THE MULTI-MARGINAL OPTIMAL PARTIAL TRANSPORT PROBLEM

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Abstract. We introduce and study a multi-marginal optimal partial transport problem. Under a natural and sharp condition on the dominating marginals, we establish uniqueness of the optimal plan. Our strategy of proof establishes and exploits a connection with another novel problem, which we call the Monge-Kantorovich partial barycenter problem (with quadratic cost). This latter problem has a natural interpretation as a variant of the mines and factories description of optimal transport. We then turn our attention to various analytic properties of these two problems. Of particular interest, we show that monotonicity of the active marginals can fail, a surprising difference from the two marginal case.

1. Introduction

Throughout this paper, whenever we write "measure" it will tacitly be assumed that we are referring to a positive, Borel measure on the relevant space in question. In all but the last section, we also assume a measure μ, when it is defined on $\mathbb{R}^n$, has finite second moment; i.e. that $\int_{\mathbb{R}^n}|x|^2\mu(dx) < \infty$. Also, "absolutely continuous", "a.e.", "null set", and "zero measure" without any further qualifiers will always be with respect to the Lebesgue measure. Finally, for any measure μ, we will write

$$\mathcal{M}(\mu) := \mu(X),$$

where X is the entire space that μ is defined on.

Recall the classical optimal transport problem with quadratic cost: let μ and ν be measures on $\mathbb{R}^n$ satisfying the mass constraint $\mathcal{M}(\mu) = \mathcal{M}(\nu) < \infty$, and write $\Pi(\mu, \nu)$ for the collection of all measures on $\mathbb{R}^n \times \mathbb{R}^n$ whose left and right marginals equal μ and ν, respectively. Then a solution of the optimal transport problem (with quadratic cost) is a measure $\gamma \in \Pi(\mu, \nu)$

2010 *Mathematics Subject Classification.* 35J96.

This material is based upon work supported by the National Science Foundation under Grant No. 0932078 000, while both authors were in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Fall 2013 program on Optimal Transport: Geometry and Dynamics. We would like to thank both MSRI and the organizers of the program for their generous hospitality. In addition, B.P. is pleased to acknowledge the support of a University of Alberta start-up grant and National Sciences and Engineering Research Council of Canada Discovery Grant number 412779-2012.

achieving the minimum value in

$$\mathcal{MK}_2^2(\mu,\nu) := \min_{\gamma'\in\Pi(\mu,\nu)} \int_{\mathbb{R}^n\times\mathbb{R}^n} |x-y|^2\gamma'(dx,dy). \tag{OT}$$

We will denote the collection of solutions to (OT) above as $\mathrm{Opt}(\mu,\nu)$. Existence of an optimizer is not difficult to show; a famous theorem of Brenier implies that if the first measure is absolutely continuous, and both measures have finite second moments, the solution is unique and is in fact concentrated on the graph $\{(x,T(x))\}$ of a function over the first variable [2, 3]. This result has been extended to a wide class of other costs (see, for example, [11, 12, 5]).

We will be concerned here with two natural extensions of (OT) above. The first is the optimal partial transport problem: let μ and ν be measures on $\mathbb{R}^n$ each with finite total mass (not necessarily equal), fix any $0\le m\le \min\{\mathcal{M}(\mu),\ \mathcal{M}(\nu)\}$, and write $\Pi_{\le}(\mu,\nu)$ for the collection of all measures on $\mathbb{R}^n\times\mathbb{R}^n$ whose left and right marginals are dominated by μ and ν respectively, that is, $(\pi_1)_\#\gamma(E)\le\mu(E)$ and $(\pi_2)_\#\gamma(E)\le\nu(E)$ for any measurable set E, where π_j denotes projection onto the jth coordinate (for ease of notation we will simply write $\mu_\le\le\mu$ to indicate that $\mu_\le$ is dominated by μ). Then a solution of the optimal partial transport problem (again with quadratic cost) is a measure $\gamma\in\Pi_\le(\mu,\nu)$ with $\mathcal{M}(\gamma)=m$ achieving the minimum value in

$$\mathcal{MK}_{2,m}^2(\mu,\nu) := \min_{\gamma'\in\Pi_\le(\mu,\nu),\ \mathcal{M}(\gamma')=m} \int_{\mathbb{R}^n\times\mathbb{R}^n} |x-y|^2\gamma'(dx,dy). \tag{$\mathrm{OT_m}$}$$

We will denote the collection of solutions to ($\mathrm{OT_m}$) above as $\mathrm{Opt}_m(\mu,\nu)$. Again, existence of an optimal measure can be established in a straightforward way. Uniqueness is much more involved; however, when the supports of the two measures are separated by a hyperplane, Caffarelli and McCann established a uniqueness result (in addition to several properties of the minimizer, see [6]). This assumption on the measures was weakened by Figalli in [10]; he assumed only that the pointwise minimum of the two measures has total mass not greater than m. This is easily seen to be a sharp condition for uniqueness; if it fails, then for any measure $\mu_\le$ with mass m satisfying both $\mu_\le\le\mu$ and $\mu_\le\le\nu$, the diagonal coupling $(\mathrm{Id}\times\mathrm{Id})_\#\mu_\le$ is clearly optimal. In addition, Figalli extended his results to a larger class of cost functions.

On the other hand, one can also consider the multi-marginal optimal transport problem: let μ_j for $j=1,\dots,N$ be measures on $\mathbb{R}^n$ all with equal, finite mass, and write $\Pi(\mu_1,\dots,\mu_N)$ for the collection of all measures on $(\mathbb{R}^n)^N$ whose jth marginal equals μ_j. Then a solution of the multi-marginal optimal transport problem, with Gangbo-Święch cost:

$$c(x_1,\dots,x_N) := \sum_{j\ne k}^N |x_j-x_k|^2,$$

(see [13]), is a measure $\sigma \in \Pi(\mu_1, \dots, \mu_N)$ achieving the minimum value in

$$\min_{\sigma' \in \Pi(\mu_1, \dots, \mu_N)} \mathcal{C}(\sigma'), \tag{MM}$$

where

$$\mathcal{C}(\sigma') := \int_{(\mathbb{R}^n)^N} c(x_1, \dots, x_N) \sigma'(dx_1, \dots, dx_N).$$

Once more, existence can be established in a straightforward way. Assuming the first measure is absolutely continuous, Gangbo and Święch proved that the optimizer is unique and, like in the two marginal case, is concentrated on a graph $\{(x_1, T_2(x_1), \dots, T_N(x_1))\}$ over the first variable [13]. This result has been extended to certain other cost functions [7, 15, 18, 14]; these costs are very special, however, and for a variety of other costs, counterexamples to uniqueness and the graphical structure are known [9, 17, 16], indicating that these properties depend delicately on the cost function for multi-marginal problems.

In this paper, we combine these two extensions ($\mathrm{OT_m}$) and (MM) and consider the <u>multi-marginal optimal partial transport problem</u>: let μ_j for $j = 1, \dots, N$ be measures on $\mathbb{R}^n$ all with finite (but not necessarily equal) total mass, fix any $0 \le m \le \min_{1 \le j \le N} \{\mathcal{M}(\mu_j)\}$, and write $\Pi_{\le}(\mu_1, \dots, \mu_N)$ for the collection of all measures on $(\mathbb{R}^n)^N$ whose jth marginal is dominated by μ_j for each j. Then a solution of the multi-marginal optimal partial transport problem (with Gangbo-Święch cost) can be defined as a measure $\sigma \in \Pi_{\le}(\mu_1, \dots, \mu_N)$ with $\mathcal{M}(\sigma) = m$ achieving the minimum value in

$$\min_{\sigma' \in \Pi_{\le}(\mu_1, \dots, \mu_N),\ \mathcal{M}(\sigma') = m} \mathcal{C}(\sigma'). \tag{$\mathrm{MM_m}$}$$

Analogously to the above, we will denote the collection of solutions to ($\mathrm{MM_m}$) as $\mathrm{Opt}_m(\mu_1, \dots, \mu_N)$. In informal exposition, we will sometimes refer to any marginal of a minimizer in either ($\mathrm{OT_m}$) or ($\mathrm{MM_m}$) as an "active submeasure."

As in ($\mathrm{OT_m}$), existence of a minimizer in ($\mathrm{MM_m}$) is not difficult to see; the first issue one encounters is that of uniqueness, which will be the focus of this paper. Our main goal is to identify conditions under which the multi-marginal problem ($\mathrm{MM_m}$) admits a unique solution; it turns out that a condition analogous to the one given by Figalli in [10] is sufficient, see Theorem 1.2 below.

Our approach here involves the analysis of another problem, which turns out to be essentially equivalent to ($\mathrm{MM_m}$) and which we call the <u>(Monge-Kantorovich) partial barycenter problem</u>. This is a natural extension of the usual (Monge-Kantorovich) barycenter problem, which is, given measures $\mu_1, \dots, \mu_N$, all with mass m, to find a minimizer of

$$\min_{\nu \in \mathcal{P}^m} \sum_{j=1}^N \mathcal{MK}_2^2(\mu_j, \nu), \tag{BC}$$

where

$$\mathcal{P}^m := \{\text{all measures } \nu \mid \mathcal{M}(\nu) = m,\ \int_{\mathbb{R}^n} |x|^2 \nu(dx) < \infty\}.$$

This problem was introduced by Agueh and Carlier, who showed it is essentially equivalent to (MM) (see [1]).

