Generalized $H(\mathrm{div})$ geodesics and solutions of the Camassa-Holm equation

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Joint work with FX Vialard and T Gallouët

The $H(\mathrm{div})$ geodesic problem

Notation:

- $M \subset \mathbb{R}^d$ is a compact domain
- ρ_0 is the normalized Lebesgue measure $\rho_0(M) = 1$
- $\mathrm{Diff}(M)$ is the diffeomorphism group and $\mathrm{Id} \in \mathrm{Diff}(M)$ is the identity map
- $\mathrm{SDiff}(M)$ volume-preserving, i.e. $\varphi_{\#}\rho_0 = \rho_0$
- A (deterministic) flow on M is a curve on $\mathrm{Diff}(M)$, i.e. $\varphi : [0, T] \times M \to M$.

Deterministic CH problem: Given $h \in \mathrm{Diff}(M)$ and $a, b > 0$, find the flow φ that minimizes the action

$$\int_M \int_0^T a\|u_t\|^2 + b|\mathrm{div} u_t|^2 \mathrm{d}t \, \mathrm{d}\rho_0 \,.$$

with $u_t = \dot{\varphi}_t \circ \varphi_t^{-1}$, and verifying $\varphi_0 = \mathrm{Id}$, $\varphi_T = h$.

For $d = 1 \implies$ Camassa-Holm equation: approximation of ideal fluid with free boundary in shallow water

For $d = 2 \implies$ higher-dimensional generalization (Kruse, Scheurle, and Du, 2001)

The L^2 case:

- Degenerate metric on $\mathrm{Diff}(M)$ (Michor and Mumford, 2005)
- On $\mathrm{SDiff}(M)$ the geodesic equations are the incompressible Euler equations
- Non-degeneracy and local well-posedness: for data h in H^s, $s > d/2 + 1$, and close to Id (Ebin and Marsden, 1970)
- Relaxation: Brenier introduced relaxation based on generalized incompressible flows, i.e. measure on the space of continuous paths (Brenier, 1989)
- Tightness: no for $d = 2$, yes for $d = 3$ (Shnirelman, 1994)

The $H(\mathrm{div})$ case:

- For $d = 1$ Camassa-Holm equation: completely integrable, bi-Hamiltonian
- Peakons: (weak) travelling wave solutions
- Collision of peakons $\implies$ blow up; solution not unique afterwards
- Non-degeneracy and local well-posedeness: (Michor and Mumford, 2005; Mumford and Michor, 2012)
- Correct definition for minimizing flows? Occurrence of blow up? Need for relaxation? Tightness? ...

A Lagrangian formulation on the cone

Given a flow $\varphi : [0, T] \times M \to M$ define

$$\lambda = \sqrt{\mathrm{Jac}(\varphi)} : [0, T] \times M \to \mathbb{R}_{>0}$$

Then the $H(\mathrm{div})$ action with $a = 1$ and $b = 1/4$, can be written as

$$\mathcal{A}([\varphi, \lambda]) = \int_M \int_0^T \lambda^2 \|\dot{\varphi}\|^2 + |\dot{\lambda}|^2 \mathrm{d}t \, \mathrm{d}\rho_0$$

This can be seen as the L^2 metric on the cone $\mathcal{C} = (M \times \mathbb{R}_{\geq 0})/(M \times \{0\})$ restricted to the subset $\mathrm{Aut}(\mathcal{C}) \subset \mathrm{Diff}(\mathcal{C})$, i.e. maps in the form $(\varphi, \lambda) : \mathcal{C} \to \mathcal{C}$,

$$(\varphi, \lambda)([x, r]) = [\varphi(x), \lambda(x) r]$$

Note:

- The CH problem consists in minimizing $\mathcal{A}([\varphi, \lambda])$ with $\varphi_{\#} \lambda^2 \rho_0 = \rho_0$ (Gallouët and Vialard, 2017)
- As for incompressible Euler, having an L^2 metric allows to decouple particle trajectories in the minimization problem $\Longrightarrow$ generalized flows

Generalized compressible flows

Notation:

- $\Omega = C^0([0,T];\mathcal{C})$, i.e. $\mathrm{z} \in \Omega$ then $\mathrm{z} : t \in [0,T] \to [x_t, r_t] \in \mathcal{C}$ continuous path
- $\mathcal{M}(\Omega)$ and $\mathcal{P}(\Omega)$: positive finite Borel measures and probability measures, resp., on space of paths

