Homogenization of Metric Hamilton-Jacobi equations

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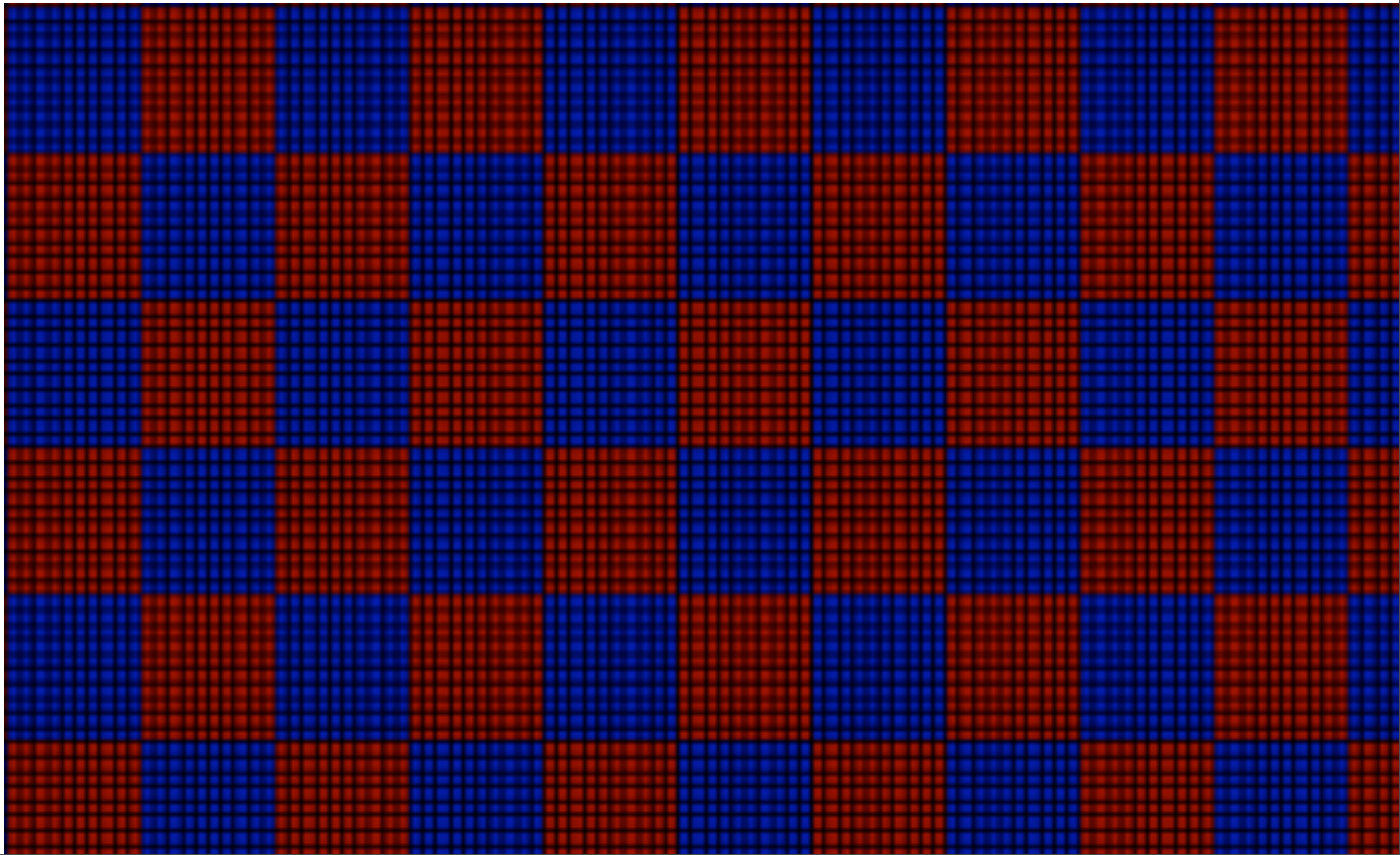
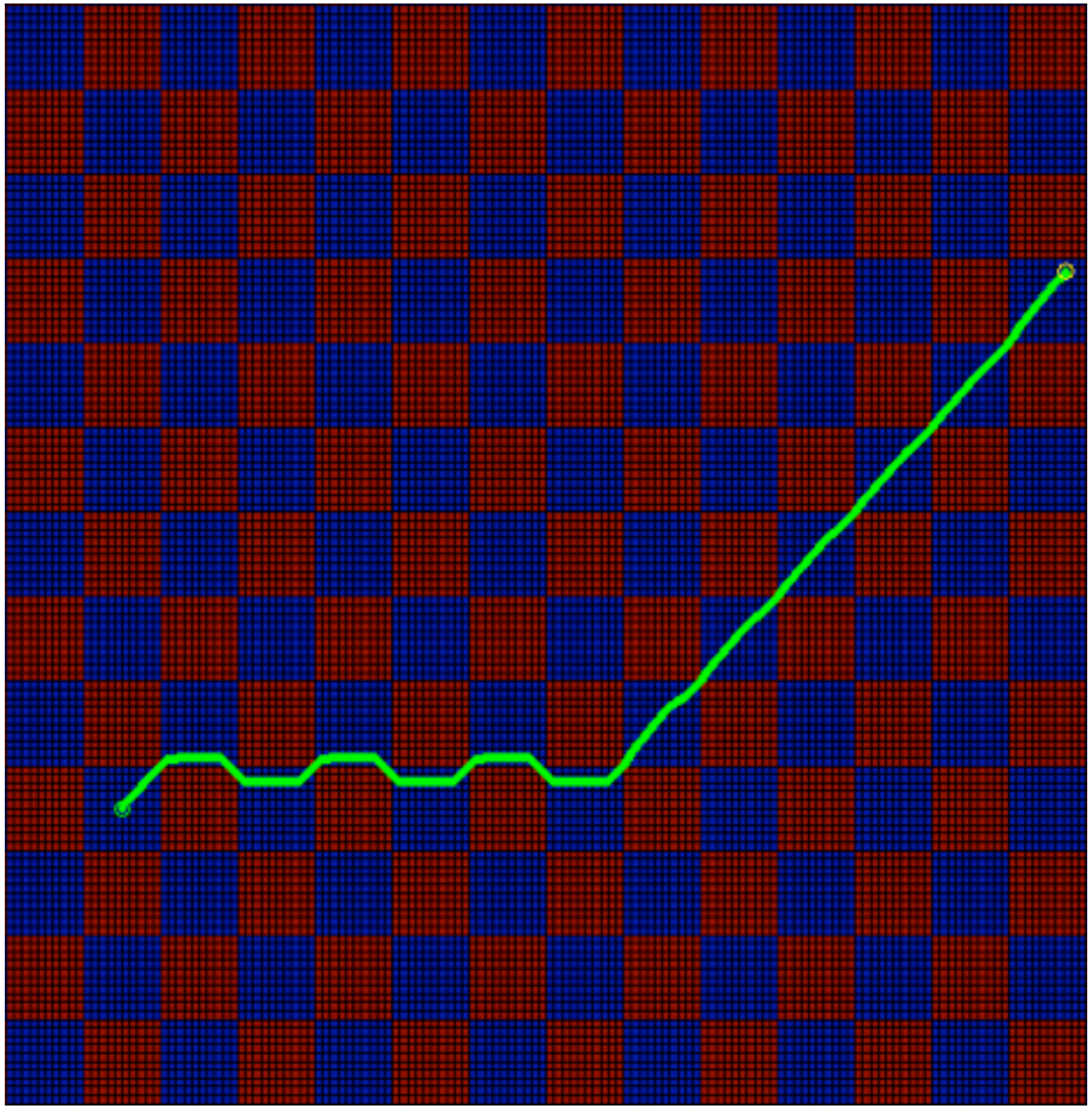
Optimal Path Checkerboard

