Nearly Tight Convergence Bounds for Semi-discrete Entropic Optimal Transport

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Introduction

Quadratic optimal transport (with entropic regularization)

Let $\mathcal{X}, \mathcal{Y} \subset \mathbb{R}^d$ compact, $\rho \in \mathcal{P}(\mathcal{X}), \mu \in \mathcal{P}(\mathcal{Y})$, $c(x,y) = \|x-y\|^2$, $\varepsilon \geq 0$:

$$\min_{\pi \in \Pi(\rho,\mu)} \langle c|\pi \rangle + \varepsilon \mathrm{KL}(\pi|\rho \otimes \mu). \qquad (\mathrm{P}_\varepsilon)$$

	Classical OT ($\varepsilon = 0$)	*Entropic* OT ($\varepsilon > 0$)
Numerical resolution?	Possibly difficult Network simplex/Auction/Newton/... "Bad" dependence on *support size* and/or d.	Generally less difficult Matrix scaling/Stochastic algorithms/... Dependence on ε.
Sample complexity? $\mathbb{E}\left\lvert (\mathrm{P}_\cdot)_{\hat{\rho}^n,\hat{\mu}^n} - (\mathrm{P}_\cdot)_{\rho,\mu} \right\rvert$	$O(n^{-1/d})$	$\lesssim \frac{1}{\varepsilon^{2(d+1)}} n^{-1/2}$
Geometry?	$\mathrm{W}_2(\rho,\mu) := \sqrt{\langle c\|\pi^{(\mathrm{P}_0)}\rangle}$ is a distance	$\mathrm{W}_{2,\varepsilon}(\rho,\mu) := \sqrt{\langle c\|\pi^{(\mathrm{P}_\varepsilon)}\rangle}$ is not a distance

How well does $\mathrm{W}_{2,\varepsilon}$ approximate W_2?

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- General non-quantitative convergence:

Theorem (Mikami 2004; Léonard 2012):

$$\mathrm{W}_{2,\varepsilon} \xrightarrow[\varepsilon\to 0]{} \mathrm{W}_2.$$

- When ρ and μ are **absolutely continuous**: $1^{st}/2^{nd}$ **order asymptotics**.

Theorem (Adams et al. 2011; ...; Conforti and Tamanini, 2021):
If the densities of ρ, μ are bounded, then

$$\mathrm{W}_{2,\varepsilon}^2(\rho,\mu) + \varepsilon \mathrm{KL}(\pi^{(\mathrm{P}_\varepsilon)}|\rho\otimes\mu) = \mathrm{W}_2^2(\rho,\mu) - \frac{\varepsilon}{2}\left(\mathrm{KL}(\rho|\lambda) + \mathrm{KL}(\mu|\lambda)\right) - \frac{\varepsilon}{2} d \log(\pi\varepsilon) + \frac{\varepsilon^2}{16} I(\rho,\mu) + o(\varepsilon^2).$$

- When ρ and μ are **discrete**: **exponential convergence rate**.

Theorem (Cominetti and San Martín 1994; Niles-Weed, 2018):
If ρ, μ discrete, $\exists C_{\rho,\mu}, \tilde{C}_{\rho,\mu}$ explicit s.t.

$$0 \le \mathrm{W}_{2,\varepsilon}^2(\rho,\mu) - \mathrm{W}_2^2(\rho,\mu) \le C_{\rho,\mu} \exp\left(-\tilde{C}_{\rho,\mu}/\varepsilon\right).$$

What about the semi-discrete setting?
(ρ absolutely continuous and μ discrete)

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Semi-discrete (entropic) optimal transport

- **Semi-discrete setting:**
  - Let $\mathcal{X} \subset \mathbb{R}^d$ compact and $\rho \in \mathcal{P}_{a.c.}(\mathcal{X})$ absolutely continuous.
  - Let $\mathcal{Y} = \{y_1, \ldots, y_N\} \subset \mathbb{R}^d$ and $\mu = \sum_{i=1}^N \mu_i \delta_{y_i} \in \mathcal{P}(\mathcal{Y})$.

  $\rightarrow$ Natural framework in **statistics** and **numerical analysis**.

- Remark:

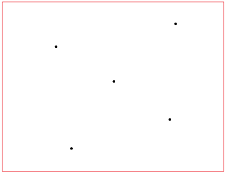
Theorem (Brenier, 1987): If $\rho \in \mathcal{P}(\mathcal{X})$ is **absolutely continuous**, then the optimal transport solution $\pi^{(\mathrm{P}_0)}$ is **unique**. It is induced by a map $T_{\rho\to\mu} : \mathcal{X} \to \mathcal{Y}$ satisfying $(T_{\rho\to\mu})_\# \rho = \mu$ characterized by $T_{\rho\to\mu} = \nabla\phi_{\rho\to\mu}$ with $\phi_{\rho\to\mu}$ convex.

$\rightarrow \pi^{(\mathrm{P}_0)} = (\mathrm{id}, T_{\rho\to\mu})_\# \rho.$

Introduction

Semi-discrete (entropic) optimal transport

- A two-dimensional example: $\rho = \mathbb{1}_{[a,b]\times[c,d]}$ and $\mu = \sum_{i=1}^{5} \mu_i \delta_{y_i}$:



Each color represents one of $\left(x \mapsto \pi^{(\mathrm{P}_\varepsilon)}(x, y_i)\right)_{i=1,\dots,5}$, level of transparency = value.