We introduce the appropriate analogue, the partial barycenter, as a minimizer in

$$\min_{\nu \in \mathcal{P}^m} F_m(\nu, \mu_1, \dots, \mu_N), \tag{$\mathrm{BC_m}$}$$

where m and the μ_i are as in ($\mathrm{MM_m}$) and,

$$F_m(\nu, \mu_1, \dots, \mu_N) := \sum_{j=1}^{N} \mathcal{MK}^2_{2,m}(\mu_j, \nu).$$

When the collection of measures $\mu_1, \dots, \mu_N$ and the mass constraint m is clear, we will suppress them and simply write $F(\nu)$ in place of $F_m(\nu, \mu_1, \dots, \mu_N)$. Also, we may sometimes refer to the submeasures of μ_j that are actually coupled to a minimizer of ($\mathrm{BC_m}$) as "active submeasures" as well.

We will first show there is a connection between the problems ($\mathrm{MM_m}$) and ($\mathrm{BC_m}$) (which is analogous to the relationship between (MM) and (BC) in [1]), expressed by the following theorem:

Proposition 1.1 (Equivalence of ($\mathrm{BC_m}$) and ($\mathrm{MM_m}$))**.** *For $1 \leq j \leq N$, fix absolutely continuous measures μ_j, and some $0 < m \leq \min_{1\leq j\leq N} \mathcal{M}(\mu_j)$.*

Then for any optimal measure σ in ($\mathrm{MM_m}$)*, $A_{\#}\sigma$ is optimal in* ($\mathrm{BC_m}$)*, where*

$$A(x_1, \dots, x_N) := \frac{1}{N}\sum_{j=1}^{N} x_j. \tag{1}$$

Conversely, for any minimizer ν in ($\mathrm{BC_m}$)*, the measure $(S^\nu_1, \dots, S^\nu_N)_{\#}\nu$ is optimal in* ($\mathrm{MM_m}$)*, where S^ν_j is the optimal mapping such that $\left(S^\nu_j \times \mathrm{Id}\right)_{\#}\nu \in \mathrm{Opt}_m(\mu_j, \nu)$ for each $1 \leq j \leq N$.*

Furthermore, the minimizer of ($\mathrm{MM_m}$) *is unique if and only the minimizer of* ($\mathrm{BC_m}$) *is unique.*

Then, we will turn to the question of uniqueness in ($\mathrm{BC_m}$). We establish the following theorem, which shows that under conditions analogous to those in [10], we indeed obtain uniqueness in ($\mathrm{BC_m}$):

Theorem 1.2 (Uniqueness of partial barycenters)**.** *For $1 \leq j \leq N$, fix absolutely continuous measures μ_j, each with finite mass and with densities g_j. Writing $\mu_\wedge$ for the absolutely continuous measure with density $g_\wedge := \min_{1\leq j\leq N} g_j$, fix some $m \geq 0$ satisfying*

$$\mathcal{M}(\mu_\wedge) \leq m \leq \min_{1\leq j\leq N}\{\mathcal{M}(\mu_j)\}.$$

Then there exists a unique minimizer in $\mathcal{P}^m$ *of* $(\mathrm{BC_m})$.

Finally, by combining Proposition 1.1 with Theorem 1.2, we immediately obtain the following corollary:

Corollary 1.3. *Under the assumptions of Theorem 1.2, the multi-marginal optimal partial transport problem* $(\mathrm{MM_m})$ *has a unique solution.*

Surprisingly, several of the monotonicity properties enjoyed by solutions of $(\mathrm{OT_m})$ are not exhibited by solutions of $(\mathrm{MM_m})$; we will briefly demonstrate this fact with some examples later.

One might expect that an alternature approach, following the work of Figalli [10], could be used to establish Corollary 1.3; more precisely, that one could show that the function $m \mapsto \min_{\sigma' \in \Pi_{\leq}(\mu_1,\ldots,\mu_N),\ \mathcal{M}(\sigma')=m} \mathcal{C}(\sigma')$ is strictly convex on $[\mathcal{M}(\mu_\wedge), \min_{1\leq j\leq N}\{\mathcal{M}(\mu_j)\}]$ and use this fact to deduce uniqueness of the optimal plan in $(\mathrm{MM_m})$. As one of our examples illustrates (see Remark 4.2 below), it turns out that the natural multi-marginal analogue of a key preliminary result of Figalli (Proposition 2.4 in [10]) *does not* hold. As this proposition is used in a crucial way in the proof of Figalli's main result ([10, Theorem 2.10]), a direct extension of his techniques cannot be used to prove Corollary 1.3.

We pause now to describe an economic interpretation of the partial barycenter problem, in the context of the well known factories-and-mines interpretation of the classical optimal transport problem.

1.1. **Interpretation of the partial barycenter problem.** The optimal transport problem is frequently interpreted as the problem of matching the production of a resource (say iron ore) by a distribution of mines over a landscape $M \subset \mathbb{R}^n$ (represented by the measure μ) with consumption of that resource by a distribution of factories over the same landscape (represented by a measure ν). The cost function $c(x, y)$ $(= |x-y|^2$ in our setting) represents the cost to move one unit of iron from a mine at position x to a factory at position y. If the total production capacity of the mines matches the total consumption capacity by the factories (that is, the total masses of μ and ν coincide), and one would like to use all of the produced resources, the problem of determining which mine should supply which factory to minimize the total transportation cost is represented by (OT). More realistically, the total production capacity of the mines may not match the total consumption capacity of the factories, and one may only wish to consume a smaller portion m of the total capacity; in this case, the analogous problem is represented by $(\mathrm{OT_m})$, as is discussed in [6].

Suppose now that production of a certain good requires several resources; for example, iron, aluminum, and nickel, and that the company has not yet built their factories (and so is free to build them at any locations they choose). Production capacity of the resources are given by distributions μ_j of mines over a landscape $M \subseteq \mathbb{R}^n$, for $j = 1, 2, 3, \ldots, N$. Given costs $c_j(x_j, y)$ $(|x_j - y|^2$ here, but see also the extension to more general costs in Section 5) to move a unit of resource j from a mine at position x_j to a (potential) factory at position y, the company now wishes to *build* a distribution of factories, ν, where these resources will be consumed, in order to minimize the sum of all the total transportation costs; if the total production of each resource is the same and all produced resources are to be consumed, this amounts to the barycenter problem (BC).

However, if, perhaps because of limited demand for the good in question, only a fixed portion m of each resource is to be consumed (less than the smallest total production capacity of the resources, which may now differ for different j), one obtains the partial barycenter problem ($\mathrm{BC_m}$).

1.2. Organization of the paper.

The remainder of this paper is organized as follows. In Section 2 we will establish Proposition 1.1. Section 3 is then devoted to the proof of Theorem 1.2. In Section 4, we discuss some other properties of interest of minimizers of ($\mathrm{MM_m}$) and ($\mathrm{BC_m}$). Namely, we first present two somewhat surprising counterexamples to the monotonicity property, followed by a discussion of points where the active submeasures fail to saturate the prescribed measures μ_j. We close the section with a brief remark on regularity properties of the "free boundary". Finally, in Section 5 we discuss an extension of our main results to more general cost functions.

2. Connection between multi-marginal optimal partial transport and the partial barycenter

For technical reasons, we will find it more convenient to work with absolutely continuous measures, hence we define the following notation:

$$\mathcal{P}^m_{ac} := \{\text{absolutely continuous } \nu \mid \mathcal{M}(\nu) = m,\ \int_{\mathbb{R}^n} |x|^2 \nu(dx) < \infty\}.$$

A simple argument now shows that any minimizer in ($\mathrm{BC_m}$) is actually absolutely continuous, hence it is equivalent to make the minimization over $\mathcal{P}^m_{ac}$ in the problem, rather than $\mathcal{P}^m$.

Lemma 2.1.

$$\min_{\nu\in\mathcal{P}^m} F_m(\nu, \mu_1, \dots, \mu_N) = \min_{\nu\in\mathcal{P}^m_{ac}} F_m(\nu, \mu_1, \dots, \mu_N).$$

Proof. We show that any optimal $\nu \in \mathcal{P}^m$ at which the minimum in the left hand side is attained must actually be absolutely continuous; the result will then follow immediately.

Note that any such ν is necessarily optimal in the classical barycenter problem (BC) for the active submeasures. As these are necessarily absolutely continuous by the absolute continuity of the μ_j, the result in [18, Theorem 3.3] (see also [1, Theorem 5.1]) implies the absolute continuity of ν. □

With the above result in hand, we can now show Proposition 1.1.