Deterministic to generalized: We can associated to any $(\varphi, \lambda) \in \mathrm{Aut}(\mathcal{C})$ a probability measure $\boldsymbol{\mu} = (\varphi, \lambda)_{\#}\rho_0$, i.e. for any $\mathcal{F} \in C_b^0(\Omega)$

$$\int_\Omega \mathcal{F}(\mathrm{z})\, \mathrm{d}\boldsymbol{\mu}(\mathrm{z}) = \int_\Omega \mathcal{F}([\varphi(x), \lambda(x)])\, \mathrm{d}\rho_0(x)$$

where $\varphi(x) : t \to \varphi_t(x)$ and $\lambda(x) : t \to \lambda_t(x)$.

Marginal constraint: The constraint $\lambda = \sqrt{\mathrm{Jac}(\varphi)}$ can be expressed as: for any $f \in C^0([0,T] \times M)$

$$\int_\Omega \int_0^T f(t, x_t) r_t^2\, \mathrm{d}t\, \mathrm{d}\boldsymbol{\mu}(\mathrm{z}) = \int_\Omega \int_0^T f(t,x)\, \mathrm{d}t\, \mathrm{d}\rho_0(x) \qquad (1)$$

which corresponds to change of variable formula

A **generalized compressible flow** is a probability measure $\boldsymbol{\mu} \in \mathcal{P}(\Omega)$ satisfying (1).

The action of a generalized compressible flow $\boldsymbol{\mu}$ is given by

$$\mathcal{A}(\boldsymbol{\mu}) = \int_\Omega \mathcal{E}(\mathrm{z})\,\mathrm{d}\boldsymbol{\mu}(\mathrm{z})\,, \quad \mathcal{E}(\mathrm{z}) = \begin{cases} \int_0^T r_t^2\|\dot{x}_t\|^2 + \dot{r}_t^2\,\mathrm{d}t & \text{if } \mathrm{z} \text{ is abs. continuous} \\ +\infty & \text{otherwise} \end{cases}$$

Lemma

The set of generalized compressible flows with uniformly bounded action $\mathcal{A}(\boldsymbol{\mu}) \leq C$ is relatively sequentially compact for the narrow topology.

Proof (sketch).

Use Prokhorov's theorem: we need to prove tightness.

1. Consider the set of paths $\Omega_R \subset \Omega$ bounded in the radial direction ($r_t \leq R$). We have

$$\Omega_{R,K} = \Omega_R \cap \{\mathrm{z} \in \Omega\,;\, \mathcal{A}(\mathrm{z} \leq K)\} \text{ is contained in a compact subset of } \Omega$$

   by Ascoli-Arzelá theorem.
2. $\boldsymbol{\mu}(\Omega \setminus \Omega_R) \leq C/R^2$ using $\mathcal{A}(\boldsymbol{\mu}) \leq C$
3. $\forall\, \epsilon > 0\ \exists\, K_\epsilon, R_\epsilon > 0$ such that $\boldsymbol{\mu}(\Omega \setminus \Omega_{R_\epsilon,K_\epsilon}) \leq \epsilon$

□

Generalized CH problem: Find the generalized compressible flow $\boldsymbol{\mu}$ minimizing the action $\mathcal{A}(\boldsymbol{\mu})$ with what boundary conditions?

In the deterministic CH problem we are looking for paths such that $\varphi_0 = \mathrm{Id}$ and $\varphi_T = h \in \mathrm{Diff}(\Omega)$. In the generalized setting this corresponds to

Strong coupling:

$$(e_0, e_T)_\# \boldsymbol{\mu} = [(\mathrm{Id}, 1), (h, \sqrt{\mathrm{Jac}(h)})]_\# \rho_0$$

where $e_t(\mathrm{z}) = z_t$ is the evaluation map at time t on paths. That is, $\forall\, f \in C_b^0(\mathcal{C} \times \mathcal{C})$

$$\int_\Omega f(z_0, z_T)\, \mathrm{d}\boldsymbol{\mu}(\mathrm{z}) = \int_M f([x, 1], [h(x), \sqrt{\mathrm{Jac}(h(x))}])\, \mathrm{d}\rho_0(x)$$

Taking these boundary conditions:

- Assuming finite action, by compactness, there is a minimizing sequence $\boldsymbol{\mu}_n \rightharpoonup \boldsymbol{\mu}^*$
- $\boldsymbol{\mu}^*$ might not be a generalized compressible flow: marginal constraint not stable under narrow convergence
- Paths with unboundedly large r (Jacobian) can be charged