Figures inspired from (Peyré and Cuturi, 2019)

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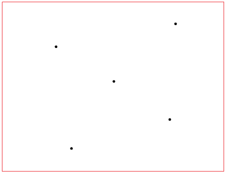
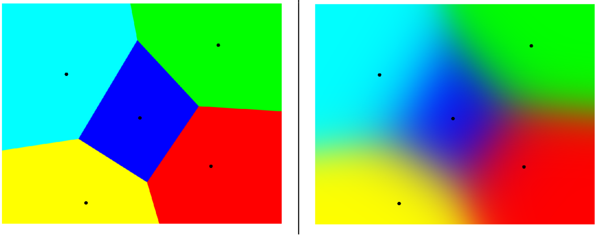


$\varepsilon = 0$ | $\varepsilon = 7.5 \times 10^{-2}$



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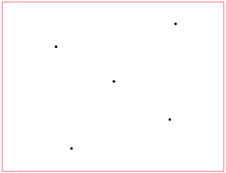

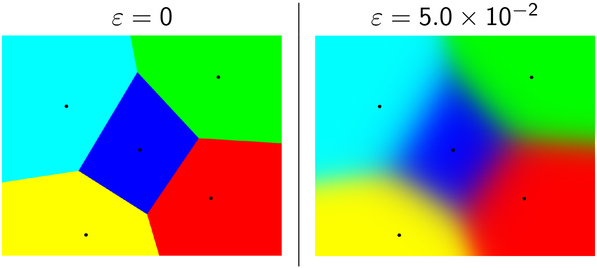

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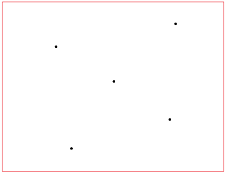

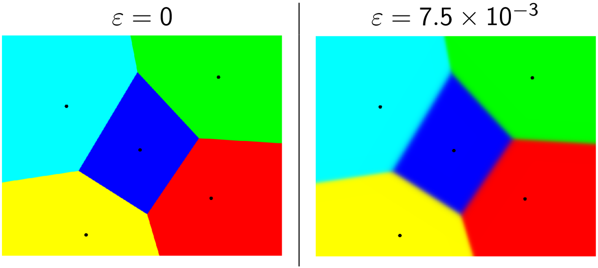

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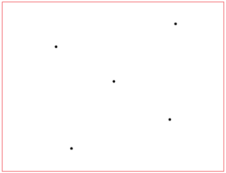

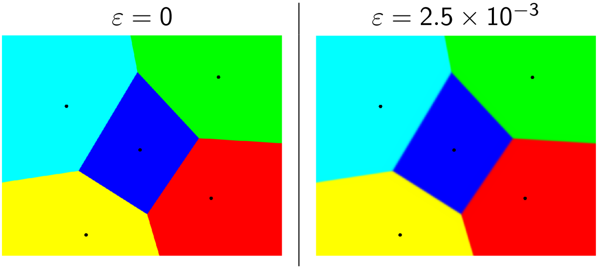

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Asymptotics for semi-discrete entropic optimal transport

Theorem (Altschuler, Niles-Weed and Stromme, 2021):
Under regularity assumptions on ρ,

$$\mathrm{W}_{2,\varepsilon}^2(\rho,\mu) = \mathrm{W}_2^2(\rho,\mu) + \varepsilon^2 \frac{\pi^2}{12} \sum_{i<j} \frac{w_{ij}}{\|y_i - y_j\|} + o(\varepsilon^2),$$

where $w_{ij} = \int_{T_{\rho\to\mu}^{-1}(y_i)\cap T_{\rho\to\mu}^{-1}(y_j)} \rho(x)\mathrm{d}\mathcal{H}^{d-1}(x)$.

Proof idea: introduce the (semi-)dual problem:

$$\min_{\psi\in\mathbb{R}^N, \langle\psi|\mathbb{1}_N\rangle=0} \int_{\mathcal{X}} \psi^{c,\varepsilon}\mathrm{d}\rho + \langle\psi|\mu\rangle + \varepsilon. \qquad (\mathrm{D}_\varepsilon)$$

where $\psi^{c,\varepsilon}$ is the Legendre/(c,ε) transform of ψ:

$$\psi^{c,\varepsilon}(x) = \begin{cases} \psi^*(x) = \max_i \langle x|y_i\rangle - \psi_i & \text{if } \varepsilon = 0, \\ \varepsilon \log \sum_i e^{\frac{\langle x|y_i\rangle - \psi_i}{\varepsilon}} & \text{if } \varepsilon > 0. \end{cases}$$

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Remark 1: $(\mathrm{P}_{2\varepsilon}) = M_2(\rho) + M_2(\mu) - 2\varepsilon\mathcal{H}(\mu) - 2\times(\mathrm{D}_\varepsilon)$.
Remark 2: (D_ε) admits a unique solution ψ^ε and

$$\frac{\mathrm{d}\pi^{(\mathrm{P}_{2\varepsilon})}}{\mathrm{d}\rho\otimes\sigma}(x,y_i) = \begin{cases} \mathbb{1}_{\mathrm{Lag}_i(\psi^0)}(x) & \text{if } \varepsilon = 0,\\ e^{\frac{\langle x|y_i\rangle - \psi_i^\varepsilon}{\varepsilon}} / \sum_j e^{\frac{\langle x|y_j\rangle - \psi_j^\varepsilon}{\varepsilon}} & \text{if } \varepsilon > 0,\end{cases}$$

where $\mathrm{Lag}_i(\psi^0) = T_{\rho\to\mu}^{-1}(y_i) = \{x\in\mathcal{X} | \forall j, \langle x|y_i\rangle - \psi_i^0 \geq \langle x|y_j\rangle - \psi_j^0\}$ and $\sigma = \Sigma_i\delta_{y_i}$.