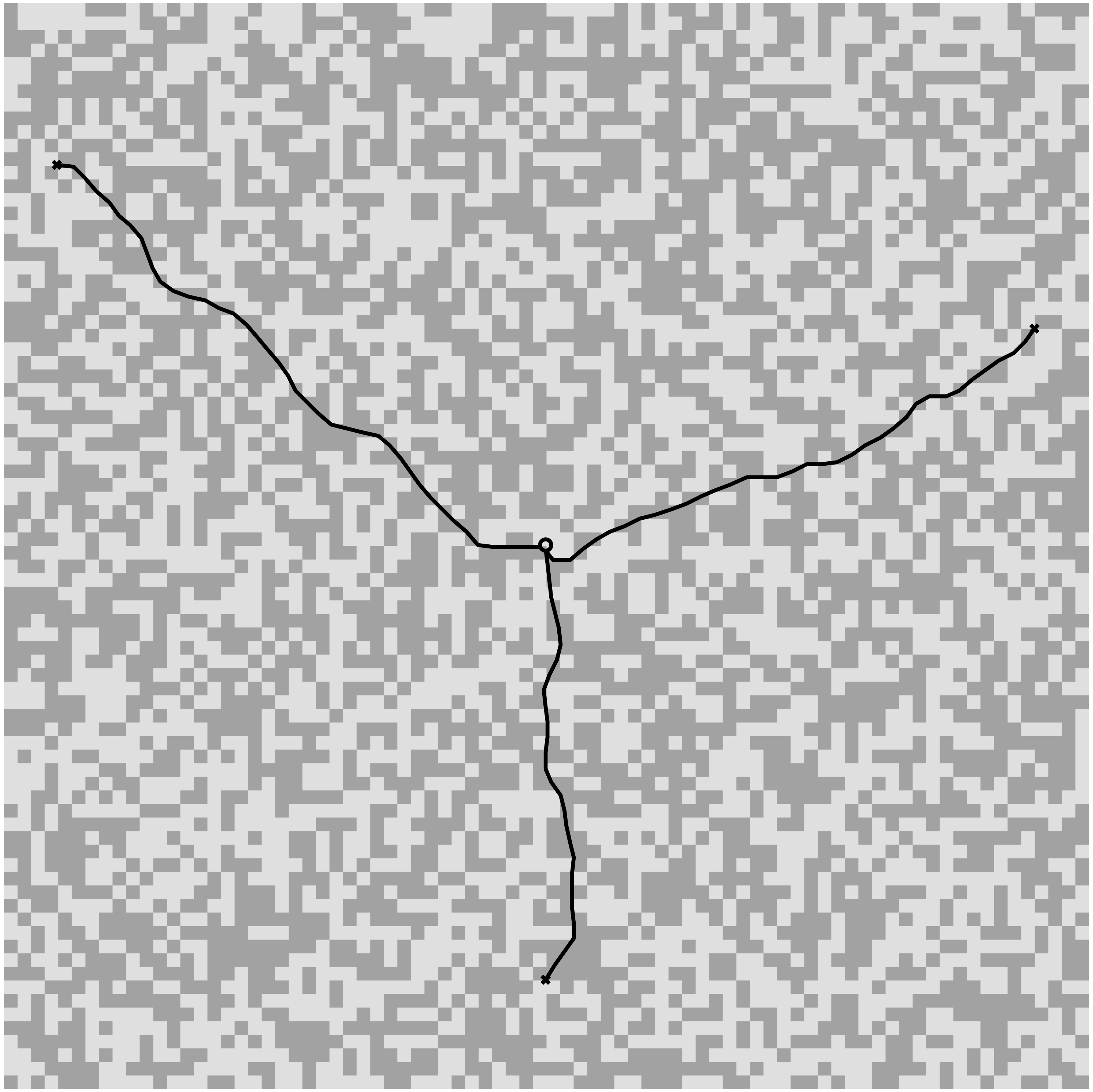


Optimal Path in Checkerboard



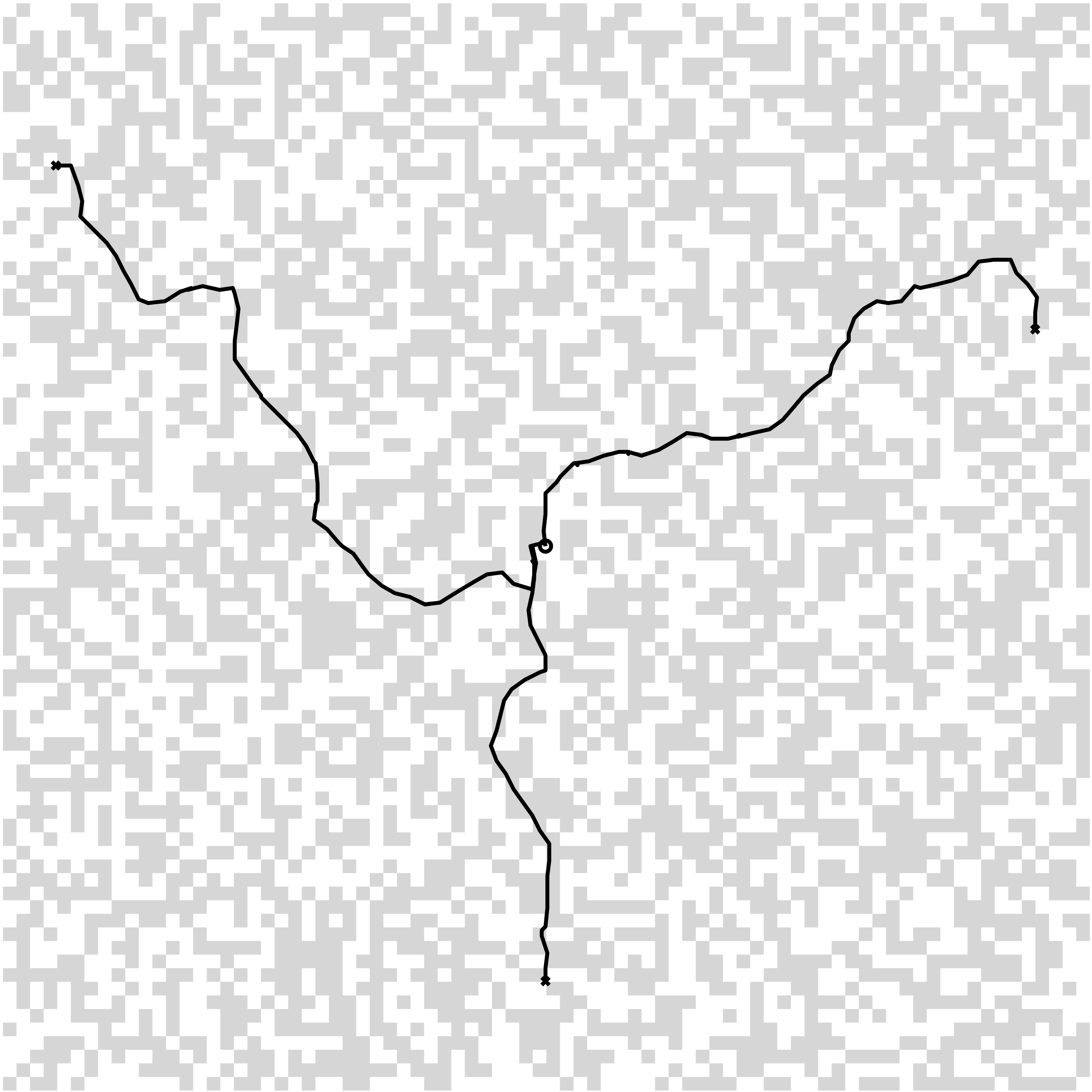
Optimal Paths in Random Media

$c = 1, 2$



Optimal Paths in Random Media

c =1, 10



Optimal Paths via Dynamic Programming

- Finding optimal paths via direct search is too costly.
- Finding optimal paths via Euler-Lagrange equation is not informative.
- Cell problem requires one problem for each direction.
