A GRADIENT FLOW INTERPRETATION OF THE KELLER-SEGEL SYSTEMS

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Numerical Optimal Transportation in (Mathematical) Economics

Joint works with V. Calvez, E. Carlen, J. Carrillo, J. Dolbeault, D. Kinderlherer, M. Kowalczyk, N. Masmoudi, Ph. Laurençot, S. Lisini & B. Perthame

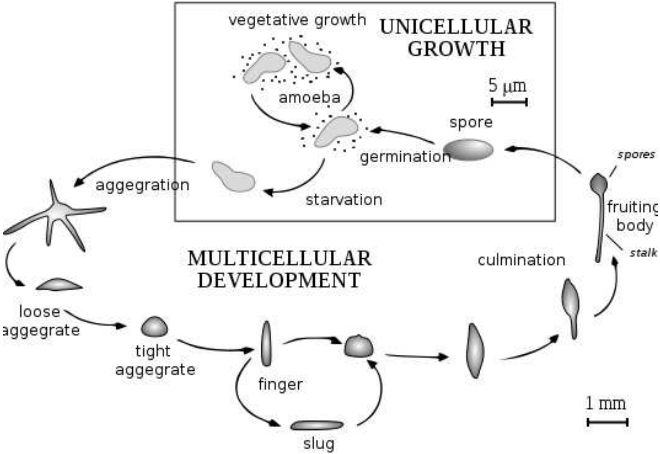

FIGURE : *Dictyostelium discoideum cycle (source: Wikipedia).*

GENERAL MODEL

$$\begin{cases} \partial_t \rho = \Delta \rho^m - \chi \nabla \cdot [\rho \nabla \phi] \,, \\ \tau \partial_t \phi = \Delta \phi - \alpha\, \phi + \rho \,, \end{cases} \qquad (t,x) \in (0,\infty) \times \mathbb{R}^d \,,$$

with $d \geq 1$, $m \in [1,2)$, $\alpha \geq 0$.

Remark:

$$\int_{\mathbb{R}^d} \rho(x,t)\, \mathrm{d}x = \int_{\mathbb{R}^d} \rho_0(x)\, \mathrm{d}x =: 1$$

The diffusion and interaction term “balance” if $(m-1)d = d-2$.

KNOWN RESULTS [SUGIYAMA, 2006 & 2007]

- if $m > m_d$ then all the solutions to (gKS) exist globally in time,
- if $m < m_d$ then for any χ there are solutions to (gKS) blowing-up in finite time.

The classical Keller-Segel system ($d = 2$, $m = 1$, $\tau = 0$ and $\alpha = 0$)

$$\begin{cases} \dfrac{\partial \rho}{\partial t} = \Delta\rho - \chi\nabla \cdot (\rho\nabla\Phi) & \text{in } (0, +\infty) \times \mathbb{R}^2 \\ \Delta\Phi = -\rho & \text{in } (0, +\infty) \times \mathbb{R}^2 \,. \end{cases} \tag{KS}$$

THE CLASSICAL KELLER-SEGEL SYSTEM ($d = 2$, $m = 1$, $\tau = 0$ AND $\alpha = 0$)

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We compute

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^2} |x|^2 \rho(t, x) \, \mathrm{d}x = 4 \left(1 - \frac{\chi}{8\pi} \right) .$$

BLOW-UP CRITERION

If $\chi > 8\pi$ and

$$\int_{\mathbb{R}^2} |x|^2 \rho_0(x) < \infty$$

then the solutions to (KS) blowup in finite time.