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Main technical result:

Theorem (Altschuler et al., 2021): For ψ^ε solution to (D_ε), under regularity assumptions on ρ:

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}(\psi^\varepsilon - \psi^0) = \dot{\psi}^\varepsilon\Big|_{\varepsilon=0} = 0.$$

These results can be extended and quantified with a non-asymptotic analysis.

Convergence bounds for entropic SDOT

Non-asymptotic behavior of potentials

Assumption: *The compact set $\mathcal{X}$ is convex. The source density ρ is α-Hölder continuous for some $\alpha \in (0,1]$ and verifies on $\mathcal{X}$:*

$$0 < m_\rho \leq \rho \leq M_\rho < +\infty.$$

Theorem (D., 2022):
The mapping $\varepsilon \mapsto \psi^\varepsilon$ is $\mathcal{C}^1$. For any $\varepsilon > 0$ and $\alpha' \in (0, \alpha)$,

$$\left\|\dot{\psi}^\varepsilon\right\|_2 \leq C_{\mathcal{X},\rho,\mathcal{Y},\mu} \min(\varepsilon^{\alpha'}, 1).$$

- **Remarks:**
  1. $\mathcal{X}$ convex can be relaxed to some extent (e.g. $\mathcal{X}$ = connected union of convex sets s.t. ρ satisfies a Poincaré-Wirtinger inequality).
  2. Constant is *explicit*:

$$C_{\mathcal{X},\rho,\mathcal{Y},\mu} = C(d) \times \frac{N}{\underline{\mu}} \frac{M_\rho}{m_\rho} e^{R_\mathcal{Y} \mathrm{diam}(\mathcal{X})} \Bigg(N R_\mathcal{X} \mathrm{diam}(\mathcal{Y}) + \log \frac{1}{\underline{\mu}} + N^2 M_\rho \mathrm{diam}(\mathcal{X})^{d-1} (1 + \frac{C_\rho}{\delta \alpha} + R_\mathcal{X} \mathrm{diam}(\mathcal{Y}) + \log \frac{1}{\underline{\mu}}) + N^3 M_\rho \frac{\mathrm{diam}(\mathcal{X})^{d-2} \mathrm{diam}(\mathcal{Y})^4}{\cos(\theta/2)\delta^4} (1 + R_\mathcal{X} \mathrm{diam}(\mathcal{Y}) + \log \frac{1}{\underline{\mu}}) \Bigg).$$

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Corollary (D., 2022):
Let $0 < \varepsilon' \leq \varepsilon$. For any $\alpha' \in (0, \alpha)$,

$$\left\| \psi^{\varepsilon} - \psi^{\varepsilon'} \right\|_{\infty} \lesssim \varepsilon^{\alpha'} (\varepsilon - \varepsilon').$$

- **Remark:** May "justify" ε-scaling heuristic, where ε is decreased over the iterations of an algorithm that estimates ψ^0 (Kosowsky and Yuille, 1994; Schmitzer, 2019; Feydy, 2020).

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- **Remark:** May "justify" ε-scaling heuristic[1], where ε is decreased over the iterations of an algorithm that estimates ψ^0 (Kosowsky and Yuille, 1994; Schmitzer, 2019; Feydy, 2020).

[1] *Originally introduced for Bertsekas' auction algorithm (Bertsekas and Eckstein, 1988) for the N-assignment problem. Reduced worst case complexity:*

$$O\left(\frac{N^2}{\varepsilon}\right) \to O\left(N^3 \log\left(\frac{1}{\varepsilon}\right)\right)$$

in order to get an ε-approximate solution.

Convergence bounds for entropic SDOT

Non-asymptotic behavior of potentials

Corollary (D., 2022):
Let $\varepsilon > 0$. For any $\alpha' \in (0, \alpha)$,

$$\left\|\psi^\varepsilon - \psi^0\right\|_\infty \lesssim \varepsilon^{1+\alpha'}.$$

Additionally, for ρ-a.e. $x \in \mathcal{X}$, $\pi^\varepsilon = \frac{\mathrm{d}\pi^{(\mathrm{P}_\varepsilon)}}{\mathrm{d}\rho\otimes\sigma}$ verifies

$$\left|\pi^\varepsilon(x, \cdot) - \pi^0(x, \cdot)\right| \lesssim e^{-c_x/\varepsilon},$$

where $c_x = \min_i\{(\psi^0)^*(x) - \langle x|y_i\rangle + \psi_i^0 \quad | \quad \langle x|y_i\rangle - \psi_i^0 \neq (\psi^0)^*(x)\} > 0$.

- **Remarks:**
  1. Second bound is reminiscent of the *large deviations principle* of (Bernton, Ghosal, Nutz, 2022).
  2. Second bound used in (Pooladian, Divol, Niles-Weed, 2023) to show:

$$||\sum_i y_i \pi^\varepsilon(\cdot, y_i) - T_{\rho\to\mu}||_{\mathrm{L}^2(\rho;\mathbb{R}^d)} \lesssim \varepsilon^{1/2}.$$

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- **Remarks:**
  1. Second bound is reminiscent of the *large deviations principle* of (Bernton, Ghosal, Nutz, 2022).
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$$|| \sum_i y_i \pi^\varepsilon(\cdot, y_i) - T_{\rho\to\mu} ||_{\mathrm{L}^2(\rho;\mathbb{R}^d)} \lesssim \varepsilon^{1/2}.$$

Convergence bounds for entropic SDOT

Non-asymptotic expansion of the difference of costs

Theorem (D., 2022):
For any $\alpha' \in (0, \alpha)$ and $\varepsilon > 0$,

$$\left| \mathrm{W}_{2,\varepsilon}^2(\rho, \mu) - \mathrm{W}_2^2(\rho, \mu) - \varepsilon^2 \frac{\pi^2}{12} \sum_{i<j} \frac{w_{ij}}{\|y_i - y_j\|} \right| \lesssim \varepsilon^{2+\alpha'}.$$

This inequality is tight.