Proof of Proposition 1.1. Fix m as in the statement of the proposition. It is straightforward to verify that, for any $x_1, \ldots, x_N \in \mathbb{R}^n$,

$$\sum_{j \neq k} |x_j - x_k|^2 = \min_{y \in \mathbb{R}^n} \sum_{j=1}^{N} |x_j - y|^2 \tag{2}$$

and that the minimum on the right hand side is attained uniquely at $y = A(x_1, \ldots, x_N)$ (recall the definition (1)). The proof of the first two assertions is then a straightforward adapatation of the argument of Carlier and Ekeland in [8, Proposition 3] (in fact, the three bullet points below closely mirror the three assertions in the proof of [8, Proposition 3]), and we only list the main steps:

- For any $\nu \in \mathcal{P}_{ac}^m$, let S_j^ν be the optimal mapping satisfying $\left(S_j^\nu \times \mathrm{Id}\right)_\# \nu \in \mathrm{Opt}_m(\mu_j, \nu)$. The measure $\sigma := (S_1^\nu, \ldots, S_N^\nu)_\# \nu$ is admissible in ($\mathrm{MM_m}$), and $\mathcal{C}(\sigma) \leq F(\nu)$, hence (also recalling Lemma 2.1) the minimum value in ($\mathrm{MM_m}$) is less than the minimum value in ($\mathrm{BC_m}$).
- For any minimizing σ in ($\mathrm{MM_m}$), if we define $\nu := A_\# \sigma$, we then have $\mathcal{C}(\sigma) = F(\nu)$, and in light of the above, the minimum values in ($\mathrm{MM_m}$) and ($\mathrm{BC_m}$) are equal, and ν is a minimizer in ($\mathrm{BC_m}$). It also follows that $(\pi_j \times A)_\# \sigma$ is a minimizer in the partial transport problem ($\mathrm{OT_m}$) between μ_j and ν, where we move mass $m = \mathcal{M}(\nu)$; i.e. it belongs to $\mathrm{Opt}_m(\mu_j, \nu)$.
- For any minimizing $\nu \in \mathcal{P}_{ac}^m$ in ($\mathrm{BC_m}$), the measure $\sigma := (S_1^\nu, \ldots, S_N^\nu)_\# \nu$ must now be minimizing in ($\mathrm{MM_m}$) by the above two points.

Turning to the uniqueness assertion, we first assume that the solution ν to ($\mathrm{BC_m}$) is unique. Note that solutions to ($\mathrm{MM_m}$) are in particular optimal in the regular multi-marginal problem (MM) for their marginals. Uniqueness of minimizers in (MM) (see [13]) then implies that, if σ and $\bar{\sigma}$ are distinct minimizers in ($\mathrm{MM_m}$), at least one of their marginals must differ.

Since $A_\# \sigma$ and $A_\# \bar{\sigma}$ are minimizers in ($\mathrm{BC_m}$) by the above, by our uniqueness assumption we have $A_\# \sigma = A_\# \bar{\sigma} = \nu$. Additionally, $\gamma_j := (\pi_j \times A)_\# \sigma$ and $\bar{\gamma}_j := (\pi_j \times A)_\# \bar{\sigma}$ both belong to $\mathrm{Opt}_m(\mu_j, \nu)$. Since clearly m satisfies the hypothesis of the uniqueness result [10, Proposition 2.2 and Theorem 2.10], we then have $\gamma_j = \bar{\gamma}_j$ for all j, and in particular, for all j the marginals $(\pi_j)_\# \sigma$ and $(\pi_j)_\# \bar{\sigma}$ must coincide, which is a contradiction.

Conversely, suppose the solution σ to ($\mathrm{MM_m}$) is unique. If ν and $\bar{\nu} \in \mathcal{P}_{ac}^m$ both minimize ($\mathrm{BC_m}$), we have that $(S_1^\nu, \ldots, S_N^\nu)_\# \nu = \sigma = (S_1^{\bar{\nu}}, \ldots, S_N^{\bar{\nu}})_\# \bar{\nu}$. In particular, note that $\left(S_j^\nu\right)_\# \nu = (\pi_j)_\# \sigma = \left(S_j^{\bar{\nu}}\right)_\# \bar{\nu}$ and both ν and $\bar{\nu}$ solve the regular barycenter problem (BC), for the measures $\left(S_j^\nu\right)_\# \nu = \left(S_j^{\bar{\nu}}\right)_\# \bar{\nu}$ in place of the μ_j. Thus it follows from the uniqueness result [1, Proposition 3.5] that $\nu = \bar{\nu}$. □

3. Uniqueness of the partial barycenter

We now turn to the proof of Theorem 1.2, uniqueness of the partial barycenter in the minimization problem ($\mathrm{BC_m}$). Throughout this section, for any measure γ on $\mathbb{R}^n \times \mathbb{R}^n$, we will use the following notation:

$$\mathcal{C}_2(\gamma) := \int_{\mathbb{R}^n\times\mathbb{R}^n} |x-y|^2\gamma(dx,dy).$$

In the first lemma, we show that for a fixed μ, the functional $\mathcal{MK}^2_{2,m}(\mu,\cdot)$ is convex with respect to linear interpolation. Additionally, we show that non-strict convexity along a segment connecting two measures ν^0 and ν^1 implies some structure of the optimal mappings pushing ν^i forward to a submeasure of μ: namely that the optimal mappings must match on the support of the "pointwise minimum" of ν^0 and ν^1, while both mappings must be the identity mapping when this "pointwise minimum" fails to saturate.

Lemma 3.1. *Let μ be an absolutely continuous measure with $\mathcal{M}(\mu) \geq m$, let ν^0, $\nu^1 \in \mathcal{P}^m_{ac}$, and define $\nu^t := (1-t)\nu^0 + t\nu^1$. Then*

$$\mathcal{MK}^2_{2,m}\left(\mu,\nu^t\right) \leq (1-t)\mathcal{MK}^2_{2,m}\left(\mu,\nu^0\right) + t\mathcal{MK}^2_{2,m}\left(\mu,\nu^1\right)$$

for all $t\in[0,1]$.

Now suppose equality holds for all $t\in[0,1]$, and for $i=0$ or 1, let S^i be any measurable mapping satisfying $\left(S^i\times\mathrm{Id}\right)_\#\nu^i \in \mathrm{Opt}_m\left(\mu,\nu^i\right)$. Also, let $\nu^\wedge$ be the measure with density $f^\wedge := \min\{f^0,f^1\}$, where f^i is the density of ν^i. Then we have that

$$S^0(x) = S^1(x) \text{ a.e. on } \{f^\wedge > 0\}, \tag{3}$$

$$S^i(x) = x \text{ a.e. on } \{f^\wedge < f^i\}. \tag{4}$$

Proof. For $i = 0$, 1 suppose that $\mu^i_\leq \leq \mu$ with total mass m, and $\gamma^i \in \mathrm{Opt}\left(\mu^i_\leq,\nu^i\right)$ satisfy $\mathcal{C}_2\big(\gamma^i\big) = \mathcal{MK}^2_{2,m}\left(\mu,\nu^i\right)$. For any $t\in[0,1]$, it is clear that for $\gamma^t := (1-t)\gamma^0 + t\gamma^1$ we have

$$(\pi_1)_\#\gamma^t = (1-t)\mu^0_\leq + t\mu^1_\leq \leq \mu,$$
$$(\pi_2)_\#\gamma^t = (1-t)\nu^0 + t\nu^1 = \nu^t,$$

thus we easily see that

$$\begin{aligned}\mathcal{MK}^2_{2,m}\left(\mu,\nu^t\right) \leq \mathcal{C}_2\big(\gamma^t\big) &= (1-t)\mathcal{C}_2\big(\gamma^0\big) + t\mathcal{C}_2\big(\gamma^1\big)\\ &= (1-t)\mathcal{MK}^2_{2,m}\left(\mu,\nu^0\right) + t\mathcal{MK}^2_{2,m}\left(\mu,\nu^1\right).\end{aligned}$$

We now turn to the proof of (3) and (4). Suppose that there is non-strict convexity along $\mu^t_\leq$, i.e.

$$\mathcal{MK}^2_{2,m}\left(\mu,\nu^t\right) = (1-t)\mathcal{MK}^2_{2,m}\left(\mu,\nu^0\right) + t\mathcal{MK}^2_{2,m}\left(\mu,\nu^1\right), \qquad \forall t\in[0,1],$$

in particular $\mathcal{MK}^2_{2,m}\left(\mu,\nu^t\right) = \mathcal{C}_2\big(\gamma^t\big)$ and hence $\gamma^t \in \mathrm{Opt}_m\left(\mu,\nu^t\right)$. Note that all ν^i and $\mu^i_\leq$ are absolutely continuous; we denote their densities by f^i and $g^i_\leq$ respectively. Also we may apply the classical result of Brenier (see [3]) to see there exist a.e. defined mappings $T^i : \operatorname{spt}\mu^i_\leq \to \operatorname{spt}\nu^i$ and $S^i : \operatorname{spt}\nu^i \to \operatorname{spt}\mu^i_\leq$ such that $T^i_\# \mu^i_\leq = \nu^i$, $\left(\mathrm{Id}\times T^i\right)_\# \mu^i_\leq = \gamma^i = \left(S^i \times \mathrm{Id}\right)_\# \nu^i$, and $S^i = \left(T^i\right)^{-1}$ a.e. on $\operatorname{spt}\nu^i$.