The free energy functional

$$\mathcal{F}_{\mathrm{PKS}}[\rho] = \int_{\mathbb{R}^2} \rho(x) \log \rho(x)\, \mathrm{d}x + \frac{\chi}{4\pi} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \rho(x) \log|x-y| \rho(y)\, \mathrm{d}x\, \mathrm{d}y \,.$$

If ρ is a smooth solution to (KS) then

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{F}_{\mathrm{PKS}}[\rho(t)] = -\int_{\mathbb{R}^2} \rho \left|\nabla \left(\log \rho - \chi\, c\right)\right|^2 \mathrm{d}x \leq 0 \,.$$

Logarithmic Hardy-Littlewood-Sobolev's inequality [Carlen-Loss, 1992]

Let $f \in L^1_+(\mathbb{R}^2)$ such that $f \log f$ and $f \log(1+|x|^2)$ are bounded in $L^1(\mathbb{R}^2)$. If $\int_{\mathbb{R}^2} f\, \mathrm{d}x = 1$, then

$$\int_{\mathbb{R}^2} f \log f + 2 \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f(x) \log|x-y| f(y)\, \mathrm{d}x\, \mathrm{d}y \geq -C \,. \tag{logHLS}$$

Let $\lambda \geq 0$, the minimisers of (logHLS) are the translations of

$$\bar{\rho}_\lambda(x) := \frac{1}{\pi} \frac{\lambda}{\left(\lambda + |x|^2\right)^2} \,.$$

Global existence:

$$\left(1 - \frac{\chi}{8\pi}\right) \int_{\mathbb{R}^2} \rho \log \rho \leq \mathcal{F}_{\mathrm{PKS}}[\rho_0] + C \frac{\chi}{8\pi} < \infty \quad \text{if } \chi < 8\pi.$$

KNOWN RESULTS [B., CARRILLO, DOLBEAULT, PERTHAME, ...]

Under the assumptions

$$\rho_0 \in \mathcal{P}(\mathbb{R}^2)\,, \quad \rho_0 \log \rho_0 \in L^1(\mathbb{R}^2) \quad \text{and} \quad |x|^2 \rho_0 \in L^1(\mathbb{R}^2)\,. \tag{H}$$

- If $\chi < 8\pi$, solutions to (KS) exist globally in time and converge to the self-similar profile.
- If $\chi = 8\pi$, solutions to (KS) exist globally in time and blowup as a Dirac mass of mass $8\,\pi$ centred at the centre of mass in infinite time.
- If $\chi > 8\pi$, solutions to (KS) blowup in finite time.

Open questions:

- Can this model be derived from a microscopic model?
- How does the solution blowup?
- What happens after blowup?

FRAMEWORK

Consider the classical parabolic-elliptic Keller-Segel system when $\chi = 8\pi$ and the 2-moment is unbounded.

GLOBAL EXISTENCE AND LARGE TIME BEHAVIOUR [B., CARLEN, CARRILLO, 2012]

If $\chi = 8\pi$ and there exists $\lambda > 0$ with

$$\mathcal{W}_2[\rho_0, \bar{\rho}_\lambda] < \infty.$$

Then there exists $\rho \in \mathcal{AC}^0([0,T], \mathcal{P}_2(\mathbb{R}^2))$, with $\rho(t) \in L^1(\mathbb{R}^2)$ for all $t \geq 0$ being a global-in-time weak solution of (KS). Moreover,

$$\lim_{t\to\infty} \mathcal{F}_{\mathrm{PKS}}[\rho(t)] = \mathcal{F}_{\mathrm{PKS}}[\bar{\rho}_\lambda] \qquad \lim_{t\to\infty} \|\rho(t) - \bar{\rho}_\lambda\|_{L^1(\mathbb{R}^2)} = 0 \ .$$

And the system satisfies the hypercontractivity property *i.e.* for any $t^* > 0$, the constructed solution ρ is bounded in $L^\infty(t^*, \infty, L^p(\mathbb{R}^2))$, for any $p \in (1, \infty)$.