Homogeneous coupling and rescaling

A function $f : \mathcal{C}^n$ is p-**homogeneous** (in the radial direction) iff for every $\alpha > 0$, $f([x_i, \alpha r_i]) = \alpha^p f([x_i, \alpha r_i])$

Homogeneous coupling: $\forall$ 2-homogeneous $f \in C^0(\mathcal{C} \times \mathcal{C})$

$$\int_\Omega f(z_0, z_T)\,\mathrm{d}\boldsymbol{\mu}(\mathrm{z}) = \int_M f([x,1],[h(x), \sqrt{\mathrm{Jac}(h(x))}])\,\mathrm{d}\rho_0(x)$$

Given a functional $\theta : \Omega \to \mathbb{R}$, define the **dilation** map by

$$\mathrm{dil}_{\theta,2} = \mathrm{prod}_{\theta\#}(\theta^2 \boldsymbol{\mu})\,, \quad \text{where} \quad \mathrm{prod}_\theta(\mathrm{z}) = (t \mapsto [x_t, r_t/\theta(\mathrm{z})])$$

Lemma (rescaling)

Consider a measure $\boldsymbol{\mu} \in \mathcal{M}(\Omega)$ and a 1-homogeneous functional $\sigma : \Omega \to \mathbb{R}$ such that $\sigma(\mathrm{z}) > 0$ for $\boldsymbol{\mu}$-almost every path z. Assume $C = \left(\int_\Omega \sigma^2\,\mathrm{d}\boldsymbol{\mu}\right)^{1/2} < +\infty$. Then $\tilde{\boldsymbol{\mu}} = \mathrm{dil}_{\sigma/C,2}\boldsymbol{\mu} \in \mathcal{P}(\Omega)$ satisfies

- *$\forall$ 2-homogeneous functionals $\mathcal{F}$, $\int_\Omega \mathcal{F}\,\mathrm{d}\boldsymbol{\mu} = \int_\Omega \mathcal{F}\,\mathrm{d}\tilde{\boldsymbol{\mu}}$*
- $\tilde{\boldsymbol{\mu}}(\{\mathrm{z} \in \Omega\,;\, \sigma(\mathrm{z}) = C\}) = 1$

Ex: if $\boldsymbol{\mu}(\{\mathrm{z} \in \Omega\,;\, r_0 = 0\}) = 0$ then for $\sigma = r_0$, $C = 1$ (marginal constraint) and $\mathrm{dil}_{r_0,2}\boldsymbol{\mu}$ is concentrated on path such that $r_0 = C = 1$.

Theorem

Provided that there exists a generalized flow $\boldsymbol{\mu}^*$ such that $\mathcal{A}(\boldsymbol{\mu}^*) < +\infty$, the minimum of the action among generalized compressible flows satisfying the homogeneous coupling constraint is attained.

Proof (sketch).

Consider a minimizing sequence $\boldsymbol{\mu}_n$

1. rescale support using dilation
2. action and constraints preserved: compactness $\implies$ limit in narrow topology
3. check uniform integrability constraints (now holds on rescaled support)

□

Note:

- This proves existence for any h in the connected component of $\mathrm{Diff}(M)$
- Solution allowed to charge **cone apex** ($r = 0$): shocks, vanishing Jacobian
- Is coupling meaningful?

A decomposition result for deterministic boundary conditions

Lemma

For all measures $\boldsymbol{\mu} \in \mathcal{M}(\Omega)$ *satisfying the homogeneous coupling constraint*

$$(*) \quad \boldsymbol{\mu}(\{\mathrm{z}, r_0 = 0\}) = 0 \quad \Longleftrightarrow \quad (**) \quad \boldsymbol{\mu}(\{\mathrm{z}, r_0 = r_T = 0\}) = 0$$

Proof.

