Z-mappings

for mathematicians

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Prelude

Nonvariational, nonlinear equations discrete and continuous

$$\partial_t u = -Q(u) \qquad -\operatorname{div}(A\nabla u) = f \qquad \operatorname{Ric}(g) = f$$

$$H(x, \nabla u) = f \qquad \det D^2 u = f$$

Common structure to problems in network flows, optimal transport, optimal control, interface dynamics, submodular functions...

Dates back to Rheinboldt ('70s)

Ideal to prove **comparison principles**

Natural **algorithm**: Jacobi

Persists through regularization and continuous $\leftrightarrow$ discrete

Outline

1. Z-mappings

2. M-mappings

3. The Jacobi algorithm

4. Examples

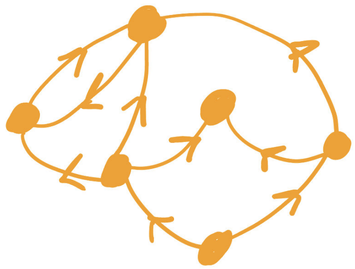
1. Z-mapping

Want to solve equations of the form

$$Q(u) = 0,$$

$u\colon X \to \mathbb{R}$, X finite or $X \subseteq \mathbb{R}^d$

$Q\colon \mathbb{R}^X \to \mathbb{R}^X$



X

DEFINITION

Q is a *Z-mapping* if for all $u, \tilde{u}$, for all x,

$$\begin{cases} u \le \tilde{u} \\ u(x) = \tilde{u}(x) \end{cases} \implies Q_x(\tilde{u}) \le Q_x(u).$$

Toy example: $Q_i(u) = 2u_i - u_{i+1} - u_{i-1}$.

Examples

Square matrix A, solve

$$Au = f$$

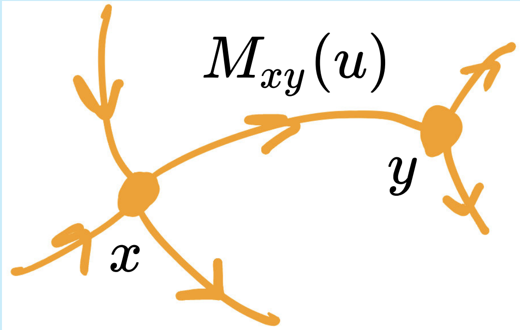
Let $Q(u) = Au - f$. Then

$$Q \text{ is a Z-mapping} \quad \Longleftrightarrow \quad a_{ij} \leq 0 \text{ for all } i \neq j.$$

Z-matrix

On a network (X, A),

$$-\mathrm{div} M(u) = f$$



$M : \mathbb{R}^X \to \mathbb{R}^A$ "flow mapping" such that:

$M_{xy}(u)$ increasing in u_y decreasing in u_x.

Ex: $M_{xy}(u) = u_y - u_x$,

$$M_{xy}(u) = e^{u_y - u_x}$$

Examples

$$\partial_t u = -Q(u)$$

If Q is a Z-mapping then so is the spacetime mapping

$$\hat{Q}_{t,x}(u) := \frac{u_{t,x} - u_{t-1,x}}{\tau} + Q_x(u_t)$$

Dynamic programming principle:

$$u(x) = \min_y c(x,y) + u(y) =: T(u)$$

$Q(u) := u - T(u)$ is a Z-mapping.