- Instead, use Hamilton-Jacobi equation for function whose level sets are arrival times from the origin.

$$c(x)|\nabla T| = 1$$

$$T(0) = 0$$

Hamilton-Jacobi Eqn for General Speeds

$$v_n(x) = \max_{\|\alpha\| \leq 1} \{n \cdot f(x, \alpha)\}$$

$f(x, \alpha)$ particle speed in direction α

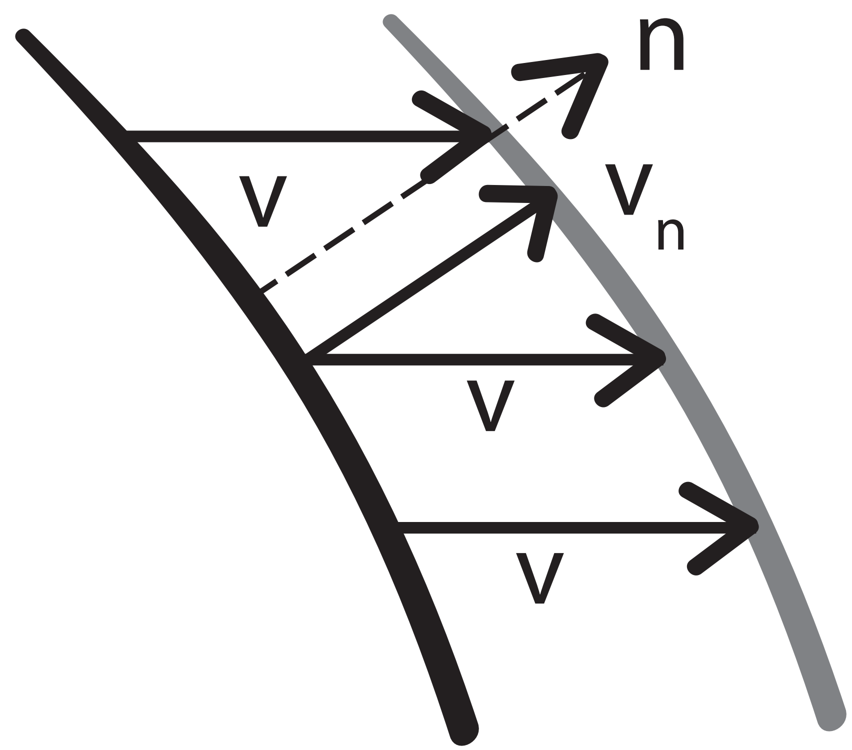
Then the function $T(x)$ whose level sets $\{T(x) = s\}$ are the position of the front at time s solves the Hamilton-Jacobi equation