We will now show (3). Indeed, by the assumption of non-strict convexity along ν^t, for any $t \in [0,1]$ we have that $\gamma^t \in \operatorname{Opt}\left(\mu^t_\leq, \nu^t\right)$, where $\mu^t_\leq := (1-t)\mu^0_\leq + t\mu^1_\leq$, and in particular $\gamma^{1/2}$ is concentrated on the graph of a mapping $S^{1/2} : \operatorname{spt}\mu^{1/2}_\leq \to \operatorname{spt}\nu^{1/2}$. On the other hand, $\gamma^{1/2}$ is clearly supported on the union of the graphs of S^0 and S^1, and therefore we must have $S^0 = S^{1/2} = S^1$ a.e. on the set where both S^0 and S^1 are defined, which includes $\{f^\wedge > 0\}$. This immediately implies (3).

We next work toward (4). As a result of (3), we can unambiguously define

$$\gamma^\wedge := \left(S^0 \times \mathrm{Id}\right)_\# \nu^\wedge = \left(S^1 \times \mathrm{Id}\right)_\# \nu^\wedge$$

with mass $m^\wedge := \mathcal{M}\left(\gamma^\wedge\right) \leq m$. Here we claim that

$$(\pi_1)_\# \gamma^\wedge = \mu^\wedge_\leq$$

where $\mu^\wedge_\leq$ is the absolutely continuous measure with density $g^\wedge_\leq := \min\{g^0_\leq,\ g^1_\leq\}$. To see this, first note it is clear that

$$(\pi_1)_\# \gamma^\wedge = S^0_\# \nu^\wedge = S^1_\# \nu^\wedge \leq \mu^\wedge_\leq.$$

Next, we can apply the arguments leading up to (3) with T^i replacing S^i to find that for a.e. $x \in \{g^\wedge_\leq > 0\}$ we have $T^0(x) = T^1(x)$, hence

$$\begin{aligned} &T^0_\# \mu^\wedge_\leq = T^1_\# \mu^\wedge_\leq \leq \nu^\wedge \\ \implies\ &\mu^\wedge_\leq = S^0_\# T^0_\# \mu^\wedge_\leq \leq S^0_\# \nu^\wedge = (\pi_1)_\# \gamma^\wedge, \end{aligned}$$

finishing the claim.

Next we claim that for any $t \in [0,1]$ we have

$$\begin{aligned} (1-t)(\gamma^0 - \gamma^\wedge) &\in \operatorname{Opt}_{(1-t)(m-m^\wedge)}\left(\mu^0_\leq - \mu^\wedge_\leq, (1-t)(\nu^0 - \nu^\wedge)\right), \\ t(\gamma^1 - \gamma^\wedge) &\in \operatorname{Opt}_{t(m-m^\wedge)}\left(\mu^1_\leq - \mu^\wedge_\leq, t(\nu^1 - \nu^\wedge)\right). \end{aligned} \tag{5}$$

Suppose by contradiction that the claim fails, then there exist

$$\begin{aligned} \bar\gamma^0 &\in \operatorname{Opt}_{(1-t)(m-m^\wedge)}\left(\mu^0_\leq - \mu^\wedge_\leq, (1-t)(\nu^0 - \nu^\wedge)\right), \\ \bar\gamma^1 &\in \operatorname{Opt}_{t(m-m^\wedge)}\left(\mu^1_\leq - \mu^\wedge_\leq, t(\nu^1 - \nu^\wedge)\right) \end{aligned}$$

with

$$\begin{aligned} \mathcal{C}_2\left(\bar\gamma^0\right) + \mathcal{C}_2\left(\bar\gamma^1\right) &< \mathcal{C}_2\left((1-t)(\gamma^0 - \gamma^\wedge)\right) + \mathcal{C}_2\left(t(\gamma^1 - \gamma^\wedge)\right) \\ &= \mathcal{C}_2\left(\gamma^t - \gamma^\wedge\right), \end{aligned}$$

(where we have used linearity of $\mathcal{C}_2(\cdot)$), which then implies

$$\mathcal{C}_2\left(\bar\gamma^0 + \bar\gamma^1 + \gamma^\wedge\right) < \mathcal{C}_2\left(\gamma^t\right). \tag{6}$$

Now note that

$$\mathcal{M}\left(\bar\gamma^0 + \bar\gamma^1 + \gamma^\wedge\right) = (1-t)(m - m^\wedge) + t(m - m^\wedge) + m^\wedge = m.$$

Also,

$$\begin{aligned}(\pi_1)_\#(\bar\gamma^0+\bar\gamma^1+\gamma^\wedge) &\le (\mu^0_\le-\mu^\wedge_\le)+(\mu^1_\le-\mu^\wedge_\le)+\mu^\wedge_\le\\ &=\mu^0_\le+\mu^1_\le-\mu^\wedge_\le\\ &=(g^0_\le+g^1_\le-g^\wedge_\le)dx\\ &=\max\{g^0_\le,g^1_\le\}dx\\ &\le\mu.\end{aligned}$$

On the other hand, the second marginal satisfies

$$(\pi_2)_\#(\bar\gamma^0+\bar\gamma^1+\gamma^\wedge)=(1-t)(\nu^0-\nu^\wedge)+t(\nu^1-\nu^\wedge)+\nu^\wedge=\nu^t,$$

hence combined with (6) this would contradict that $\gamma^t\in\mathrm{Opt}_m\left(\mu,\nu^t\right)$, and we obtain the claim.

Finally, for $t\in(0,1)$ (in terms of densities),

$$(\pi_1)_\#(1-t)(\gamma^0-\gamma^\wedge)=(1-t)(\mu^0_\le-\mu^\wedge_\le)<\mu^0_\le-\mu^\wedge_\le$$

a.e on its support, hence by combining [10, Theorem 2.6] with (5) we see that $\gamma^0-\gamma^\wedge=\left(S^0\times\mathrm{Id}\right)_\#(\nu^0-\nu^\wedge)$ is supported on the diagonal. This immediately implies that for $(\nu^0-\nu^\wedge)$-a.e. x we must have $S^0(x)=x$, and with a symmetric argument applied to $\gamma^1-\gamma^\wedge$ we obtain (4). $\square$

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Fix m as in the statement of the Theorem, and suppose by contradiction that $\nu^0\neq\nu^1$ are both minimizers in ($\mathrm{BC_m}$); again by Lemma 2.1 we may assume both ν^0, $\nu^1\in\mathcal{P}^m_{ac}$. We will construct a $\overline{\nu}\in\mathcal{P}^m_{ac}$ which achieves a lower value than one of ν^0 or ν^1 in ($\mathrm{BC_m}$), contradicting their minimality.

Since each summand in F is convex under linear interpolation by Lemma 3.1, so is F. In particular, as $F(\nu^0)=F(\nu^1)$, we see that $F(\nu^t)=(1-t)F(\nu^0)+tF(\nu^1)$ for all $t\in[0,1]$ with ν^t defined as in Lemma 3.1, which in turn implies

$$\mathcal{MK}^2_{2,m}\left(\mu_j,\nu^t\right)=(1-t)\mathcal{MK}^2_{2,m}\left(\mu_j,\nu^0\right)+t\mathcal{MK}^2_{2,m}\left(\mu_j,\nu^1\right)$$

for all $t\in[0,1]$ and $1\le j\le N$. For each $i=0$, 1 and $1\le j\le N$ we again obtain an a.e. defined collection of maps $S^i_j:\mathrm{spt}\,\nu^i\to\mathrm{spt}\,\mu_j$ such that $\left(S^i_j\times\mathrm{Id}\right)_\#\nu^i\in\mathrm{Opt}_m\left(\mu_j,\nu^i\right)$, and $\mu^i_{\le j}:=\left(S^i_j\right)_\#\nu^i\le\mu_j$. We will extend each S^i_j to all of $\mathbb{R}^n$ by taking it to be the identity mapping where it is not defined (in particular, on $\mathbb{R}^n\setminus\mathrm{spt}\,\nu^i$). With this extension, by using Lemma 3.1 (4) and (3) we can see that for every $1\le j\le N$,

$$S^0_j(x)=S^1_j(x),\ \text{a.e. } x\in\mathbb{R}^n,$$

and additionally,

$$S^i_j(x)=x,\ \text{a.e. } x\in I\cup\{f^i=0\} \tag{7}$$

where

$$I := \{x \in \mathbb{R}^n \mid f^0(x) \neq f^1(x)\},$$

and f^i is the density of ν^i. We also note here that S^i_j is injective a.e. on $\operatorname{spt} \nu^i$ by the absolute continuity of the μ_j.

By these observations and the a.e. injectivity of S^i_j on $\operatorname{spt} \nu^i$, we immediately obtain

$$f^i \leq g_\wedge \tag{8}$$

a.e. on I. Now by the assumption $\mathcal{M}\left(\nu^i\right) = m > \mathcal{M}\left(\mu_\wedge\right)$, there exists a set $\mathcal{A}_{\{g_\wedge < f^i\}}$ of strictly positive measure on which $g_\wedge < f^i$; by (8) we must have $\mathcal{A}_{\{g_\wedge < f^i\}} \subset \{f^i > 0\} \setminus I$, after possibly discarding a null set, and as a result we can even see that $g_\wedge < f^0 = f^1$ on $\mathcal{A}_{\{g_\wedge < f^i\}}$.