If (*), $\boldsymbol{\mu}^1 = \mathrm{dil}_{r_0,2}\boldsymbol{\mu}$ is concentrated on paths with $r_0 = 1$. Then,

$$\int_\Omega (r_T - \sqrt{\mathrm{Jac}(h)(x_0)})^2 \, \mathrm{d}\boldsymbol{\mu}^1(\mathrm{z}) = \int_\Omega (r_T - r_0\sqrt{\mathrm{Jac}(h)(x_0)})^2 \, \mathrm{d}\boldsymbol{\mu}^1(\mathrm{z}) = 0$$

Therefore $\boldsymbol{\mu}^1$ satisfies the strong coupling and (**) holds. □

There exists a **decomposition**

$$\boldsymbol{\mu} = \boldsymbol{\mu}^0 + \tilde{\boldsymbol{\mu}}$$

with

- $\tilde{\boldsymbol{\mu}} = \boldsymbol{\mu} \llcorner \{\mathrm{z}; r_0 \neq 0 \text{ and } r_T \neq 0\}$: can be rescaled to satisfy strong coupling
- $\boldsymbol{\mu}^0 = \boldsymbol{\mu} \llcorner \{\mathrm{z}; r_0 = r_T = 0\}$: meaning?

Smooth solutions are minimizers

Euler-Lagrange equations in terms of (φ, λ):

$$\begin{cases} \lambda\ddot{\varphi} + 2\dot{\lambda}\dot{\varphi} + \frac{1}{2}\lambda\nabla P \circ \varphi = 0\,, \\ \ddot{\lambda} - \lambda\|\dot{\varphi}\|^2 + \lambda P \circ \varphi = 0\,, \end{cases}$$

where $P : M \to \mathbb{R}$ is the pressure: Lagrange multiplier for constraint $\lambda = \sqrt{\mathrm{Jac}(\varphi)}$.

Theorem

On $M = S^1_1$, unit radius circle, if for all $t \in [0, T]$

$$\left\| \begin{pmatrix} 2P + (\nabla)^2 P & \nabla P \\ (\nabla P)^T & 2P \end{pmatrix} \right\|_2 \le \frac{2\pi^2}{T^2} \tag{2}$$

then $(\varphi, \lambda)_\# \rho_0$ is a minimizer and it is unique (up to dilation) if the inequality is strict.

Note:

- for $M = S^1_1$ we identify $\mathcal{C}$ with $\mathbb{R}^2$
- result holds on $M \subset \mathbb{R}^d$ compact, but on shorter time for given P
- proof uses Poincaré inequality as for incompressible Euler also to prove that $\boldsymbol{\mu}^0 = 0$ (paths starting at the apex not charged)
- $P < \pi^2/T^2$ is sharp for uniqueness

Generalized CH problem on S^1_R, circle of radius R, with coupling given in polar coordinates by

$$h : \theta \in \mathbb{R}/\mathbb{Z}_{2\pi} \to \theta + \pi$$

so that $\mathrm{Jac}(h) = 1$.

Theorem

When $R = 1$ the dynamic plan

$$\boldsymbol{\mu}^* = \frac{1}{2}(\mathrm{Id}, \zeta^0)_{\#}\rho_0 + \frac{1}{2}(\psi^1, \zeta^1)_{\#}\rho_0 \,,$$

with

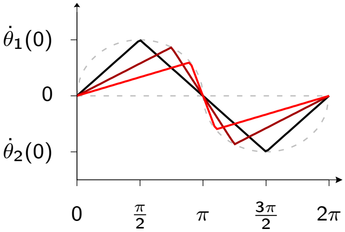
$$\zeta^0_t(\theta) = \sqrt{2}\sin(\sqrt{P^*}\,t)\,, \quad \zeta^1_t(\theta) = \sqrt{2}|\cos(\sqrt{P^*}\,t)|\,, \quad \psi^1_t(\theta) = \begin{cases} \theta & t \le T/2\,, \\ \theta + \pi & t > T/2\,, \end{cases}$$

as well as the dynamic plan induced by constant speed rotation are minimizers corresponding to the constant pressure $P^* = (\pi/T)^2$; when $R > 1$ the constant speed rotation is not a minimizer.

Next, we show that for no rotation ($\psi^1_1 = \mathrm{Id}$) this flow arises as narrow limit of deterministic flows.

Peakon collisions

- Peakon collision means we can compress particles to occupy the same position in finite time and at finite cost
- At small scales the optimal way to do this is using **linear peakons** collision



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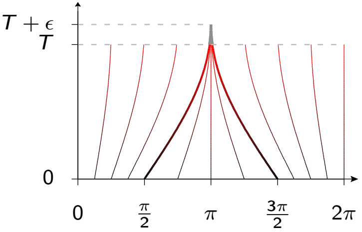
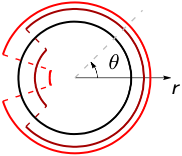
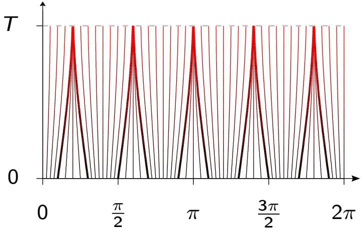
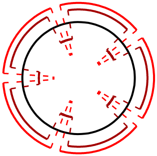