- **Remark:** Possibly no third-order term in this expansion.

Sketch of proof for $\left\|\dot{\psi}^\varepsilon\right\|_2 \lesssim \min(\varepsilon^{\alpha'}, 1)$

A governing O.D.E.

- Dual formulation:

$$\min_{\psi\in\mathbb{R}^N, \langle\psi|\mathbb{1}_N\rangle=0} \int_{\mathcal{X}} \psi^{c,\varepsilon}\mathrm{d}\rho + \langle\psi|\mu\rangle + \varepsilon. \tag{D_ε}$$

Definition: Regularized Kantorovich's functional:

$$\mathcal{K}_\rho^\varepsilon : \begin{cases} \mathbb{R}^N & \to \mathbb{R}, \\ \psi & \mapsto \int_{\mathcal{X}} \psi^{c,\varepsilon}\mathrm{d}\rho + \varepsilon. \end{cases}$$

- $\mathcal{K}_\rho^\varepsilon$ strictly convex on $(\mathbb{1}_N)^\perp$. First-order condition for (D_ε):

$$\boxed{\nabla\mathcal{K}_\rho^\varepsilon(\psi^\varepsilon) = -\mu.}$$

Implicit function theorem:

Proposition (D., 2022):
$\varepsilon \mapsto \psi^\varepsilon$ is a $\mathcal{C}^1$ mapping from $\mathbb{R}_+^*$ to $(\mathbb{1}_N)^\perp$ and

$$\nabla^2\mathcal{K}_\rho^\varepsilon(\psi^\varepsilon)\dot{\psi}^\varepsilon + \frac{\partial}{\partial\varepsilon}(\nabla\mathcal{K}_\rho^\varepsilon)(\psi^\varepsilon) = 0.$$

Sketch of proof for $\left\|\dot{\psi}^{\varepsilon}\right\|_2 \lesssim \min(\varepsilon^{\alpha'}, 1)$

A governing O.D.E.

- Dual formulation:

$$\min_{\psi \in \mathbb{R}^N, \langle \psi | \mathbb{1}_N \rangle = 0} \int_{\mathcal{X}} \psi^{c,\varepsilon} \mathrm{d}\rho + \langle \psi | \mu \rangle + \varepsilon. \qquad (\mathrm{D}_\varepsilon)$$

Definition: Regularized Kantorovich's functional:

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Sketch of proof for $\left\|\dot{\psi}^\varepsilon\right\|_2 \lesssim \min(\varepsilon^{\alpha'}, 1)$

A governing O.D.E.

The potential ψ^ε satisfies

$$\boxed{\nabla^2 \mathcal{K}_\rho^\varepsilon(\psi^\varepsilon)\dot{\psi}^\varepsilon + \frac{\partial}{\partial \varepsilon}(\nabla \mathcal{K}_\rho^\varepsilon)(\psi^\varepsilon) = 0.}$$

$\Longrightarrow$ An upper bound on $\left\|\dot{\psi}^\varepsilon\right\|$ may be obtained from:

1. A lower bound on $\nabla^2 \mathcal{K}_\rho^\varepsilon(\psi^\varepsilon) \to$ **Prékopa-Leindler inequality**.
2. An upper bound on $\frac{\partial}{\partial \varepsilon}(\nabla \mathcal{K}_\rho^\varepsilon)(\psi^\varepsilon) \to$ **"Laplace's method"**.

Sketch of proof for $\left\|\dot{\psi}^\varepsilon\right\|_2 \lesssim \min(\varepsilon^{\alpha'}, 1)$

Strong convexity estimate for $\mathcal{K}_\rho^\varepsilon$

Theorem (D., 2022): For any $\varepsilon > 0$ and $v \in \mathbb{R}^N$,

$$\mathbb{V}\mathrm{ar}_\mu(v) \leq \left(e^{R_\mathcal{Y} \mathrm{diam}(\mathcal{X})} \frac{M_\rho}{m_\rho} + \varepsilon \right) \langle v | \nabla^2 \mathcal{K}_\rho^\varepsilon(\psi^\varepsilon) v \rangle.$$

- **Remarks:**
  1. $\mathcal{X}$ may not be convex and ρ may not be Hölder continuous. Target(s) may not be discrete.
  2. As $\varepsilon \to 0$, recovers (D. and Mérigot, 2021):
  $$\mathbb{V}\mathrm{ar}_\mu(v) \leq \left(e^{R_\mathcal{Y} \mathrm{diam}(\mathcal{X})} \frac{M_\rho}{m_\rho} \right) \langle v | \nabla^2 \mathcal{K}^0(\psi^0) v \rangle.$$
  3. **Proof idea:** using the Prékopa-Leindler inequality[2], show the concavity of
  $$\psi \mapsto \log \int_\mathcal{X} e^{-\psi^{c,\varepsilon}}.$$

[2] *Let $0 < \lambda < 1$ and $f, g, h : \mathbb{R}^N \to \mathbb{R}_+$. Assume that $h((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} g(y)^\lambda$ for all $x, y \in \mathbb{R}^N$. Then $\|h\|_1 \geq \|f\|_1^{1-\lambda} \|g\|_1^\lambda$.*