We now claim that for either $i = 0$ or 1, there exists a Borel set $\mathcal{A}_{\{g_\wedge > f^i\}} \subset I$ with strictly positive measure on which $g_\wedge > f^i$, that also satisfies

$$\int_{\mathcal{A}_{\{g_\wedge < f^i\}}} (f^i - g_\wedge) dx = \int_{\mathcal{A}_{\{g_\wedge > f^i\}}} (g_\wedge - f^i) dx > 0 \tag{9}$$

and

$$|S^i_j(\mathbb{R}^n \setminus \mathcal{A}_{\{g_\wedge > f^i\}}) \cap \mathcal{A}_{\{g_\wedge > f^i\}}|_{\mathcal{L}} = 0 \tag{10}$$

for all $1 \leq j \leq N$. Note here that we can write $\mathbb{R}^n \setminus \mathcal{A}_{\{g_\wedge > f^i\}}$ as the disjoint union of the sets $\left\{f^i = 0\right\} \setminus \mathcal{A}_{\{g_\wedge > f^i\}}$, $\left\{f^i > 0\right\} \cap \left(I \setminus \mathcal{A}_{\{g_\wedge > f^i\}}\right)$, and $\left\{f^i > 0\right\} \setminus I$. Since we choose $\mathcal{A}_{\{g_\wedge > f^i\}} \subset I$, by (7) we have

$$\begin{aligned}
&|S^i_j\left(\left\{f^i = 0\right\} \setminus \mathcal{A}_{\{g_\wedge > f^i\}}\right) \cap S^i_j((\mathcal{A}_{\{g_\wedge > f^i\}})|_{\mathcal{L}} \\
&= |\left(\left\{f^i = 0\right\} \setminus \mathcal{A}_{\{g_\wedge > f^i\}}\right) \cap \mathcal{A}_{\{g_\wedge > f^i\}}|_{\mathcal{L}} = 0
\end{aligned}$$

and likewise

$$|S^i_j\left(\left\{f^i > 0\right\} \cap \left(I \setminus \mathcal{A}_{\{g_\wedge > f^i\}}\right)\right) \cap S^i_j\left(\mathcal{A}_{\{g_\wedge > f^i\}}\right)|_{\mathcal{L}} = 0.$$

Thus to guarantee (10) it would be sufficient to show that

$$|S^i_j(\left\{f^i > 0\right\} \setminus I) \cap S^i_j(\mathcal{A}_{\{g_\wedge > f^i\}})|_{\mathcal{L}} = 0. \tag{11}$$

Now, by (8) and since $\nu^0 \neq \nu^1$, there must exist a positive measure subset of I on which $g_\wedge > f^i$ for either of $i = 0$ or 1. By the definition of I, we can take $\mathcal{A}_{\{g_\wedge > f^i\}}$ to be a subset of this aforementioned set, in such a way that $f^{i'} > 0$ on $\mathcal{A}_{\{g_\wedge > f^i\}}$, for either $i' = 0$ or 1 (independent of i). At this point, by shrinking $\mathcal{A}_{\{g_\wedge < f^i\}}$ as necessary, we can ensure (9) holds. Finally, using that $\{f^i > 0\} \setminus I = \{f^{i'} > 0\} \setminus I$ by definition of I, the fact that $S^i_j = S^{i'}_j$ a.e., and that $S^{i'}_j$ is injective a.e. on $\operatorname{spt} \nu^{i'}$, we see

$$\begin{aligned}
&|S^i_j(\{f^i > 0\} \setminus I) \cap S^i_j(\mathcal{A}_{\{g_\wedge > f^i\}})|_{\mathcal{L}} \\
&= |S^{i'}_j(\{f^{i'} > 0\} \setminus I) \cap S^{i'}_j(\mathcal{A}_{\{g_\wedge > f^i\}})|_{\mathcal{L}} = 0
\end{aligned}$$

and we obtain (11).

Note that clearly $\mathcal{A}_{\{g_\wedge<f^i\}}\cap\mathcal{A}_{\{g_\wedge>f^i\}}=\emptyset$ in both cases. Let us now define

$$\overline{\nu}:=\nu^i+(g_\wedge-f^i)\mathbb{1}_{\mathcal{A}_{\{g_\wedge<f^i\}}\cup\mathcal{A}_{\{g_\wedge>f^i\}}}dx,$$

which is a positive measure by the disjointness of $\mathcal{A}_{\{g_\wedge<f^i\}}$ and $\mathcal{A}_{\{g_\wedge>f^i\}}$. It is clear $\overline{\nu}$ is absolutely continuous, and by (9),

$$\begin{aligned}\mathcal{M}\left(\overline{\nu}\right)&=\mathcal{M}\left(\nu^i\right)+\int_{\mathcal{A}_{\{g_\wedge>f^i\}}}(g_\wedge-f^i)dx-\int_{\mathcal{A}_{\{g_\wedge<f^i\}}}(f^i-g_\wedge)dx\\&=m,\end{aligned}$$

i.e. $\overline{\nu}\in\mathcal{P}^m_{ac}$.

Next we claim that $\left(S^i_j\right)_\#\overline{\nu}\leq\mu_j$ for $1\leq j\leq N$. To this end fix an arbitrary measurable set E, then

$$\begin{aligned}\left(S^i_j\right)_\#\overline{\nu}(E)&=\nu^i(\left(S^i_j\right)^{-1}(E))+\int_{\mathcal{A}_{\{g_\wedge>f^i\}}\cap\left(S^i_j\right)^{-1}(E)}(g_\wedge-f^i)dx-\int_{\mathcal{A}_{\{g_\wedge<f^i\}}\cap\left(S^i_j\right)^{-1}(E)}(f^i-g_\wedge)dx\\&\leq\mu^i_{\leq j}(E)+\mu_j(\mathcal{A}_{\{g_\wedge>f^i\}}\cap\left(S^i_j\right)^{-1}(E))-\nu^i(\mathcal{A}_{\{g_\wedge>f^i\}}\cap\left(S^i_j\right)^{-1}(E))\end{aligned}$$

Since $\mathcal{A}_{\{g_\wedge>f^i\}}\subset I$, by (7) and (10), we see that

$$\mathcal{A}_{\{g_\wedge>f^i\}}\cap\left(S^i_j\right)^{-1}(E)=\mathcal{A}_{\{g_\wedge>f^i\}}\cap E=\left(S^i_j\right)^{-1}(\mathcal{A}_{\{g_\wedge>f^i\}}\cap E)$$

up to null sets, hence

$$\begin{aligned}&\mu^i_{\leq j}(E)+\mu_j(\mathcal{A}_{\{g_\wedge>f^i\}}\cap\left(S^i_j\right)^{-1}(E))-\nu^i(\mathcal{A}_{\{g_\wedge>f^i\}}\cap\left(S^i_j\right)^{-1}(E))\\&=\mu^i_{\leq j}(E)+\mu_j(\mathcal{A}_{\{g_\wedge>f^i\}}\cap E)-\nu^i(\left(S^i_j\right)^{-1}(\mathcal{A}_{\{g_\wedge>f^i\}}\cap E))\\&=\mu^i_{\leq j}(E)+\mu_j(\mathcal{A}_{\{g_\wedge>f^i\}}\cap E)-\mu^i_{\leq j}(\mathcal{A}_{\{g_\wedge>f^i\}}\cap E)\\&=\mu^i_{\leq j}((\mathbb{R}^n\setminus\mathcal{A}_{\{g_\wedge>f^i\}})\cap E)+\mu_j(\mathcal{A}_{\{g_\wedge>f^i\}}\cap E)\\&\leq\mu_j(E),\end{aligned}$$

proving our claim. In particular, this shows that for each i and j, the measure $\left(S^i_j\times\mathrm{Id}\right)_\#\overline{\nu}$ is an admissible competitor in $\mathcal{MK}^2_{2,m}\left(\mu_j,\overline{\nu}\right)$.