- Jacobian is piecewice constant, collision for $\epsilon \to 0$
- Marginals on the cone at fixed time $(\varphi_t, \lambda_t)_{\#}\rho_0$: at collision concentrated on circle of radius $\sqrt{2}$

Concatenating peakon collisions in space and time yields a sequence of deterministic flows convergenging to a non-deterministic generalized compressible flow





An approximation result on the circle

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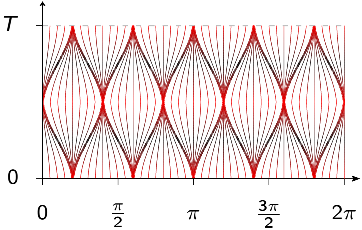

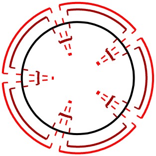

Note: same fixed-time marginals if we rescale the paths to have $r_{T/4} = 1$.

Theorem

There exists a sequence φ^n such that $\boldsymbol{\mu}^n = (\varphi^n, \sqrt{\mathrm{Jac}(\varphi^n)})_{\#}\rho_0$ can be rescaled to $\tilde{\boldsymbol{\mu}}^n \rightharpoonup \boldsymbol{\mu}^*$ (def. as before with no rotation) and $\mathcal{A}(\tilde{\boldsymbol{\mu}}^n) \to \mathcal{A}(\boldsymbol{\mu}^*) = \pi^2/T$.

- Paths starting at and reaching the apex correspond to creation of voids (unbounded Jacobian) in the limit
- Cannot reach final configuration due to topology of S^1 (it would be very inexpensive to rotate most of points at $T/2$)

Rotation on the torus: a non-deterministic solution

Generalized CH problem on $T^2_{1,R} = S^1_1 \times S^1_R$, with coupling in polar coordinates

$$h : (\theta, \phi) \in \mathbb{R}^2/\mathbb{Z}^2_{2\pi} \to (\theta + \pi, \phi + \pi)$$

so that $\mathrm{Jac}(h) = 1$.

Theorem

The dynamic plan

$$\boldsymbol{\mu}^* = \frac{1}{2}(\mathrm{Id}, \zeta^0)_{\#}\rho_0 + \frac{1}{2}(\psi^1, \zeta^1)_{\#}\rho_0 \,,$$

with $\zeta^0_t = \sqrt{2}\sin(\sqrt{P^*}t)$, $\zeta^1_t = \sqrt{2}|\cos(\sqrt{P^*}t)|$ *where* $P^* = (\pi/T)^2$*, and*

$$\psi^1_t(\theta, \phi) = \begin{cases} (\theta, \phi) & t \leq T/2 \,, \\ (\theta + \pi, \phi + \pi) & t > T/2 \,, \end{cases}$$

is a minimizer, whereas the constant speed rotation is not a minimizer.

Note:

- we can prove the result since the action for the rotation on $T^2_{1,R}$ is larger than that on S^1_1 (where we know the minimizer's action)

Adapting 1d construction $\implies$ approximation result holds on the torus with rotation

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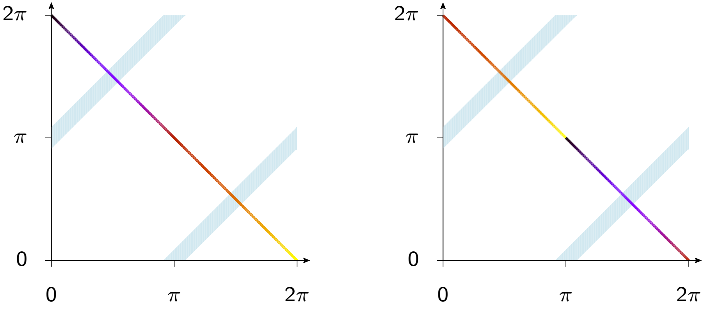
Proof (sketch).

