Examples of gradient flows based on optimal transport of differential forms

Xianglong Duan

joint work of Prof. Yann Brenier

Centre de mathémathiques Laurent Schwartz, École Polytechnique

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Curve-shortening flow

Gradient flow structure

Dissipative solutions

Curve-shortening flow

We start with the curve-shortening flow in $\mathbb{R}^d$, which is the 1-dimensional case of the mean-curvature flow:

$$\partial_t X = \frac{1}{|\partial_s X|}\partial_s \left(\frac{\partial_s X}{|\partial_s X|}\right), \tag{1}$$

where $s \in \mathbb{R}/\mathbb{Z} \to X(t,s)$ describes a time-dependent curve in $\mathbb{R}^d$. We introduce a singular vector-valued measure

$$(t,x) \to B(t,x) = \int_{\mathbb{R}/\mathbb{Z}} \delta(x - X(t,s))\partial_s X(t,s) ds \in \mathbb{R}^d,$$

which automatically satisfies $\nabla \cdot B = 0$. Let $v(t,x)$ be the smooth vector field such that

$$\partial_t X(t,s) = v(t, X(t,s)).$$

"Eulerian version" of curve-shortening flow

If the loop $X(t,s)$ solves the curve-shortening flow (1), then (B,v) satisfies the following parabolic type PDEs (in the sense of distributions)

$$\partial_t B + \nabla\cdot(B\otimes v - v\otimes B) = 0, \quad \nabla\cdot B = 0, \tag{2}$$

$$|B|v = \nabla\cdot\left(\frac{B\otimes B}{|B|}\right), \tag{3}$$

which can be interpreted as the "Eulerian version" of the curve-shortening flow. In the framework of optimal transport, these PDEs can be viewed as the gradient flow based on optimal transport of closed $(d-1)-$forms in $\mathbb{R}^d$ with suitable transportation metrics.

Divergence-free vector fields as closed $(d-1)-$forms

The divergence-free vector field B can be seen as a closed $(d-1)-$form. For instance, as $d=3$,

$$\beta = B^1 dx^2 \wedge dx^3 + B^2 dx^3 \wedge dx^1 + B^3 dx^1 \wedge dx^2,$$

$$0 = d\beta = (\partial_1 B^1 + \partial_2 B^2 + \partial_3 B^3) dx^1 \wedge dx^2 \wedge dx^3 = \mathrm{div} B \; dx^1 \wedge dx^2 \wedge dx^3,$$

and these formulae easily extend to arbitrary dimensions d. For simplicity, let's work on the torus $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$.

Transportation of closed $(d-1)-$forms

The concept of transport involves time-dependent closed $(d-1)-$forms β_t and vector fields $v(t,x)$. Let ϕ_t be the group of diffeomorphisms generated by $v(t,x)$ such that

$$\frac{d}{dt}\phi_t = v_t \circ \phi_t, \quad \phi_0 = Id.$$

At any time t, β_t is the pushforward of β_0 by ϕ_t:

$$\beta_t = (\phi_t)_* \beta_0.$$

This implies

$$\frac{d}{dt}\beta_t + \mathcal{L}_{v_t}\beta_t = 0.$$

In terms of divergence-free vector fields, this gives the induction equations

$$\partial_t B + \nabla \cdot (B \otimes v - v \otimes B) = 0.$$

Transportation cost

Mimicking the case of volume forms, we define a transportation cost by introducing, for each fixed fields B, a Hilbert norm depending on B:

$$\|v\|_B = \sqrt{\int_{\mathbb{T}^d} |v|^2 |B|}.$$

We look at the gradient flow of the convex functional

$$\mathcal{F}[B] = \int_{\mathbb{T}^d} |B|.$$

Steepest descent I

By the induction equation (2), we have

$$\frac{d}{dt}\mathcal{F}[B] = \int_{\mathbb{T}^d} \frac{B}{|B|} \cdot \partial_t B = -\int_{\mathbb{T}^d} \frac{B_i}{|B|} \partial_j (B^i v^j - v^i B^j)$$

$$= \int_{\mathbb{T}^d} v^i \left(B^j \partial_i \left(\frac{B_j}{|B|} \right) - B^j \partial_j \left(\frac{B_i}{|B|} \right) \right) = -\int_{\mathbb{T}^d} v \cdot G$$

where

$$G = \nabla \cdot \left(\frac{B \otimes B}{|B|} \right).$$

Steepest descent II

So we have

$$\frac{d}{dt}\mathcal{F}[B] = -\int_{\mathbb{T}^d} v\cdot G = \int_{\mathbb{T}^d} \frac{|G-|B|v|^2}{2|B|} - \frac{1}{2}\int_{\mathbb{T}^d} |v|^2|B| - \frac{1}{2}\int_{\mathbb{T}^d} \frac{|G|^2}{|B|},$$

$$\geq -\frac{1}{2}\int_{\mathbb{T}^d} |v|^2|B| - \frac{1}{2}\int_{\mathbb{T}^d} \frac{|G|^2}{|B|},$$

To get the steepest descent according to norm $||\cdot||_B$, it is enough to saturate this inequality. In other words, choose v such that

$$|B|v = G = \nabla\cdot\left(\frac{B\otimes B}{|B|}\right).$$

This gives exactly the Eulerian version of the curve-shortening flow equations (2),(3).