Finally, we will show that

$$\sum_{j=1}^N\mathcal{C}_2\Big(\left(S^i_j\times\mathrm{Id}\right)_\#\overline{\nu}\Big)<\sum_{j=1}^N\mathcal{C}_2\Big(\left(S^i_j\times\mathrm{Id}\right)_\#\nu^i\Big).\tag{12}$$

Since $F(\overline{\nu})\leq\sum_{j=1}^N\mathcal{C}_2\left(\left(S^i_j\times\mathrm{Id}\right)_\#\overline{\nu}\right)$ and $F(\nu^i)=\sum_{j=1}^N\mathcal{C}_2\left(\left(S^i_j\times\mathrm{Id}\right)_\#\nu^i\right)$, this contradicts the fact that ν^i is a minimizer in ($\mathrm{BC_m}$), and will finish the proof. First we calculate for each $1 \leq j \leq N$,

$$\begin{aligned}\mathcal{C}_2\Big(\big(S^i_j \times \mathrm{Id}\big)_\# \overline{\nu}\Big) &= \mathcal{C}_2\Big(\big(S^i_j \times \mathrm{Id}\big)_\# \nu^i\Big) + \int_{\mathcal{A}_{\{g_\wedge > f^i\}}} |S^i_j(x) - x|^2 (g_\wedge - f^i) dx \\ &\quad - \int_{\mathcal{A}_{\{g_\wedge < f^i\}}} |S^i_j(x) - x|^2 (f^i - g_\wedge) dx \\ &= \mathcal{MK}^2_{2,m}\left(\mu_j, \nu^i\right) - \int_{\mathcal{A}_{\{g_\wedge < f^i\}}} |S^i_j(x) - x|^2 (f^i - g_\wedge) dx,\end{aligned} \tag{13}$$

where we have again used (7) and that $\mathcal{A}_{\{g_\wedge > f^i\}} \subset I$. Now by definition of $\mathcal{A}_{\{g_\wedge < f^i\}}$, we must have $-\int_{\mathcal{A}_{\{g_\wedge < f^i\}}} |S^i_j(x) - x|^2 (f^i - g_\wedge) dx \leq 0$ for each j. Suppose that there is equality for every $1 \leq j \leq N$, this would imply that S^i_j is the identity map a.e. on $\mathcal{A}_{\{g_\wedge < f^i\}}$ for every j as well. However, there exists some set $\mathcal{A}_{j'} \subset \mathcal{A}_{\{g_\wedge < f^i\}}$ with strictly positive measure on which $g_\wedge \equiv g_{j'}$ for some index $1 \leq j' \leq N$. This would imply that (using the a.e. injectivity of $S^i_{j'}$ on $\operatorname{spt} \nu^i$)

$$\begin{aligned}\mu^i_{\leq j'}(\mathcal{A}_{j'}) &= \nu^i\big(\big(S^i_{j'}\big)^{-1}(\mathcal{A}_{j'})\big) \\ &= \int_{\mathcal{A}_{j'}} f^i dx \\ &> \int_{\mathcal{A}_{j'}} g_\wedge dx \\ &= \mu_{j'}(\mathcal{A}_{j'}),\end{aligned}$$

contradicting that $\mu^i_{\leq j'} \leq \mu_{j'}$. Thus by summing (13) over $1 \leq j \leq N$, we obtain (12), leading to the desired contradiction and finishing the proof. $\square$

4. Other properties of the multi-marginal optimal partial transport problem and the partial barycenter problem

In this section, we will discuss other analytic properties of minimizers in ($\mathrm{MM_m}$). We begin with a counterexample to the monotonicity property, in contrast to the two marginal case of ($\mathrm{OT_m}$) (see [6, Theorem 3.4] and [10, Remark 3.4]).

Proposition 4.1. *There exist measures μ_1, μ_2, μ_3 on $\mathbb{R}$ and $0 < m < \bar{m} < \min_{j=1,2,3} \mathcal{M}(\mu_j)$ for which:*

1. *The barycenters ν^m and $\nu^{\bar{m}}$, minimizing $\sum_{j=1}^3 \mathcal{MK}^2_{2,m}(\mu_j, \nu)$ and $\sum_{j=1}^3 \mathcal{MK}^2_{2,\bar{m}}(\mu_j, \nu)$ over $\mathcal{P}^m_{ac}$ and $\mathcal{P}^{\bar{m}}_{ac}$ respectively, are not monotone; that is, $\nu^m \not\leq \nu^{\bar{m}}$.*
2. *The active submarginals $\mu^m_{\leq 3} := (\pi_3)_\# (\sigma_m)$ and $\mu^{\bar{m}}_{\leq 3} := (\pi_3)_\# (\sigma_{\bar{m}})$ (where $\sigma_m \in \mathrm{Opt}_m(\mu_1, \mu_2, \mu_3)$ and $\sigma_{\bar{m}} \in \mathrm{Opt}_{\bar{m}}(\mu_1, \mu_2, \mu_3)$) are not monotone; that is, $\mu^m_{\leq 3} \not\leq \mu^{\bar{m}}_{\leq 3}$.*

In fact, as we will see in the proof below, even more is true; the barycenter of the first two measures is not monotone.

It may be helpful to see Figures 1 and 2 below while following the proof.

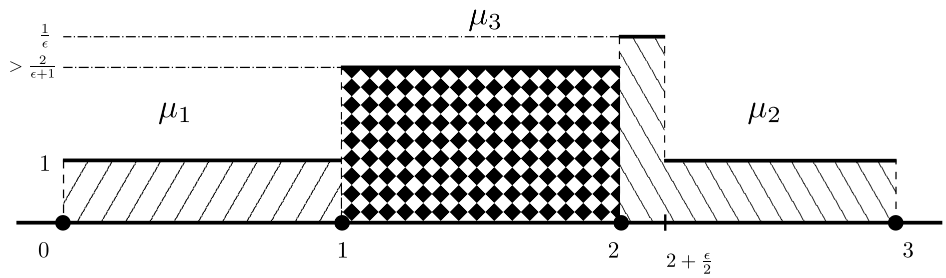

FIGURE 1. Here we have shown graphically the three dominating measures μ_1, μ_2, and μ_3 in Proposition 4.1.

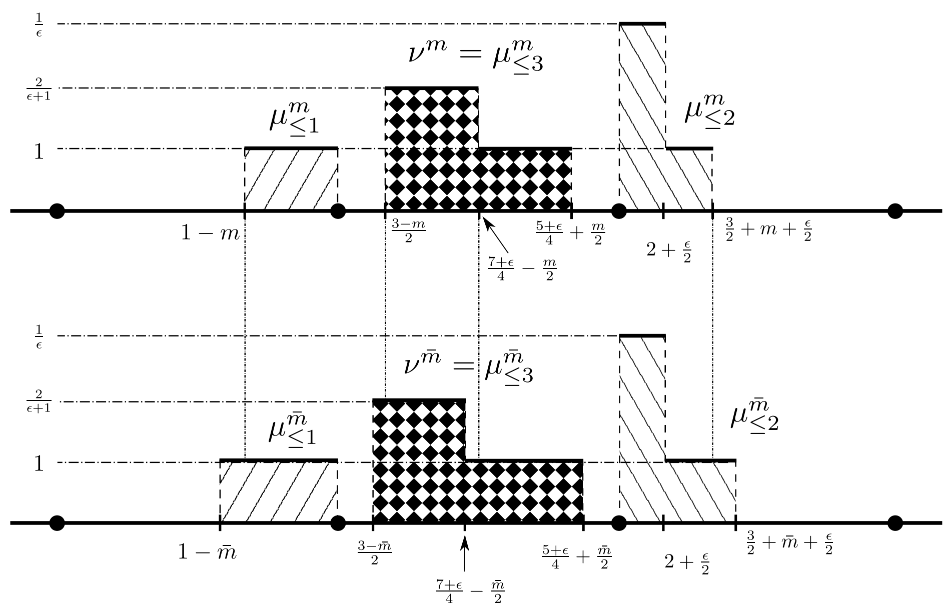

FIGURE 2. This figure illustrates the active submeasures in Proposition 4.1 for two different values $\frac{1}{2} < m < \bar{m} < 1$. Note that in each case, the partial barycenter is equal to the third active submeasure, which is not monotone (as demonstrated by the dashed vertical lines).

Proof. Take μ_1 to be uniform measure on $[0,1]$ with density 1 and take μ_2 to be absolutely continuous, supported on $[2,3]$ with density given by

$$g_2(x) = \begin{cases} \frac{1}{\varepsilon} & \text{on } [2, 2+\frac{\varepsilon}{2}], \\ 1 & \text{on } [2+\frac{\varepsilon}{2}, 3]. \end{cases}$$

First, consider the optimal partial transport problem between the two marginals μ_1 and μ_2. For $\frac{1}{2} < m < 1$, it is an easy consequence of [6, Corollary 2.4] combined with [10, Theorem 2.6] that if $\gamma_m \in \mathrm{Opt}_m(\mu_1, \mu_2)$ then the active submeasures $\mu^m_{\leq 1} := (\pi_1)_{\#}\gamma_m$ and $\mu^m_{\leq 2} := (\pi_2)_{\#}\gamma_m$ are the measures μ_1 and μ_2, restricted to the intervals $[1-m, 1]$ and $[2, \frac{3}{2}+\frac{\varepsilon}{2}+m]$, respectively (these are the "right most" piece of the first measure and the "left most" piece of the second). In addition we can write $\gamma_m = (\mathrm{Id} \times T^m)_{\#}\mu^m_{\leq 1}$, where the optimal map T^m between the two active submeasures is the unique increasing map pushing $\mu^m_{\leq 1}$ forward to $\mu^m_{\leq 2}$, given by:

$$T^m(x) = \begin{cases} \varepsilon x + 2 - \varepsilon(1-m) & \text{on } [1-m, \frac{3}{2}-m], \\ x + \frac{1+\varepsilon}{2} + m & \text{on } [\frac{3}{2}-m, 1]. \end{cases}$$

By Proposition 1.1, the partial barycenter $\nu^m := \left(\frac{x+T^m(x)}{2}\right)_{\#}\mu^m_{\leq 1}$ minimizes $\sum_{j=1}^2 \mathcal{MK}^2_{2,m}(\mu_j, \nu)$ over $\mathcal{P}^m_{ac}$; thus ν^m is supported on $[\frac{3-m}{2}, \frac{5+\varepsilon}{4}+\frac{m}{2}]$, with a density given by

$$f^m(x) = \begin{cases} \frac{2}{\varepsilon+1} & \text{on } [\frac{3-m}{2}, \frac{7+\varepsilon}{4}-\frac{m}{2}], \\ 1 & \text{on } [\frac{7+\varepsilon}{4}-\frac{m}{2}, \frac{5+\varepsilon}{4}+\frac{m}{2}]. \end{cases}$$

In particular, note that the partial barycenter of μ_1 and μ_2 is not monotone; the location of the jump in f^m moves to the left as m increases, hence $\nu^m \not\leq \nu^{\bar{m}}$ when $\frac{1}{2} < m < \bar{m} < 1$.