$$H(\nabla T(x), x) = 1$$

where H is given by

$$H(p, x) := \sup_{\|\alpha\| \leq 1} \{p \cdot f(x, \alpha)\}$$

Front Speeds vs. Particle Speeds



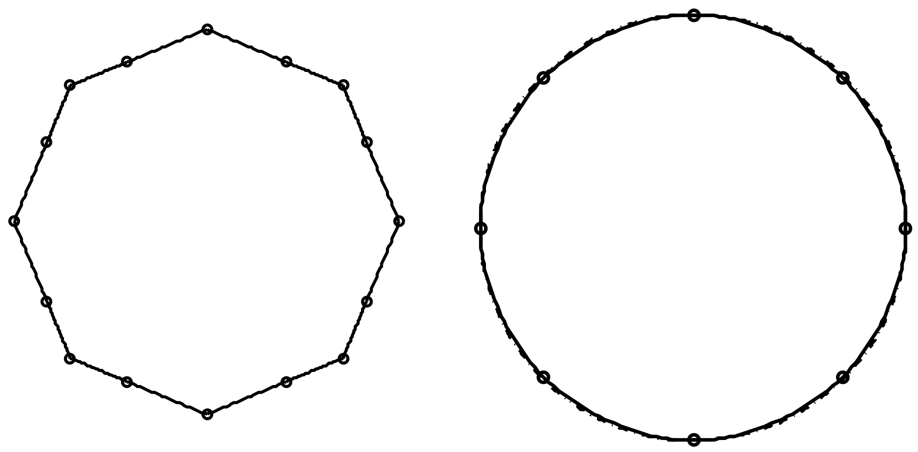
- In the anisotropic case, normal velocity of fronts is not the same as particle velocity.
- Suppose particles move horizontally with speed 1, and have no vertical speed allowed. Then front moves with normal speed given by projection of the normal onto the particle speed.

Controlled ODEs for paths

Let x denote the position, and α denote the direction of motion.
Write $\dot{x}(s,\alpha(s)) := \frac{d}{ds}x(s,\alpha(s))$. The admissible paths $x(t,\alpha(t))$ satisfy

(ODE) $$\dot{x}(s,\alpha(s)) = f(x(s),\alpha(s))$$

where $\alpha(\cdot) \in \mathcal{A}$ is the *control.*



vectogram $V_c \subset \mathbb{R}^n$.

$$V_c = \{c(x,\alpha)\alpha \mid |\alpha| \leq 1\}.$$

$$T_c(x_1,x_2) = \inf_{x(\cdot)\ \text{admissible}} \{t \mid x(0) = x_1, x(t) = x_2\},$$

this defines a metric on the ambient space

Optimal Paths via Lagrangian

Can Homogenize the Lagrangian:

$$\bar{L}(q) = \lim_{T\to\infty} \inf \frac{1}{T} \inf_{\phi\in H_0^1(0,T)} \int_0^T L(qt + \phi(t), q + \dot{\phi}(t))\, dt$$

In this case, the minimization of over curves is performed for each value of q, and the Hamiltonian $\bar{H}(p)$ is recovered via the Legendre transform.