Sketch of proof for $\left\|\dot{\psi}^\varepsilon\right\|_2 \lesssim \min(\varepsilon^{\alpha'}, 1)$

Bound on the second term $\frac{\partial}{\partial\varepsilon}(\nabla\mathcal{K}^\varepsilon_\rho)(\psi^\varepsilon)$

Theorem (D., 2022): For any $\varepsilon > 0$,

$$\left\|\frac{\partial}{\partial\varepsilon}(\nabla\mathcal{K}^\varepsilon_\rho)(\psi^\varepsilon)\right\|_\infty \lesssim \min\left(\varepsilon^{\alpha'}, \frac{1}{\varepsilon}\right).$$

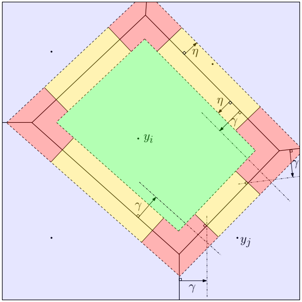
- **Proof idea:** We have

$$[\frac{\partial}{\partial\varepsilon}(\nabla\mathcal{K}^\varepsilon_\rho)(\psi^\varepsilon)]_i = \int_{\mathcal{X}} \sum_{j\neq i} \left(\frac{f_i^\varepsilon(x) - f_j^\varepsilon(x)}{\varepsilon^2}\right) \pi^\varepsilon_{x,j}\pi^\varepsilon_{x,i}\mathrm{d}\rho(x),$$

where $\forall j$,

$$f_j^\varepsilon(x) = \langle x|y_j\rangle - \psi_j^\varepsilon,$$

$$\pi^\varepsilon_{x,j} = \exp(\frac{f_j^\varepsilon(x)}{\varepsilon})/\Sigma_k \exp(\frac{f_k^\varepsilon(x)}{\varepsilon}).$$



Numerical illustrations

Behavior of $\varepsilon \mapsto \psi^\varepsilon$

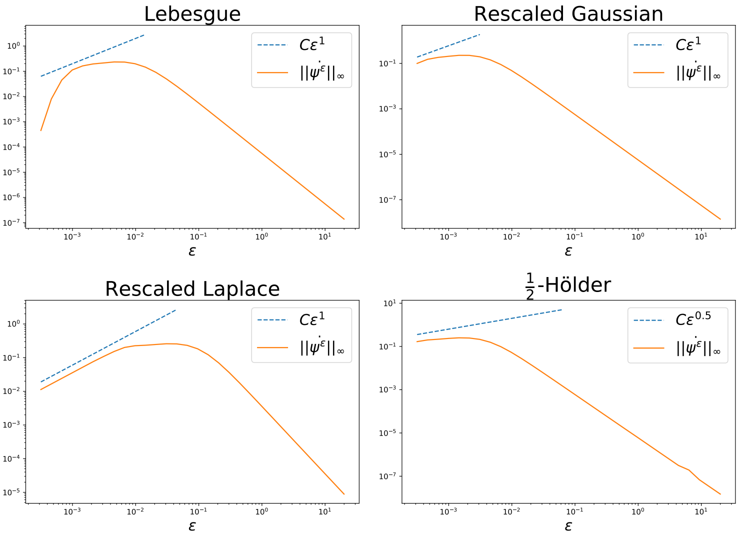
- Let $\mathcal{X} = [-1, 1]$, ρ symmetric on $\mathcal{X}$ and $\mu = \frac{1}{5} \sum_{i=1}^{5} \delta_{y_i}$, where $\{y_1, \dots, y_5\} \subset \mathcal{X}$.

- Consider 4 different sources:
  1. Lebesgue: $\rho(x) \propto \mathbb{1}_{[-1,1]}(x)$,
  2. Rescaled Gaussian: $\rho(x) \propto e^{-x^2/2\sigma^2} \mathbb{1}_{[-1,1]}(x)$,
  3. Rescaled Laplace: $\rho(x) \propto e^{-|x|} \mathbb{1}_{[-1,1]}(x)$,
  4. $\frac{1}{2}$-Hölder density: $\rho(x) \propto (1 - |x|^{1/2}) \mathbb{1}_{[-1,1]}(x)$.

Numerical illustrations

Behavior of $\varepsilon \mapsto \psi^\varepsilon$

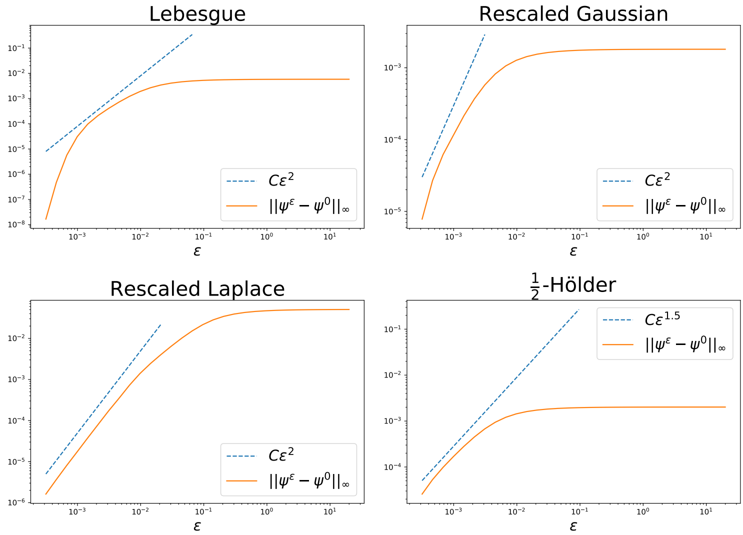
- Observe $\left\|\dot{\psi^\varepsilon}\right\|_2 \lesssim \min(\varepsilon^{\alpha'}, 1)$:



Numerical illustrations

Behavior of $\varepsilon \mapsto \psi^\varepsilon$

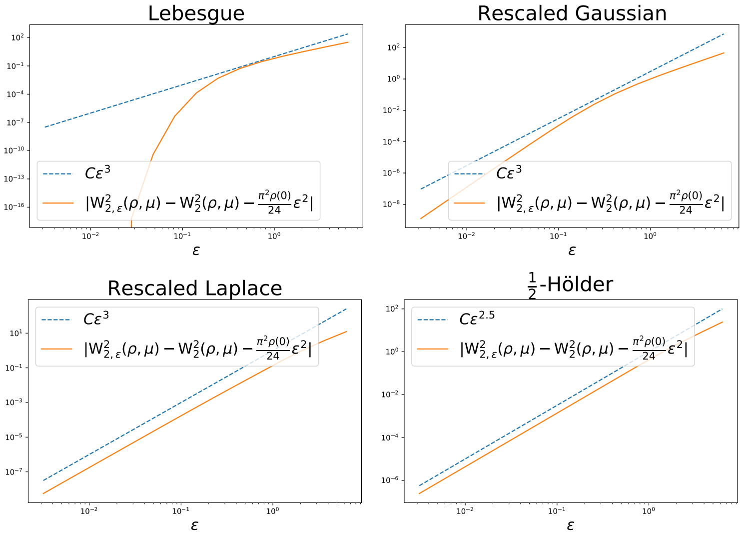
- Observe $\left\|\psi^\varepsilon - \psi^0\right\|_2 \lesssim \varepsilon^{1+\alpha'}$:



Numerical illustrations

Difference of Costs ($\mu = \frac{1}{2}(\delta_{-1} + \delta_1)$)

- Observe $\left| W_{2,\varepsilon}^2(\rho,\mu) - W_2^2(\rho,\mu) - \varepsilon^2 \frac{\pi^2}{12} \sum_{i<j} \frac{w_{ij}}{\|y_i - y_j\|} \right| \lesssim \varepsilon^{2+\alpha'}$:



Thank you for your attention!

ε-scaling

Corollary (D., 2022): *Let* $0 < \varepsilon' \leq \varepsilon$. *For any* $\alpha' \in (0, \alpha)$,

$$\left\| \psi^{\varepsilon} - \psi^{\varepsilon'} \right\|_{\infty} \lesssim \varepsilon^{\alpha'}(\varepsilon - \varepsilon').$$

- **Remark:** (ε-scaling) Assume we know an algorithm `approx_psi` such that

$$\widetilde{\psi^{\varepsilon}} := \texttt{approx\_psi}(\rho, \mu, \varepsilon, \psi^{init}) \approx \psi^{\varepsilon}.$$

Fast if ε large or $\left\| \psi^{\varepsilon} - \psi^{init} \right\|$ small.

ε-scaling

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  ε**-scaling approximates** ψ^0 **with** $\widetilde{\psi^{\varepsilon_K}}$ **for some** $K \in \mathbb{N}^*$, where:
  - $\varepsilon_0 > 0$ (large), $\widetilde{\psi^{\varepsilon_0}} := \texttt{approx\_psi}(\rho, \mu, \varepsilon_0, \psi^{init})$,
  - $\varepsilon_{k+1} = \varepsilon_k / 2$,
  - $\widetilde{\psi^{\varepsilon_{k+1}}} := \texttt{approx\_psi}(\rho, \mu, \varepsilon_{k+1}, \widetilde{\psi^{\varepsilon_k}})$.

ε-scaling

Corollary (D., 2022): *Let $0 < \varepsilon' \leq \varepsilon$. For any $\alpha' \in (0, \alpha)$,*

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  - $\widetilde{\psi^{\varepsilon_{k+1}}} := \texttt{approx\_psi}(\rho, \mu, \varepsilon_{k+1}, \widetilde{\psi^{\varepsilon_k}})$.

Hope that $\left\| \widetilde{\psi^{\varepsilon_k}} - \psi^{\varepsilon_{k+1}} \right\|$ gets small as $\varepsilon_k \to 0$.

Strong convexity estimate for $\mathcal{K}_\rho^\varepsilon$

Theorem (D., 2022): *For any* $\varepsilon > 0$ *and* $v \in \mathbb{R}^N$,

$$\mathbb{V}\mathrm{ar}_\mu(v) \leq \left(e^{R_{\mathcal{Y}} \mathrm{diam}(\mathcal{X})} \frac{M_\rho}{m_\rho} + \varepsilon \right) \langle v | \nabla^2 \mathcal{K}_\rho^\varepsilon(\psi^\varepsilon) v \rangle.$$

- Notice that

$$\nabla^2 \mathcal{K}_\rho^\varepsilon(\psi) = \frac{1}{\varepsilon} \mathbb{E}_{x\sim\rho} \left(\mathrm{diag}(\pi_x^\varepsilon(\psi)) - \pi_x^\varepsilon(\psi) \pi_x^\varepsilon(\psi)^\top \right),$$

where $\pi_x^\varepsilon(\psi) \in \mathbb{R}^N$ and $\forall i \in \{1, \ldots, N\}$,

$$\pi_x^\varepsilon(\psi)_i = \frac{\exp\left(\frac{\langle x|y_i\rangle - \psi_i}{\varepsilon}\right)}{\sum_{j=1}^N \exp\left(\frac{\langle x|y_j\rangle - \psi_j}{\varepsilon}\right)}.$$