Now, take μ_3 to be uniform on $[1,2]$, with density $g_3 > \frac{2}{\varepsilon+1}$. Then each $\nu^m \leq \mu_3$, and it is straightforward to see that ν^m minimizes $\nu \mapsto \sum_{j=1}^3 \mathcal{MK}^2_{2,m}(\mu_j, \nu)$ over $\mathcal{P}^m_{ac}$, (as ν^m minimizes $\nu \mapsto \mathcal{MK}^2_{2,m}(\mu_1, \nu) + \mathcal{MK}^2_{2,m}(\mu_2, \nu)$ while $\mathcal{MK}^2_{2,m}(\mu_3, \nu^m) = 0$, it clearly minimizes their sum).

As the active submeasure $\mu^m_{\leq 3}$ corresponding to μ_3 is precisely ν^m, this shows that the active submeasures are not monotone either. □

Remark 4.2. The example in the preceding proof also implies that the naive multi-marginal analogue of [10, Proposition 2.4] fails.

The analogous statement would be the following: if σ solves ($\mathrm{MM_m}$) with marginals $\mu_1, \ldots, \mu_N$ and $\mu_{\leq j} := (\pi_j)_{\#}\sigma$, then

$$\bar{\sigma} := \sigma + \left(\bigotimes_{j=1}^N \mathrm{Id}\right)_{\#}\left(\sum_{j=1}^N (\mu_j - \mu_{\leq j})\right)$$

solves (MM) with marginals $\mu_1 + \sum_{j\neq 1}(\mu_j - \mu_{\leq j}), \ldots, \mu_N + \sum_{j\neq N}(\mu_j - \mu_{\leq j})$.

However, we can see that already in the case $N = 3$, this statement does not hold for the example given above in Proposition 4.1. Indeed, note that for $x \in [\frac{7+\varepsilon}{4} - \frac{m}{2}, \frac{5+\varepsilon}{4} + \frac{m}{2}]$ we have $g_{\leq 3}(x) = 1 < \frac{2}{\varepsilon+1} < g_3(x)$; therefore the density $g_3 - g_{\leq 3}$ of $\mu_3 - \mu_{\leq 3}$ is strictly positive on this interval. As a result, for each $x_3 \in [\frac{7+\varepsilon}{4} - \frac{m}{2}, \frac{5+\varepsilon}{4} + \frac{m}{2}]$ the support of $\bar{\sigma}$ includes (x_3, x_3, x_3) (via $(\mathrm{Id} \times \mathrm{Id} \times \mathrm{Id})_{\#} \left(\mu_3 - \mu_{\leq 3}\right))$ as well as points of the form (x_1, x_2, x_3), where $x_j \in spt(g_{\leq j})$ for $j = 1$, 2 (via σ). In particular, $\bar{\sigma}$ is not concentrated on a graph over the third marginal, and so, by [13, Theorem 2.1] $\bar{\sigma}$ cannot be optimal in (MM).

As Proposition 2.4 of [10] plays a key role in Figalli's proof of Theorem 2.10 there, this indicates that a direct application of the techniques in [10] does not translate to the multi-marginal case.

The next example shows that, in contrast to a result of Caffarelli and McCann in the $N = 2$ case, monotonicity of the active submeasures can fail even for discrete measures (compare [6, Proposition 3.1]).

Example 4.3. *Consider the real line $\mathbb{R}$ and take*

$$\mu_1 = \delta_{-5} + \delta_{-3},$$
$$\mu_2 = \delta_{-1} + \delta_0 + \delta_1,$$
$$\mu_3 = \delta_3 + \delta_5.$$

Taking $m = 1$, it is easy to see that the optimizer is $\delta_{(-3,0,3)}$ which couples the Dirac masses at -3, 0, and 3, while for $m = 2$, the optimizer is $\delta_{(-5,-1,3)} + \delta_{(-3,1,5)}$ which couples the masses at -5, -1, and 3; and -3, 1, and 5 respectively. This shows that even the support of the active submeasure of μ_2 is not monotone.

An example with absolutely continuous measures where the supports of the active submeasures are not monotone can be constructed by replacing the Dirac masses with uniform measure on small disjoint intervals; however it is not clear to us whether an example can be constructed in which the marginals are absolutely continuous with connected supports.

As the above two examples illustrate, when there are three or more marginals in $(\mathrm{MM_m})$, an optimal coupling may move mass away from a location where the active submeasure does not saturate the dominating measure: in Proposition 4.1 above, on $\operatorname{spt}\mu_3$ we have $g_{\leq 3} < g_3$ (where $g_{\leq 3}$ is the density of $\mu_{\leq 3}$), yet under the optimal coupling none of the mass of $\mu_{\leq 3}$ remains in place. At first glance, this seems to be a sharp distinction from the two marginal case, [10, Theorem 2.6], however we now show there is an appropriate analogous statement for the multi-marginal case, in the form of Corollary 4.5.

Proposition 4.4. *Suppose that $\sigma \in \mathrm{Opt}_m(\mu_1, \ldots, \mu_N)$, and g_j is the density of μ_j. Then for a.e. x_j in $\{g_{\leq j} < g_j\}$, where $g_{\leq j}$ is the density of*

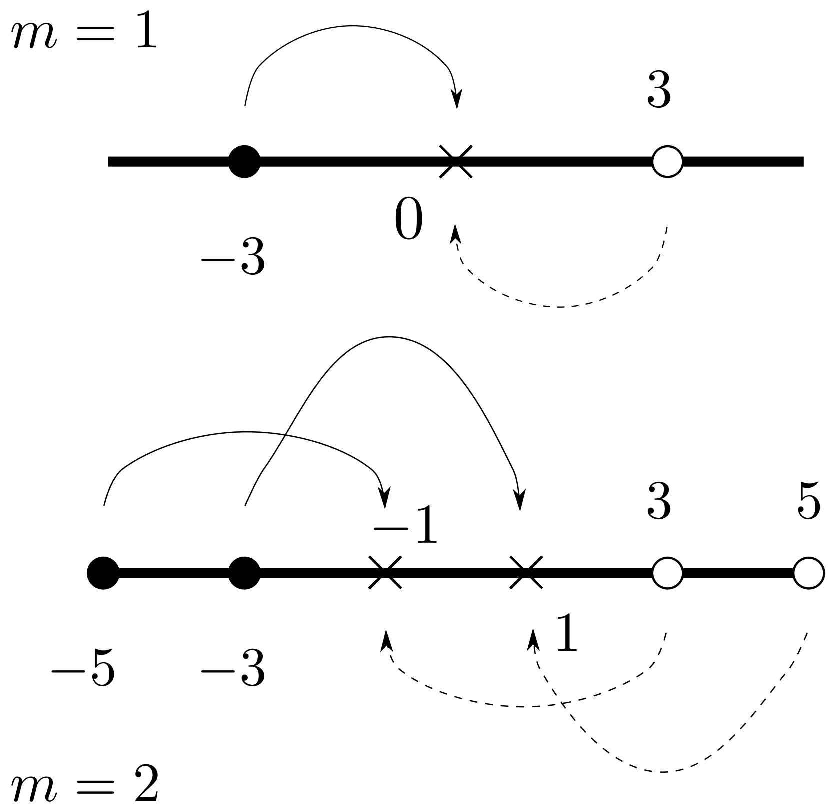

FIGURE 3. This figure illustrates the active submeasures in Example 4.3 for $m = 1$ and $m = 2$. Filled in dots represent the support of the active submeasure of μ_1, empty dots the active submeasure of μ_2, and the crosses are the active submeasure of μ_3. Each submeasure is a sum of unit Dirac measures supported at the various points. Again in each case, the partial barycenter (whose couplings are illustrated by the solid and dotted arrows) is equal to the third active submeasure, which fails to be monotone (in fact, the support itself fails to be monotone).

$(\pi_j)_{\#}\,\sigma$,

$$x_j = \frac{1}{N-1}\sum_{k\neq j} x_k$$

where $\{x_k\}_{k\neq j}$ *is the unique collection of points such that* $(x_1, \ldots, x_N) \in \operatorname{spt}\sigma$.

