}}(z) \) is built to cope with \( \vartheta(x,t) \) and \( \zeta(x,t) \) for \((109)\).

Leading to a \( \rho \)– order sliding mode for \( s(x,t) \).
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2. **Classical first order Sliding Mode**

3. **Higher order sliding mode**
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Problem setup

Reference trajectory

\[
\begin{bmatrix}
\dot{x}_{ref} \\
\dot{y}_{ref} \\
\dot{\theta}_{ref}
\end{bmatrix}
= \begin{bmatrix}
\cos \theta_{ref} & 0 \\
\sin \theta_{ref} & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
v_{ref} \\
w_{ref}
\end{bmatrix}
\]

Objective

Individual tracking of the optimal planned trajectory for each robot \(i\)

To stabilize the tracking errors:

\[
\begin{bmatrix}
e_x \\
e_y \\
e_{\theta}
\end{bmatrix}
= \begin{bmatrix}
x - x_{ref} \\
y - y_{ref} \\
\theta - \theta_{ref}
\end{bmatrix}
\]
Problem setup

Reference trajectory

\[
\begin{bmatrix}
\dot{x}_{ref} \\
\dot{y}_{ref} \\
\dot{\theta}_{ref}
\end{bmatrix}
= \begin{bmatrix}
\cos \theta_{ref} & 0 \\
\sin \theta_{ref} & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
v_{ref} \\
w_{ref}
\end{bmatrix}
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Objective

Individual tracking of the optimal planned trajectory for each robot \(i\)

To stabilize the tracking errors:

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\begin{bmatrix}
e_x \\
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\end{bmatrix}
= \begin{bmatrix}
x - x_{ref} \\
y - y_{ref} \\
\theta - \theta_{ref}
\end{bmatrix}
\]
Problem setup

Difficulties

Presence of perturbations and parametric uncertainties in the model:

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{\theta}
\end{bmatrix} =
\begin{bmatrix}
\cos \theta & 0 \\
\sin \theta & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
v \\
w
\end{bmatrix} + p(q, t)
\]
Assumptions

- Perturbations satisfy the matching condition
- Perturbations are bounded by known positive functions
- Reference velocities are continuous and bounded
- No stop point
The tracking errors asymptotically converge toward zero under:

\[ u = u_{\text{nom}} + u_{\text{disc}} \]

**Continuous term** \( u_{\text{nom}} [\text{Jiang et al., 2001}] \)

\[ u_{\text{nom}} \] stabilize the tracking errors without perturbation

\[
\begin{bmatrix}
    v_{\text{ref}} \cos e_3 + \mu_3 \tanh e_1 \\
    w_{\text{ref}} + \frac{\mu_1 v_{\text{ref}} e_2}{1+e_1^2+e_2^2} \frac{\sin e_3}{e_3} + \mu_2 \tanh e_3
\end{bmatrix}
\]

with

\[
\begin{bmatrix}
    e_1 \\
    e_2 \\
    e_3
\end{bmatrix} =
\begin{bmatrix}
    -\cos \theta & -\sin \theta & 0 \\
    \sin \theta & -\cos \theta & 0 \\
    0 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
    e_x \\
    e_y \\
    e_\theta
\end{bmatrix}
\]
The tracking errors asymptotically converge toward zero under:

$$u = u_{\text{nom}} + u_{\text{disc}}$$

**Discontinuous term** $u_{\text{disc}}$

$u_{\text{disc}}$ reject the effect of the perturbation from the initial time instance

$$u_{\text{disc}} = \begin{bmatrix} -G_1(e) \text{sign}(\sigma_1) \\ -G_2(e) \text{sign}(-e_2\sigma_1 + \sigma_2) \end{bmatrix}$$

with $\sigma = [\sigma_1, \sigma_2]^T$ given by:

- $\sigma_0(e) = [-e_1, -e_3]^T$: linear combination of state
- integral part

$$\begin{cases} \dot{e}_{\text{aux}} = \begin{bmatrix} v_{\text{ref}} \cos e_3 \\ w_{\text{ref}} \end{bmatrix} - \begin{bmatrix} 1 & -e_2 \\ 0 & 1 \end{bmatrix} u_{\text{nom}}(e) \\ e_{\text{aux}} = -\sigma_0(e(0)) \end{cases}$$
The tracking errors asymptotically converge toward zero under:

\[ u = u_{nom} + u_{disc} \]

**Discontinuous term** \( u_{disc} \)

\( u_{disc} \) reject the effect of the perturbation from the initial time instance

\[
 u_{disc} = \begin{bmatrix}
 -G_1(e) \text{sign}(\sigma_1) \\
 -G_2(e) \text{sign}(-e_2\sigma_1 + \sigma_2) 
\end{bmatrix}
\]

with \( \sigma = [\sigma_1, \sigma_2]^T \) given by:

- \( \sigma_0(e) = [-e_1, -e_3]^T \): linear combination of state
- integral part

\[
\begin{align*}
\dot{e}_{aux} &= \begin{bmatrix}
 v_{ref} \cos \theta_3 \\
 w_{ref}
\end{bmatrix} - \begin{bmatrix} 1 & -e_2 \\
 0 & 1 \end{bmatrix} u_{nom}(e) \\
\sigma &= \sigma_0(e) + e_{aux} \\
e_{aux} &= -\sigma_0(e(0))
\end{align*}
\]
The tracking errors asymptotically converge toward zero under:

\[ u = u_{nom} + u_{disc} \]

**Discontinuous term** \( u_{disc} \)

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\begin{align*}
\dot{e}_{aux} &= \begin{bmatrix} v_{ref} \cos e_3 \\ w_{ref} \end{bmatrix} - \begin{bmatrix} 1 & -e_2 \\ 0 & 1 \end{bmatrix} u_{nom}(e) \\
\end{align*}
\]

\[
\begin{align*}
e_{aux} &= -\sigma_0(e(0))
\end{align*}
\]
Experimental results: Algo. 1

Single nominal control

![Graph 1](image1)

ISM C

![Graph 2](image2)
Limitations

- conservative assumptions
- discontinuities on velocities
- perturbations must satisfy the matching condition

Solution

Practical stabilization using second order ISMC
Algo. 1

Limitations

- conservative assumptions
- discontinuities on velocities
- perturbations must satisfy the matching condition

Solution

Practical stabilization using second order ISMC

Figure 1:

\[ T \]
Experimental results

ISM of Order 1

ISM of Order 2
Experimental results

ISM of Order 1

ISM of Order 2

W. Perruquetti

1rst to HOSM
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Video
Differential Inclusion: Notion of solution
Classical first order Sliding Mode
Higher order sliding mode
FTS and homogeneity
Arbitray HOSM using ISM concept
Application to mobile robots

Video with 3 miabot

Video
Video with 7 miabots