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- Introduce $I : \mathbb{R}^N \to \mathbb{R}, \psi \mapsto \log(\int_\mathcal{X} e^{-\psi^{c,\varepsilon}})$. Notice that I is $\mathcal{C}^2$:

$$\begin{aligned} \nabla^2 I(\psi^\varepsilon) = & - \frac{1}{\varepsilon} \mathbb{E}_{x\sim\tilde{\rho}^\varepsilon} (\mathrm{diag}(\pi_x^\varepsilon(\psi^\varepsilon)) - \pi_x^\varepsilon(\psi^\varepsilon) \pi_x^\varepsilon(\psi^\varepsilon)^\top) \\ & + \mathbb{E}_{x\sim\tilde{\rho}^\varepsilon} \pi_x^\varepsilon(\psi^\varepsilon) \pi_x^\varepsilon(\psi^\varepsilon)^\top - \mathbb{E}_{x\sim\tilde{\rho}^\varepsilon} \pi_x^\varepsilon(\psi^\varepsilon) \mathbb{E}_{x\sim\tilde{\rho}^\varepsilon} \pi_x^\varepsilon(\psi^\varepsilon)^\top, \end{aligned}$$

where $\tilde{\rho}^\varepsilon := \frac{e^{-(\psi^\varepsilon)^{c,\varepsilon}}}{\int_\mathcal{X} e^{-(\psi^\varepsilon)^{c,\varepsilon}}}$.

Strong convexity estimate for $\mathcal{K}_\rho^\varepsilon$

Theorem (D., 2022): *For any* $\varepsilon > 0$ *and* $v \in \mathbb{R}^N$,

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- $I : \psi \mapsto \log(\int_{\mathcal{X}} e^{-\psi^{c,\varepsilon}})$ is concave: Let $\psi, \varphi \in \mathbb{R}^N, 0 < \lambda < 1$. For any $u, v \in \mathcal{X}$, we have:

$$\begin{aligned}
&\big(\lambda\psi + (1-\lambda)\varphi\big)^{c,\varepsilon}(\lambda u + (1-\lambda) v) \\
&= \varepsilon \log \left(\sum_{i=1}^N e^{\frac{\langle \lambda u + (1-\lambda) v | y_i \rangle - (\lambda\psi + (1-\lambda)\varphi)(y_i)}{\varepsilon}} \right) \\
&= \varepsilon \log \left(\sum_{i=1}^N \left(e^{\frac{\langle u | y_i \rangle - \psi(y_i)}{\varepsilon}} \right)^\lambda \left(e^{\frac{\langle v | y_i \rangle - \varphi(y_i)}{\varepsilon}} \right)^{1-\lambda} \right) \\
&\leq \varepsilon \log \left[\left(\sum_{i=1}^N e^{\frac{\langle u | y_i \rangle - \psi(y_i)}{\varepsilon}} \right)^\lambda \left(\sum_{i=1}^N e^{\frac{\langle v | y_i \rangle - \varphi(y_i)}{\varepsilon}} \right)^{1-\lambda} \right] \\
&= \lambda \psi^{c,\varepsilon}(u) + (1-\lambda) \varphi^{c,\varepsilon}(v).
\end{aligned}$$

Strong convexity estimate for $\mathcal{K}_\rho^\varepsilon$

Theorem (D., 2022): *For any $\varepsilon > 0$ and $v \in \mathbb{R}^N$,*

$$\mathbb{V}\mathrm{ar}_\mu(v) \leq \left(e^{R_{\mathcal{Y}} \mathrm{diam}(\mathcal{X})} \frac{M_\rho}{m_\rho} + \varepsilon \right) \langle v | \nabla^2 \mathcal{K}_\rho^\varepsilon(\psi^\varepsilon) v \rangle.$$

- $I : \psi \mapsto \log(\int_{\mathcal{X}} e^{-\psi^{c,\varepsilon}})$ is concave: Let $\psi, \varphi \in \mathbb{R}^N, 0 < \lambda < 1$. For any $u, v \in \mathcal{X}$, we have:

$$\big(\lambda\psi + (1-\lambda)\varphi\big)^{c,\varepsilon}(\lambda u + (1-\lambda)v) \leq \lambda\psi^{c,\varepsilon}(u) + (1-\lambda)\varphi^{c,\varepsilon}(v).$$

Denoting

$$h(u) = e^{-(\lambda\psi + (1-\lambda)\varphi)^{c,\varepsilon}(u)},$$
$$f(u) = e^{-\psi^{c,\varepsilon}(u)}, \quad g(u) = e^{-\varphi^{c,\varepsilon}(u)},$$

we thus have shown that

$$h(\lambda u + (1-\lambda)v) \geq f(u)^\lambda g(v)^{1-\lambda}.$$

$\Longrightarrow$ Prékopa–Leindler inequality:

$$\int_{\mathcal{X}} h \geq \left(\int_{\mathcal{X}} f\right)^\lambda \left(\int_{\mathcal{X}} g\right)^{1-\lambda}.$$

Strong convexity estimate for $\mathcal{K}_\rho^\varepsilon$

Theorem (D., 2022): *For any $\varepsilon > 0$ and $v \in \mathbb{R}^N$,*

$$\mathbb{V}\mathrm{ar}_\mu(v) \leq \left(e^{R_{\mathcal{Y}} \mathrm{diam}(\mathcal{X})} \frac{M_\rho}{m_\rho} + \varepsilon \right) \langle v | \nabla^2 \mathcal{K}_\rho^\varepsilon(\psi^\varepsilon) v \rangle.$$