Proof. Let us write

$$\gamma_j := (\pi_j \times A)_{\#}\, \sigma,$$
$$\mu_{\leq j} := (\pi_j)_{\#}\, \sigma$$

(recall the definition of A is given by (1)). By Proposition 1.1, we have $\gamma_j \in \mathrm{Opt}_m\,(\mu_j, A_{\#}\sigma)$, thus by the absolute continuity of μ_j there exists an a.e. defined mapping T_j on $\operatorname{spt}\mu_{\leq j}$ such that $\gamma_j = \left(\mathrm{Id}\times T_j\right)_{\#}\mu_{\leq j}$. By [10, Theorem 2.6], for a.e. $x_j \in \{g_{\leq j} < g_j\}$ we have

$$T_j(x_j) = x_j, \tag{14}$$

note that for a.e. $x_j \in \operatorname{spt}\mu_{\leq j}$,

$$(x_j, y) \in \operatorname{spt}\gamma_j \implies y = T_j(x_j). \tag{15}$$

On the other hand, by [13, Theorem 2.1, Corollary 2.2] for a.e. x_j, there exists a unique set of points $\{x_k\}_{k\neq j}$ such that $(x_1, \ldots, x_N) \in \operatorname{spt}\sigma$. Thus for a.e. $x_j \in \operatorname{spt}\mu_{\leq j}$, we see that

$$(x_j, A(x_1, \ldots, x_N)) \in \operatorname{spt}\gamma_j$$

which combined with (15) implies

$$A(x_1, \ldots, x_N) = T_j(x_j).$$

Finally, combining this with (14) we find that for a.e. $x_j \in \{g_{\leq j} < g_j\}$, there exists unique $\{x_k\}_{k\neq j}$ such that $(x_1, \ldots, x_N) \in \operatorname{spt}\sigma$ and

$$x_j = T_j(x_j) = A(x_1, \ldots, x_N) = \frac{1}{N}\sum_{k=1}^{N} x_k.$$

By rearranging, we obtain the conclusion of the proposition. □

Corollary 4.5. *Let σ, g_j, and $g_{\leq j}$ be as above. Also fix an integer $1 \leq K < N$, and some subcollection of indices $I := \{j_1, \ldots, j_K\} \subset \{1, \ldots, N\}$. Then for $(\mathbb{R}^n)^K$-Lebesgue a.e. $(x_{j_1}, \ldots, x_{j_K}) \in \prod_{j\in I}\{g_{\leq j} < g_j\}$ which can be completed to some $(x_1, \ldots, x_N) \in \operatorname{spt}\sigma$, the following hold:*

$$x_{j_1} = \ldots = x_{j_K}, \tag{16}$$
$$x_{j_1} = \frac{1}{N-K}\sum_{k\notin I} x_k. \tag{17}$$

Proof. The case $K = 1$ and $N = 2$ is exactly [10, Theorem 2.6], so let us assume that $N > 2$. Clearly there is a $(\mathbb{R}^n)^K$-Lebesgue full measure subset of $\prod_{j\in I}\{g_{\leq j} < g_j\}$ on which Proposition 4.4 applies to every component; fix one such point $(x_{j_1}, \ldots, x_{j_K})$ in that set and complete it to some

$(x_1, \ldots, x_N) \in \operatorname{spt}\sigma$. Then we can see, for example,

$$x_{j_1} = \frac{1}{N-1} \sum_{k \neq j_1} x_k,$$

$$x_{j_2} = \frac{1}{N-1} \sum_{k \neq j_2} x_k,$$

and by subtracting and rearranging (since $N > 2$) we find that $x_{j_1} = x_{j_2}$. Proceeding as such for other indices in I, we immediately obtain the claim (16).

Now, by another application of Proposition 4.4,

$$\begin{aligned} x_{j_1} &= \frac{1}{N-1} \sum_{k \neq j_1} x_k \\ \iff (N-1)x_{j_1} &= \left((K-1)x_{j_1} + \sum_{k \notin I} x_k \right) \\ \iff (N-K)x_{j_1} &= \sum_{k \notin I} x_k, \end{aligned}$$

and we obtain (17). □

In particular, if $K = N-1$ above, we recover an appropriate analogue of [10, Theorem 2.6] in the multi-marginal case: if $N-1$ of the active submeasures do not saturate the original measures at an N-tuple in the optimal coupling, then all of the coupled points must be the same (up to a set of measure zero).

Remark 4.6 (Semiconcavity of the free boundary)**.** Lastly we remark that under certain conditions, we can obtain the semiconcavity of the free boundary "for free" simply by applying the theory of the two marginal case. Assume that each support $\operatorname{spt}\mu_j$, $1 \leq j \leq N$ is separated by a hyperplane from their Minkowski average. Note the support of any ν that minimizes ($\mathrm{BC_m}$) is contained in this Minkowski average by [1, Proposition 4.2]. Then, the marginal of any optimizer in ($\mathrm{MM_m}$) can be thought of as the left marginal of ($\mathrm{OT_m}$) with right marginal ν, hence we may apply [6, Proposition 5.2] to conclude that the "free boundary" (as defined in [6]) in $\operatorname{spt}\mu_j$ enjoys the same semiconcavity. However, since one cannot make any assumptions about convexity of $\operatorname{spt}\nu$ and bounds on the density of ν, arguments based on Caffarelli's regularity theory (see [4]) to obtain higher regularity of the free boundary cannot be applied.

5. Extension to more general cost functions

Here we mention that our main result can be extended to a more general class of cost functions. Consider a cost function of the form

$$c(x_1,\dots,x_N)=\inf_{y\in Y}\sum_{j=1}^{N}c_j(x_j,y), \tag{18}$$

where $c_j:\Omega_j\times Y\to\mathbb{R}$ for some fixed, open domains Ω_j and Y. Also consider the generalized partial barycenter problem:

$$\min_{\nu\in\mathcal{P}_{ac}^m}\sum_{j=1}^{N}\mathcal{T}_{c_j,m}(\mu_j,\nu) \tag{19}$$

where $\mathcal{T}_{c_j,m}$ is the partial transport distance with cost c_j:

$$\mathcal{T}_{c_j,m}(\mu_j,\nu):=\min_{\gamma'\in\Pi_{\leq}(\mu_j,\nu),\ \mathcal{M}(\gamma')=m}\int_{\mathbb{R}^n\times\mathbb{R}^n}c_j(x_j,y)\gamma'(dx,dy).$$

In order to obtain the equivalent of Proposition 1.1, we will require the following assumptions:

[H1]: For all j, the costs c_j are C^2 and $\det D^2_{x_jy}c_j\neq 0$ on $\Omega_j\times Y$.
[H2]: For all j, the mappings $D_{x_j}c_j(x_j,\cdot)$ are injective for each x_j, and $D_yc_j(\cdot,y)$ are injective for each y
[H3]: For each $(x_1,x_2,\dots,x_N)$ the infimum in (18) is attained by a *unique* $y=\overline{y}(x_1,x_2,\dots,x_N)\in Y$.
[H4]: The matrix $B(x_1,x_2,\dots,x_N):=\sum_{i=1}^{N}D^2_{yy}c_j(x_j,\overline{y}(x_1,x_2,\dots,x_N))$ is non-singular.

Additionally, to obtain the equivalent of Theorem 1.2, we must also assume the following condition:

[H5]: For all j, we have $c_j(x_j,y)\geq 0$ with $c_j(x_j,y)=0$ if and only if $x_j=y$.

Under these assumptions, we can generalize the main results to a more general class of cost functions.

Theorem 5.1. *Fix compactly supported, absolutely continuous measures μ_j, for $j=1,\dots,N$, let $0\leq m\leq\min_{1\leq j\leq N}\{\mathcal{M}(\mu_j)\}$, and assume that the cost functions c_j for $1\leq j\leq N$ satisfy conditions* **[H1]-[H5]** *above. Then the following hold:*

(1) *If σ is a solution to* ($\mathrm{MM_m}$) *with cost* (18)*, then $\nu=\bar{y}_\#\sigma$ is a solution to* (19)*. On the other hand, if ν is a solution to* (19)*, then $\sigma=(S_1^\nu,\dots,S_N^\nu)_\#\nu$ is a solution to* ($\mathrm{MM_m}$)*, where $\left(S_j^\nu\times\mathrm{Id}\right)_\#\nu$ is the (unique) minimizer in $\mathcal{T}_{c_j,m}(\mu_j,\nu)$.*
(2) *Assume in addition that $m\geq\mathcal{M}(\mu_\wedge)$ where $\mu_\wedge$ is the measure with density $\min_{1\leq j\leq N}g_j$. Then both the multi-marginal optimal partial transport problem* ($\mathrm{MM_m}$) *with cost function given by* (18)*, and the generalized partial barycenter problem* (19) *admit unique solutions.*

The proof of the preceding theorem is a straightforward adaptation of the proof of our main results. The necessary ingredients are a relationship between the partial transport problem and the partial barycenter problem, established by Carlier and Ekeland [8, Proposition 3]. The conditions **[H1] - [H4]** guarantee the uniqueness of the solution to the standard multi-marginal problem with cost (18), and absolute continuity of the standard generalized barycenter [18], necessary ingredients in our argument here.

On the other hand, condition **[H5]** (together with **[H1]** and **[H2]**) are necessary to invoke the results of Figalli in the two marginal case $(\mathrm{OT_m})$ that we have relied on to prove the results of Section 3 (see [10, Remark 2.11] for details).

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