Main Result

Theorem 1. *Let $H(p,x)$ be a metric Hamiltonian which is periodic on the unit cube. The $H(p,x)$ defines a distance on $\mathbb{R}^n$. We can write*

$$H(p,x) = \|p\|_{b(x)^*}$$

where $b(x,p)$ is corresponding metric cost function. Then $H(p,x)$ homogenizes to $\bar{H}(p)$ which is a homogeneous metric Hamiltonian. Furthermore

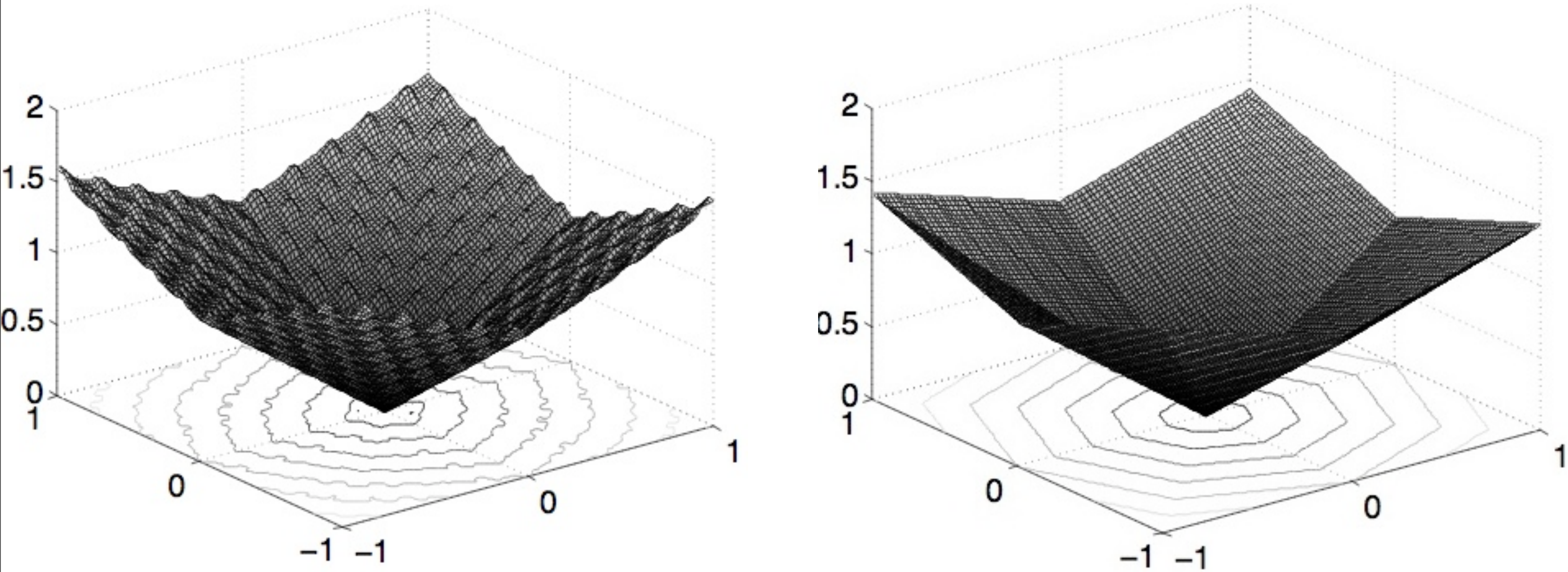
$$\bar{H}(p) = \|p\|_{\bar{b}^*} \tag{17}$$

which is the dual norm (14) *of the* $\bar{\mathrm{b}}$ *norm. The values can be obtained from*

$$\bar{b}(p) = \frac{1}{\bar{c}(p)} = \frac{T(p)}{|p|} = \lim_{\epsilon \to 0} \frac{T^\epsilon(p)}{|p|} \tag{18}$$

where T^ϵ is the solution of (HJ^ϵ).