Rotate n stripes in the domain separately when they occupy small area. For 1 stripe:



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An approximation result on the torus

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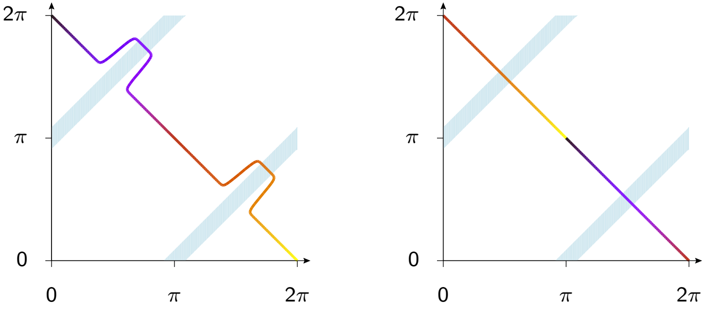

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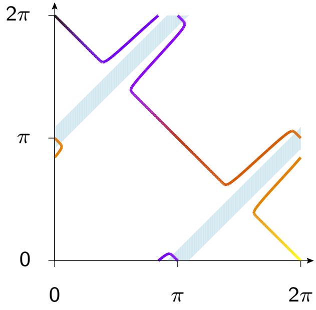

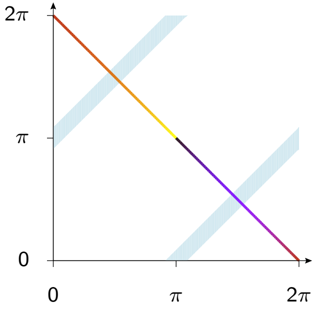

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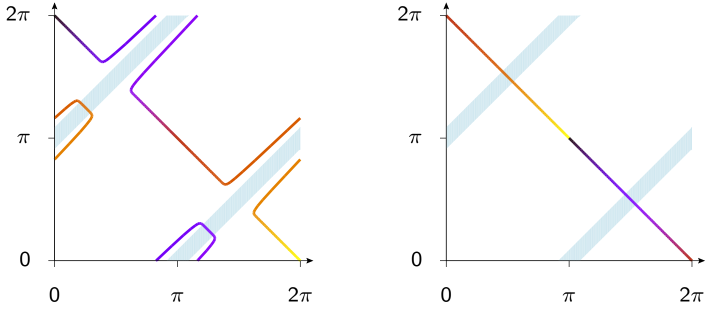

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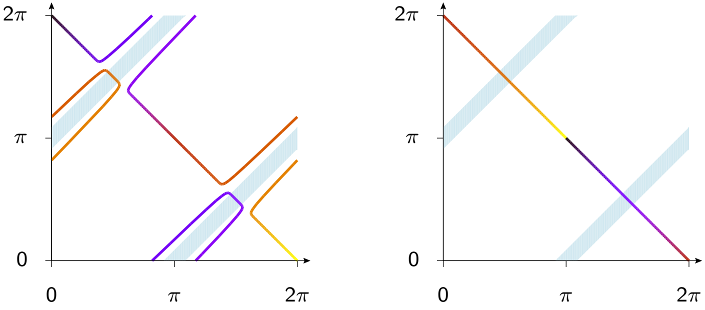

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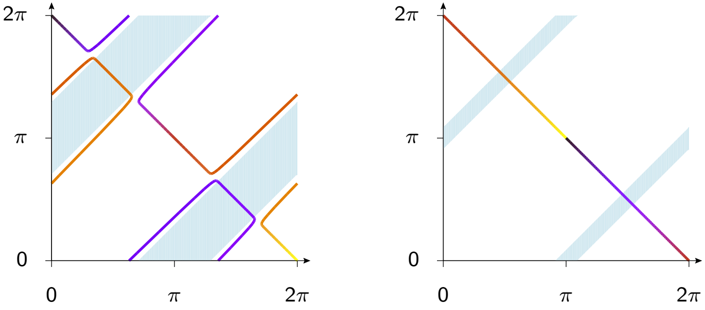

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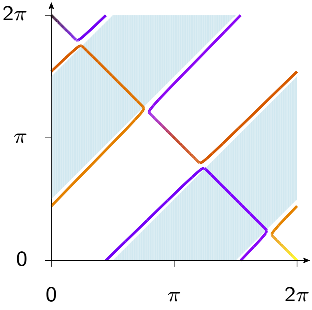