Basin on attraction: If $\lambda \neq \mu$ then

$$\mathcal{W}_2(\bar{\rho}_\mu, \bar{\rho}_\lambda) = \frac{1}{2} \int_{\mathbb{R}^2} \left| \frac{\lambda}{\mu} x - x \right|^2 \bar{\rho}_\mu = +\infty \ .$$

THE CLASSICAL KS SYSTEM SEEN AS A GRADIENT FLOW, [A. B, V. CALVEZ, J. CARRILLO, 2007]

The system (KS) can be seen as a gradient flow of the free energy in the Monge-Kantorovich metric:

$$\partial_t \rho = -\nabla_{W} \mathcal{F}_{\mathrm{PKS}}[\rho(t)] \ .$$

THE JORDAN-KINDERLEHRER-OTTO (JKO) SCHEME

Given a time step h, we define the solution by the minimising scheme:

$$\rho_\tau^{n+1} \in \operatorname{argmin}_{\rho \in \mathcal{S}} \left[\frac{\mathcal{W}_2^2(\rho, \rho_\tau^n)}{2h} + \mathcal{F}_{\mathrm{PKS}}[\rho] \right] \ ,$$

where $\mathcal{S} := \{\rho \in \mathcal{P}_2(\mathbb{R}^2) \ : \ \int_{\mathbb{R}^2} \rho \log \rho < \infty\}$.

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where $\mathcal{S} := \{\rho \in \mathcal{P}_2(\mathbb{R}^2) \ : \ \int_{\mathbb{R}^2} \rho \log \rho < \infty\}$.

It is classical, but not trivial, to prove using the logarithmic Hardy-Littlewood-Sobolev inequality that this minimisation problem has a minimiser if $\chi < 8\pi$.

Let ρ be a minimiser. Let $\zeta \in \mathcal{C}_0^\infty(\mathbb{R}^d;\mathbb{R}^d)$. For $\delta \in (0,1)$, define

$$\rho_\delta \quad := T_\delta \# \rho \quad \text{where} \quad T_\delta := \mathrm{id} + \delta\,\zeta\ .$$

THE DISCRETE EULER-LAGRANGE EQUATION

$$\frac{1}{h}\int_{\mathbb{R}^2} \zeta\left[\rho - \rho_h^n\right]\,\mathrm{d}x = \int_{\mathbb{R}^2} \Delta\zeta\,\mathrm{d}\rho - \chi \iint_{\mathbb{R}^2\times\mathbb{R}^2} \frac{[\nabla\zeta(x) - \nabla\zeta(y)]\cdot(x-y)}{|x-y|^2}\,\mathrm{d}\rho(x)\,\mathrm{d}\rho(y) + O(h^{1/2}).$$

For each positive integer n, let $\nabla\varphi^n$ be the optimal transportation plan with $\nabla\varphi^n\#\rho_h^n = \rho_h^{n-1}$. Then for $(n-1)h \le t \le nh$ we define

MCCANN'S INTERPOLANT

$$\rho_h(t) = \left(\frac{t-(n-1)h}{h}\mathrm{id} + \frac{nh-t}{h}\nabla\varphi^n\right)\#\rho_h^n\,.$$

And we have to pass to the limit in the discrete Euler-Lagrange equation:

$$\int_{\mathbb{R}^2}\zeta(x)\,[\rho_h(t_2,x)-\rho_h(t_1,x)]\,\mathrm{d}x = \int_{t_1}^{t_2}\int_{\mathbb{R}^2}\Delta\zeta(x)\,\mathrm{d}\rho_h(s,x)\,\mathrm{d}s$$
$$-\chi\int_{t_1}^{t_2}\iint_{\mathbb{R}^2\times\mathbb{R}^2}\frac{[\nabla\zeta(x)-\nabla\zeta(y)]\cdot(x-y)}{|x-y|^2}\,\mathrm{d}\rho_h(s,y)\,\mathrm{d}\rho_h(s,x)\,\mathrm{d}s + O(h^{1/2}).$$