Illustration of Main Result



Computations

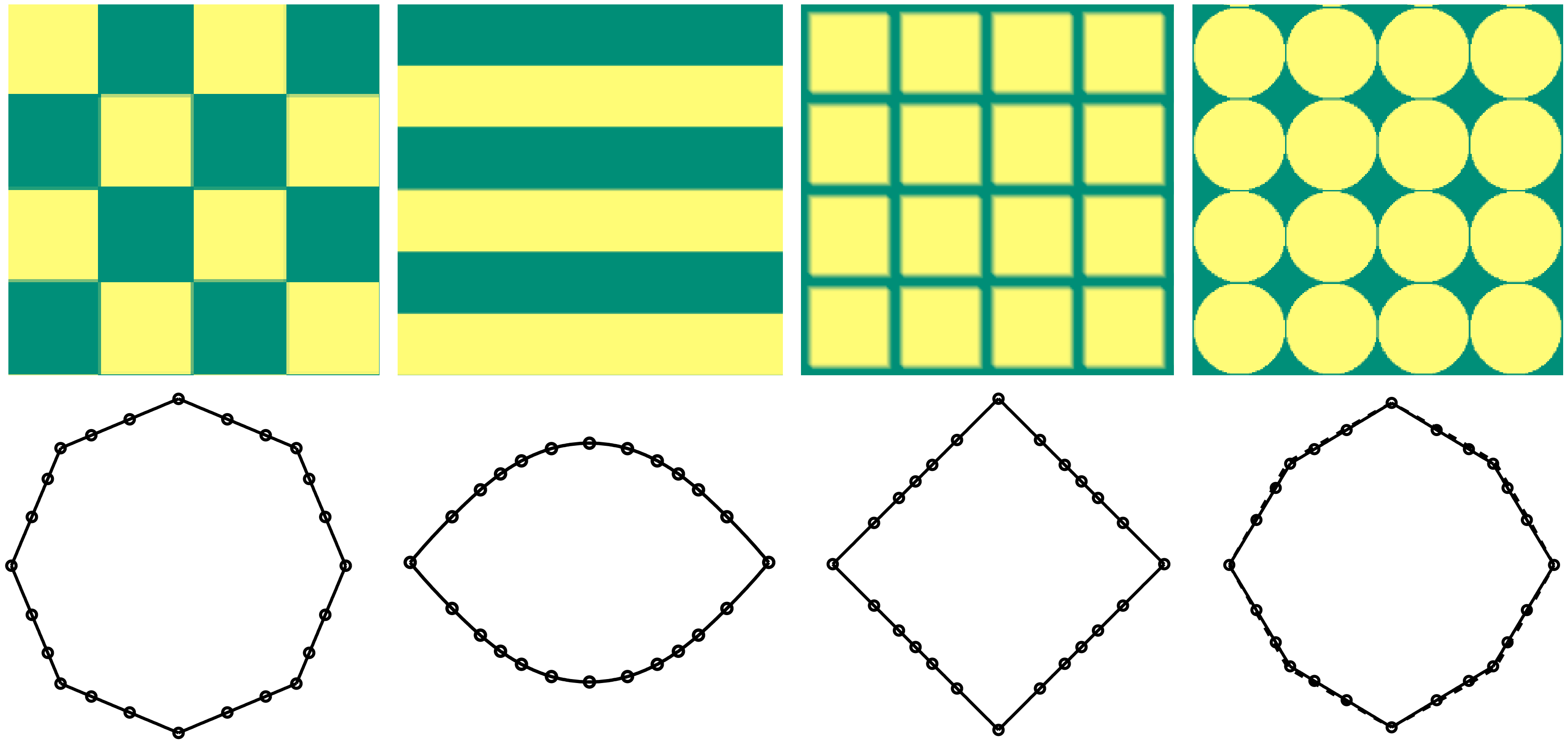


Figure 8: Period domains and computed vectograms: checkerboard, stripes, squares, and circles.