Idea of dissipative solution I

We define a concept of "dissipative solutions" related to the work of P.-L. Lions for the Euler equation of incompressible fluids or to the work of L. Ambrosio, N. Gigli, G. Savaré for the heat equation and similar to the one introduced by Y. Brenier.
Briefly speaking, for any smooth trial field b^*, v^* with $|b^*| = 1$, we look at the relative entropy

$$\eta = |B| - B \cdot b^* = \frac{(B - |B|b^*)^2}{2|B|} \geq 0.$$

For smooth solution (B, v) to (2)(3), after lengthy computations, we have

$$\frac{d}{dt}\int_{\mathbb{T}^d} \eta + \int_{\mathbb{T}^d} \frac{|P - |B|v^*|^2}{2|B|} \leq c^* \int_{\mathbb{T}^d} \eta + \int_{\mathbb{T}^d} B \cdot \mathrm{L}_2 + \int_{\mathbb{T}^d} P \cdot \mathrm{L}_3$$

where $P = |B|v$, $c^*, \mathrm{L}_2, \mathrm{L}_3$ depending only on b^* and v^*.

Idea of dissipative solution II

By writing

$$\frac{|P-|B|v^*|^2}{2|B|} = \sup_A \left\{ (P-|B|v^*)\cdot A - \frac{A^2}{2}|B| \right\}$$

and integrating from $[0,t]$, we get the family of inequalities

$$e^{-rt}\int_{\mathbb{T}^d}\eta(t) + \int_0^t e^{-r\sigma}\left[\int_{\mathbb{T}^d}\left(r - c^* - \frac{A\cdot(A+2v^*)}{2}\right)\eta \right.$$
$$\left. + P\cdot(A-L_3) - B\cdot\left(L_2 + b^*\frac{A\cdot(A+2v^*)}{2}\right)\right](\sigma)d\sigma \le \int_{\mathbb{T}^d}\eta(0). \tag{4}$$

where r is taken largely enough, so that the inequalities are stable under weakly-$*$ convergence.

Idea of dissipative solution III

This inspires us to give a notion of "very weak" solutions (we call them dissipative solutions), which do not necessarily need to satisfy the equations in the weak or strong sense, but need to satisfy the inequalities (4) for all test functions (b^*, v^*, A) and r large enough. By defining in this way, these solutions enjoy the so called **weak-strong uniqueness**, which means that any dissipative solutions must coincide with a strong solution emanating from the same initial data as long as the latter exists. In the curve-shortening case, we only get the uniqueness for the homogeneous variables $b = B/|B|$ and $v = P/|B|$. There is a lot of room left for the evolution of $|B|$ itself.

THANKS FOR YOUR ATTENTION!

Definition of dissipative solution

Definition

Let us fix $T > 0$. We say that (B, P) with $B \in C([0,T], C(\mathbb{T}^d, \mathbb{R}^d)'_{w^*})$, $\quad P \in C([0,T] \times \mathbb{T}^d, \mathbb{R}^d)'$ is a dissipative solution of the curve-shortening flow with initial data $B_0 \in C(\mathbb{T}^d, \mathbb{R}^d)'$ if and only if:

i) $\quad B(0) = B_0, \quad \nabla \cdot B = 0$ in sense of distributions;

ii) B and P are bounded, respectively in the spaces $C^{1/2}([0,T], (C^1(\mathbb{T}^d))'_{w^*})$ and $C([0,T] \times \mathbb{T}^d, \mathbb{R}^d)'$, by constants depending only on T and $\int_{\mathbb{T}^d} |B_0|$.

iii) For all $\lambda > 0$, $\theta \in [0,T]$, for all smooth trial functions (b^*, v^*, A) valued in $\mathbb{R}^d$, with $\|A\|_\infty \le \lambda$ and ${b^*}^2 = 1$, for all $r \ge c^* + \frac{\lambda^2}{2} + \lambda \|v^*\|_\infty$, (4) is satisfied.

$$\frac{d}{dt}\int_{\mathbb{T}^d}\eta+\int_{\mathbb{T}^d}\frac{(P-\rho v^*)^2}{\rho}=\int_{\mathbb{T}^d}\frac{1}{2\rho}(B_i-\rho b_i^*)(B_j-\rho b_j^*)(\partial_j v_i^*+\partial_i v_j^*)$$

$$-\int_{\mathbb{T}^d}\frac{1}{\rho}(B_i-\rho b_i^*)(P_j-\rho v_j^*)(\partial_j b_i^*-\partial_i b_j^*)$$

$$+\int_{\mathbb{T}^d}\eta\,\mathrm{L}_1+\int_{\mathbb{T}^d}B\cdot\mathrm{L}_2+\int_{\mathbb{T}^d}P\cdot\mathrm{L}_3$$

where

$$\rho=|B|,\quad \mathrm{L}_1=v^{*2}-b^*\cdot\nabla(b^*\cdot v^*),$$

$$\mathrm{L}_2=-D_t^*b^*+(b^*\cdot\nabla)v^*+\nabla(b^*\cdot v^*)+b^*(v^{*2}-b^*\cdot\nabla(b^*\cdot v^*)),$$

$$D_t^*=(\partial_t+v^*\cdot\nabla),\quad \mathrm{L}_3=-v^*+(b^*\cdot\nabla)b^*,$$