By the energy and Hölder estimates, up to the extraction of a sub-sequence, $(\rho_h)_h$ converges in $\mathcal{C}([0,T],\mathcal{P}(\mathbb{R}^2))$ and $(\rho_h(t))_h$ in w-$\mathrm{L}^1(\mathbb{R}^2)$. We can pass to the limit and obtain

WEAK SOLUTIONS, [B., CALVEZ, CARRILLO, SINUM 2008]

$$\int_{\mathbb{R}^2}\zeta(x)\,[\rho(t_2,x)-\rho(t_1,x)]\,\mathrm{d}x = \int_{t_1}^{t_2}\int_{\mathbb{R}^2}\Delta\zeta(x)\,\mathrm{d}\rho(s,x)\,\mathrm{d}s$$
$$-\chi\int_{t_1}^{t_2}\iint_{\mathbb{R}^2\times\mathbb{R}^2}\frac{[\nabla\zeta(x)-\nabla\zeta(y)]\cdot(x-y)}{|x-y|^2}\,\mathrm{d}\rho(s,y)\,\mathrm{d}\rho(s,x)\,\mathrm{d}s.$$

TOY KELLER-SEGEL SYSTEM

$$\begin{cases} \partial_t \rho = \Delta \rho - \chi \nabla \cdot (\rho \nabla \phi) & \text{dans } (0,\infty) \times \mathbb{R}\,, \\ \phi(t,x) = -\dfrac{1}{\pi} \displaystyle\int_{\mathbb{R}} \rho(y)\, \log|x-y|\, \mathrm{d}y & \text{dans } (0,\infty) \times \mathbb{R}\,. \end{cases}$$

Associated free energy

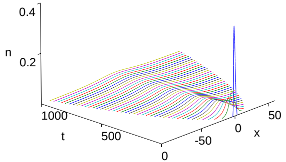
$$\mathcal{F}[\rho] = \int_{\mathbb{R}} \rho \log \rho + \frac{\chi}{2\,\pi} \iint_{\mathbb{R}\times\mathbb{R}} \rho(x)\,\rho(y)\, \log|x-y|\, \mathrm{d}x\, \mathrm{d}y$$

The scheme can be seen as the gradient flow of the inverse distribution function V_n:

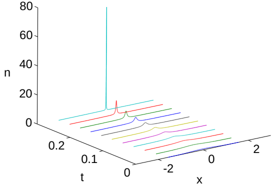
$$V_{n+1} \in \arg\inf_{\{W\,:\,(W^{-1})' \in \mathcal{K}\}} \left[\mathcal{G}[W] + \frac{1}{2\,\tau} \|W - V_n\|^2_{L^2(0,1)} \right]$$

where

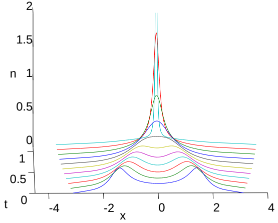
$$\mathcal{G}[W] := -\int_0^1 \log W'(w)\, \mathrm{d}w + \int_0^1 |W(w)|^2\, \mathrm{d}w + \frac{\chi}{\pi} \int_0^1 \log|W(w) - W(z)|\, \mathrm{d}w\, \mathrm{d}z\,.$$



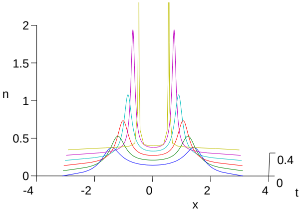
Sub-critical self-similar case.



Sub-critical case.



Two pics, case $\chi_C < \chi < 2\,\chi_C$.



Two pics, case $\chi > 2\,\chi_C$.