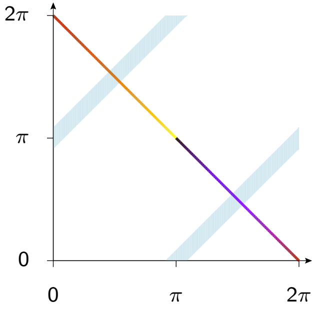

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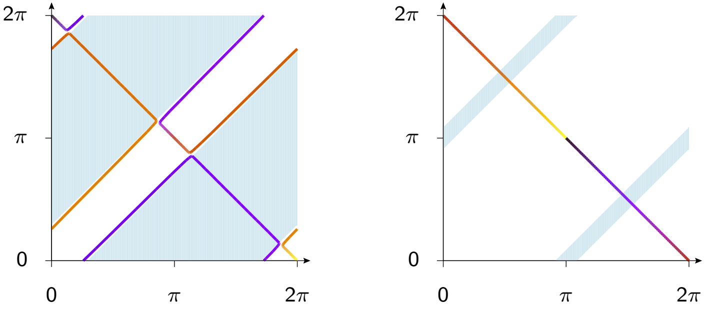

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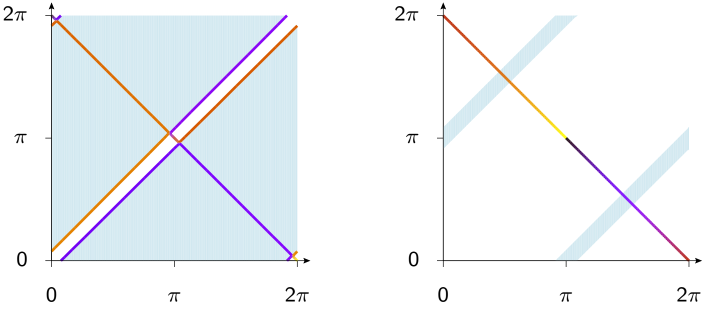

□

Adapting 1d construction $\implies$ approximation result holds on the torus with rotation

Theorem

There exists a sequence φ^n such that $\boldsymbol{\mu}^n = (\varphi^n, \sqrt{\mathrm{Jac}(\varphi^n)})_{\#}\rho_0$ can be rescaled to $\tilde{\boldsymbol{\mu}}^n \rightharpoonup \boldsymbol{\mu}^*$ (minimizer for double rotation) and $\mathcal{A}(\tilde{\boldsymbol{\mu}}^n) \to \mathcal{A}(\boldsymbol{\mu}^*) = \pi^2/T$.

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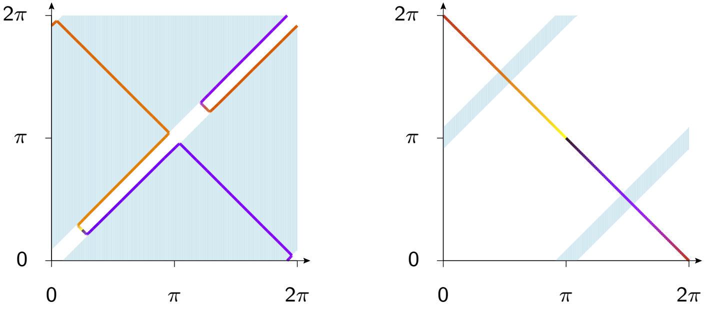

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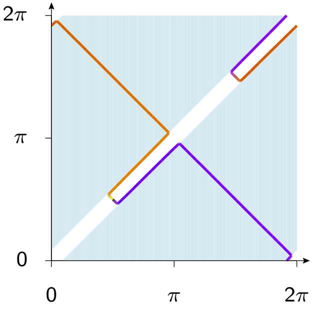

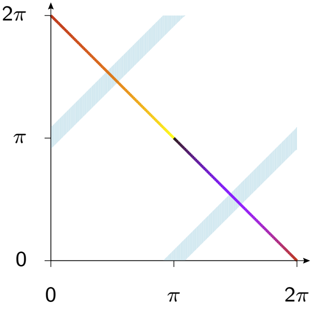

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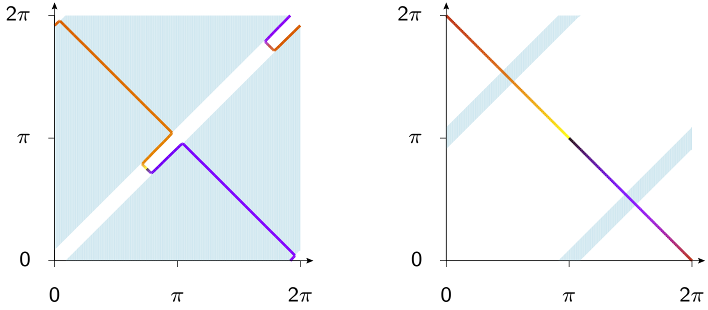