- $I : \psi \mapsto \log(\int_{\mathcal{X}} e^{-\psi^{c,\varepsilon}})$ is concave: Prékopa–Leindler inequality:

$$\int_{\mathcal{X}} h \geq \left(\int_{\mathcal{X}} f \right)^\lambda \left(\int_{\mathcal{X}} g \right)^{1-\lambda}.$$

$\implies$ I is concave:

$$\begin{aligned} I(\lambda\psi + (1-\lambda)\varphi) &= \log\left(\int_{\mathcal{X}} h \right) \\ &\geq \lambda \log\left(\int_{\mathcal{X}} f \right) + (1-\lambda) \log\left(\int_{\mathcal{X}} g \right) \\ &= \lambda I(\psi) + (1-\lambda) I(\varphi). \end{aligned}$$

Strong convexity estimate for $\mathcal{K}_\rho^\varepsilon$

Theorem (D., 2022): *For any $\varepsilon > 0$ and $v \in \mathbb{R}^N$,*

$$\mathbb{V}\mathrm{ar}_\mu(v) \leq \left(e^{R_{\mathcal{Y}} \mathrm{diam}(\mathcal{X})} \frac{M_\rho}{m_\rho} + \varepsilon \right) \langle v | \nabla^2 \mathcal{K}_\rho^\varepsilon(\psi^\varepsilon) v \rangle.$$

- Notice that

$$\nabla^2 \mathcal{K}_\rho^\varepsilon(\psi) = \frac{1}{\varepsilon} \mathbb{E}_{x \sim \rho} \left(\mathrm{diag}(\pi_x^\varepsilon(\psi)) - \pi_x^\varepsilon(\psi) \pi_x^\varepsilon(\psi)^\top \right).$$

- Introduce $I : \mathbb{R}^N \to \mathbb{R}, \psi \mapsto \log(\int_{\mathcal{X}} e^{-\psi^{c,\varepsilon}})$. Notice that I is $\mathcal{C}^2$ **and concave**:

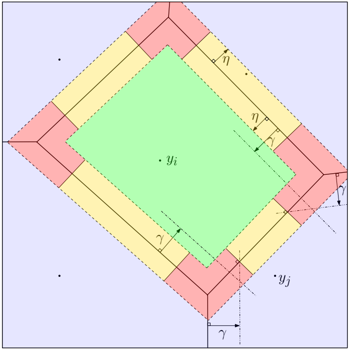
$$\nabla^2 I(\psi^\varepsilon) = -\frac{1}{\varepsilon} \mathbb{E}_{x \sim \tilde{\rho}^\varepsilon} (\mathrm{diag}(\pi_x^\varepsilon(\psi^\varepsilon)) - \pi_x^\varepsilon(\psi^\varepsilon) \pi_x^\varepsilon(\psi^\varepsilon)^\top) \\ + \mathbb{E}_{x \sim \tilde{\rho}^\varepsilon} \pi_x^\varepsilon(\psi^\varepsilon) \pi_x^\varepsilon(\psi^\varepsilon)^\top - \mathbb{E}_{x \sim \tilde{\rho}^\varepsilon} \pi_x^\varepsilon(\psi^\varepsilon) \mathbb{E}_{x \sim \tilde{\rho}^\varepsilon} \pi_x^\varepsilon(\psi^\varepsilon)^\top \leq 0,$$

where $\tilde{\rho}^\varepsilon := \frac{e^{-(\psi^\varepsilon)^{c,\varepsilon}}}{\int_{\mathcal{X}} e^{-(\psi^\varepsilon)^{c,\varepsilon}}}$.

Upper bound on $\left\|\frac{\partial}{\partial\varepsilon}(\nabla\mathcal{K}_\rho^\varepsilon)(\psi^\varepsilon)\right\|_\infty$

We have:

$$[\frac{\partial}{\partial\varepsilon}(\nabla\mathcal{K}_\rho^\varepsilon)(\psi^\varepsilon)]_i = \int_{\mathcal{X}} \sum_{j\neq i} \left(\frac{f_i^\varepsilon(x) - f_j^\varepsilon(x)}{\varepsilon^2} \right) \pi_{x,j}^\varepsilon \pi_{x,i}^\varepsilon \mathrm{d}\rho(x).$$



Key technical result: for any $\eta, \gamma > 0$,

$$\begin{aligned}[\frac{\partial}{\partial\varepsilon}(\nabla\mathcal{K}_\rho^\varepsilon)(\psi)]_i \lesssim\ & \frac{1}{\varepsilon^2} e^{-\eta/\varepsilon} + \frac{\eta^{2+\alpha}}{\varepsilon^2} \\ & + \frac{\gamma^2}{\varepsilon^2}\left(\eta + e^{-\eta/\varepsilon}\right) \\ & + \frac{1}{\varepsilon^2} e^{-\tilde{\gamma}/\varepsilon}\Big(\eta + \varepsilon\eta e^{\eta/\varepsilon} \\ & \qquad - \varepsilon^2(e^{\eta/\varepsilon} - 1)\Big),\end{aligned}$$

where $\tilde{\gamma} = \gamma\delta - \frac{\mathrm{diam}(\mathcal{Y})^2}{\delta}\eta$ and $\delta = \min_{i\neq j} \|y_i - y_j\| > 0$.