These are validations of analytic results

Open Problem: random case

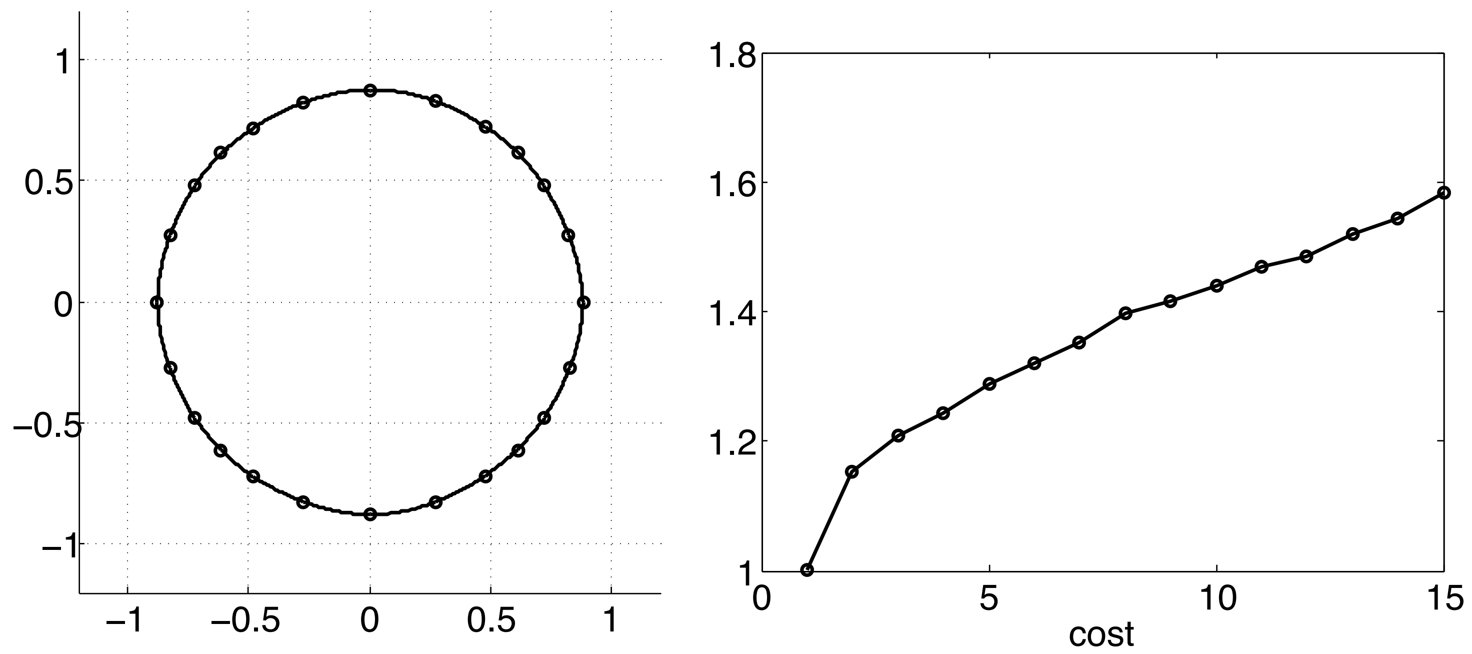
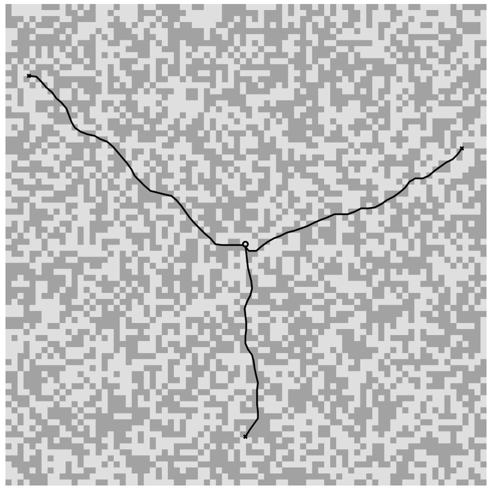
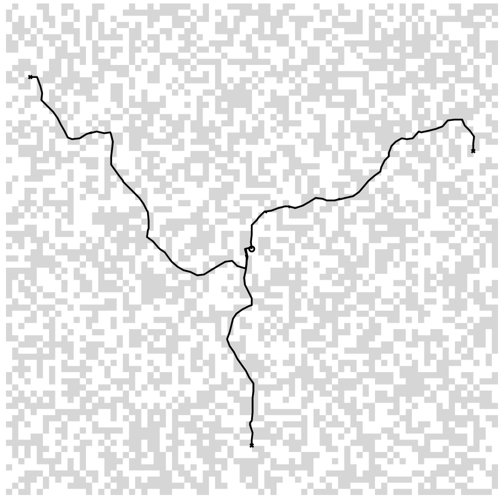


Figure 14: Illustration of the homogenized speed/cost in the random case. Left: computed vectogram, averaged over several trials, for $c = 1, 1/2$ with probability $1/2$. Right: computed homogenized cost $\bar{b}$ as a function of the random cost $b = 1$, or $b = b_0 = 1/c_0$ with probability $1/2$.





More Open Problems

- Application: wave propagation in inhomogeneous medium.
- Anisotropic eikonal equation. Similar results?
- Add an external velocity field to the equation.
  - Now measures the combination of growth with advection. Model for combustion.
  - Expect to homogenize to either: "shifted circle" (non-Finsler metric) or "non-controllable"