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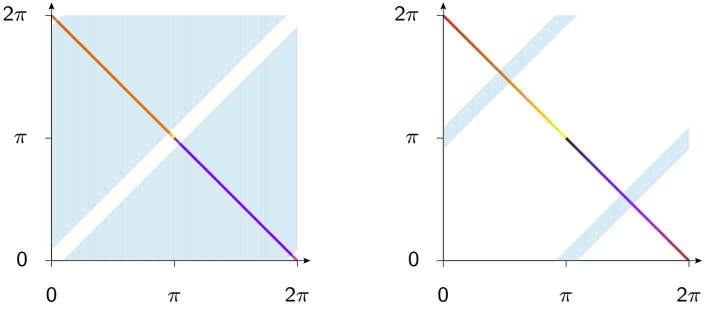

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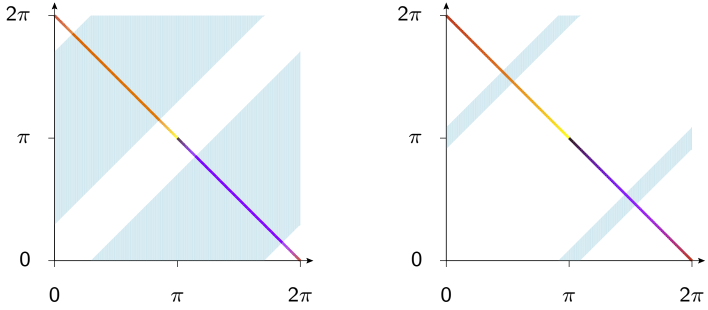

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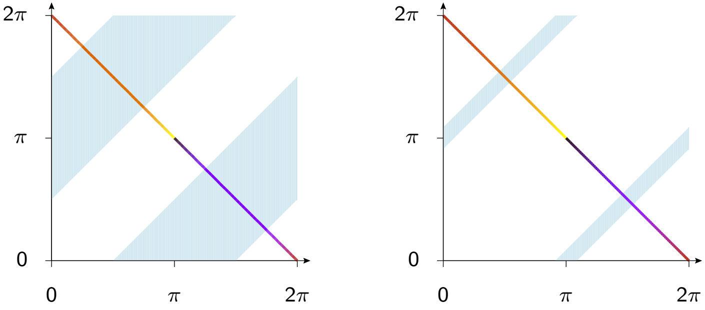

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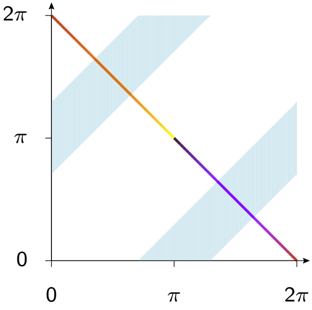

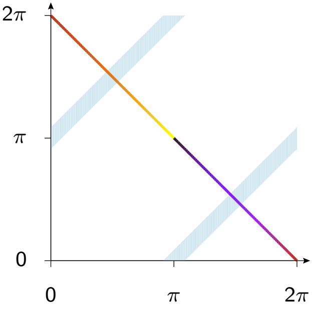

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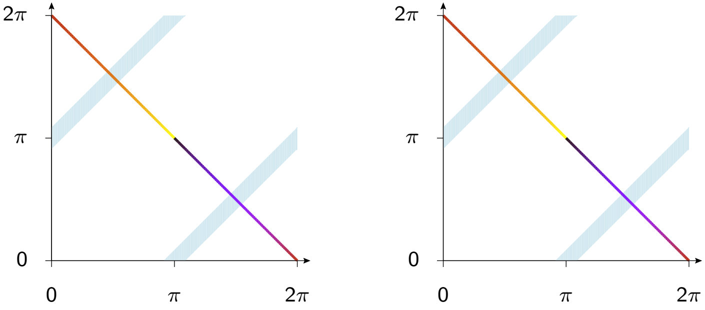

□

Some open questions

- Sharper estimate on pressure for general case ($M \neq S^1_1$)?
- Is non-deterministic rotation the unique minimizer (on S^1_R or $T^2_{1,R}$)?
- Conditions for occurrence of blow up in the general case?
- Pressure always exists as a distribution independently of blow up (Gallouët, Natale, and Vialard, 2018). Regularity?
- Tightness? In 2d no topological impediment at least for rotation. General case?

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