(Shortest path, stochastic shortest path, Markov Decision Process)
(Eikonal equation, Hamilton–Jacobi Bellman, viscosity solutions)

2. M-mappings

DEFINITION

A Z-mapping Q is an M-mapping if it satisfies a *comparison principle*

$$Q(u) \le Q(\tilde{u}) \implies u \le \tilde{u}$$

$$Q(u) = f,\, Q(\tilde{u}) = \tilde{f} \text{ with } f \le \tilde{f} \implies u \le \tilde{u}$$

Uniqueness of $Q(u) = f$

Maximum principle

Strong comparison principle: if V is a strongly connected component of an M-mapping Q then

$$Q(u) \le Q(\tilde{u}) \implies u = \tilde{u} \text{ or } u < \tilde{u} \text{ on } V.$$

Proving a comparison principle

Basic idea: combine the Z-property and "some isotonicity"

THEOREM

Berry, Gandhi, Haile '2013; Chen, Choo, Galichon, Weber '2021

Suppose that a Z-mapping Q satisfies: for all $V \subseteq X$,

$$\begin{cases} u < \tilde{u} \text{ on } V \\ u = \tilde{u} \text{ on } X \setminus V \end{cases} \implies \sum_{x \in V} Q_x(u) < \sum_{x \in V} Q_x(\tilde{u})$$

Then Q is an M-mapping.

Example: $Q(u) = -\operatorname{div}(a(x,u)\nabla u) - f, \quad a(x,u) \geq c_0 > 0$

$$\int_V Q(u)dx = \int_{\partial V} a(x,u)\langle \nabla u, -n \rangle$$

Works for quasilinear PDEs, Monge–Ampère, semi-discrete optimal transport, entropic transport...

3. The Jacobi algorithm

$$Q(u) = 0$$

DEFINITION

The Jacobi transform $J\colon \mathbb{R}^X \to \mathbb{R}^X$ is defined by

$$Q_x(J_x(u), u_{-x}) = 0.$$

“coordinate update”

Also related: Gauss–Seidel

ALGORITHM

Jacobi algorithm

$$u_{n+1} = J(u_n)$$

Properties of Jacobi

PROPOSITION Rheinbolt '70

Let Q be a continuous Z-mapping.

1. If $Q(u_n) \leq 0$ then
$$Q(u_{n+1}) \leq 0 \quad \text{and} \quad u_n \leq u_{n+1}$$
2.
$$u_n \leq \tilde{u}_n \implies u_{n+1} \leq \tilde{u}_{n+1}$$

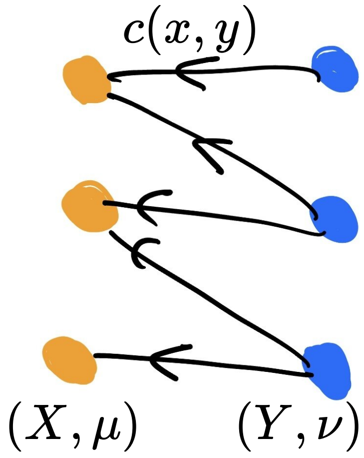
1: Useful for algorithm or showing existence of solutions

"method of subsolutions", "Perron's method"...

2: Useful when sandwich $v_0 \leq u_0 \leq w_0$, then
$$v_n \leq u_n \leq w_n$$

4. Examples

Optimal transport (continuous/discrete/semi-discret/entropic), generated Jacobian equations



$$S_u(y) := \arg\max_x u(x) - c(x,y)$$

$$\boxed{S_{u\#}\nu = \mu}$$

$Q(u) = S_{u\#}\nu$ is a Z-mapping

☐ Monge–Ampère has a comparison principle

☐ Algorithm: Jacobi = Bertsekas' (naive) auction algorithm

☐ Discrete: Q is discontinuous $\rightarrow$ need regularization.

Real auction, or replace min by softmin: Entropic OT. ☐ Jacobi = Sinkhorn

Mean curvature motion

mean curvature motion = L^2 gradient flow of

$$P(u) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|\, dx$$

($u = 1_K$). Heat content:

$$P_\varepsilon(u) = -\frac{1}{\varepsilon}\iint G_\varepsilon(x-y)u(x)(1-u(y))\, dxdy + \int \chi(u(x))\, dx$$

submodular, nonconvex

$Q(u) = DP(u)$ is a Z-mapping.

□ Jacobi = MBO (Merriman–Bence–Osher ’92)

Thank you!