If $\chi < 8\pi$. Let (u_∞, v_∞) the unique solution to

$$\frac{e^{\chi v_\infty - |x|^2/2}}{\int_{\mathbb{R}^2} e^{\chi v_\infty - |x|^2/2}\,\mathrm{d}x} = -\Delta v_\infty = u_\infty\ .$$

$$\text{Let}\quad n_\infty(x,t) := \frac{1}{1+2t}\, u_\infty\left(\frac{1}{2}\log(1+2t), \frac{x}{\sqrt{1+2t}}\right)$$

$$\text{and}\quad c_\infty(x,t) := v_\infty\left(\frac{1}{2}\log(1+2t), \frac{x}{\sqrt{1+2t}}\right)$$

THEOREM (ASYMPTOTIC BEHAVIOUR, A. B, J. DOLBEAULT, B. PERTHAME, 2006)

If $\chi < 8\pi$ and ρ is a solution to (KS) then

$$\lim_{t\to\infty} \|n(t) - n_\infty(t)\|_{L^1(\mathbb{R}^2)} = 0 \text{ and } \lim_{t\to\infty} \|\nabla c(t) - \nabla c_\infty(t)\|_{L^2(\mathbb{R}^2)} = 0\,.$$

Let $n(x,t) := \frac{1}{1+2t}\, u\left(\frac{1}{2}\log(1+2t), \frac{x}{\sqrt{1+2t}}\right)$

and $c(x,t) := v\left(\frac{1}{2}\log(1+2t), \frac{x}{\sqrt{1+2t}}\right)$

(u,v) is solution to

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + \nabla(x\,u) - \nabla\cdot(u\nabla v) & \text{in } (0,\infty)\times\mathbb{R}^2\,, \\ \Delta v = -u & \text{in } (0,\infty)\times\mathbb{R}^2\,. \end{cases}$$

The free energy associated satisfies

$$\mathcal{F}[u(t)] - \mathcal{F}[u_0] \leq \int_0^t \int_{\mathbb{R}^2} u(x,t) \left|\nabla\left(\log u(x,t) - c(x,t) + \frac{|x|^2}{2}\right)\right|^2 dx\, ds$$

THE PARABOLIC-PARABOLIC KELLER-SEGEL SYSTEM ($d = 2$, $m = 1$, $\tau > 0$ AND $\alpha > 0$)

$$\begin{cases} \partial_t \rho = \Delta \rho - \chi \nabla \cdot [\rho \nabla \phi] \,, \\ \tau \partial_t \phi = \Delta \phi - \alpha \, \phi + \rho \,, \end{cases} \quad (t, x) \in (0, \infty) \times \mathbb{R}^2 \,, \tag{1}$$

THE FREE ENERGY FUNCTIONAL

$$\mathcal{F}[\rho, \phi] := \int_{\mathbb{R}^2} \left\{ \frac{\rho \log \rho}{\chi} - \rho \, \phi + \frac{1}{2} \, |\nabla \phi|^2 + \frac{\alpha}{2} \, \phi^2 \right\} \mathrm{d}x \,.$$

MAIN RESULT, [A.B., J. CARRILLO, D. KINDERLHERER, M. KOWALCZYK, PH. LAURENÇOT & S. LISINI, 2014]

The system (1) has the following “gradient flow” structure

$$\begin{cases} \partial_t \rho = -\nabla_W \mathcal{F}[\rho, \phi] \\ \partial_t \phi = -\dfrac{\delta \mathcal{F}[\rho, \phi]}{\delta \phi} \end{cases} \,,$$

Define

$$\mathcal{K} := \{\rho \in \mathcal{P}_2(\mathbb{R}^2) \,:\, \int_{\mathbb{R}^2} \rho \log \rho < \infty\} \,.$$

THE HYBRID VARIATIONAL SCHEME

$$\begin{cases} (\rho_h^0, \phi_h^0) = (\rho_0, \phi_0)\,, \\ (\rho_h^{n+1}, \phi_h^{n+1}) \in \mathrm{Argmin}_{(\rho,\phi)\in\mathcal{K}} \dfrac{1}{2h}\left[\dfrac{\mathcal{W}_2^2(\rho, \rho_h^n)}{\chi} + \tau \, \|\phi - \phi_h^n\|_2^2\right] + \mathcal{F}[\rho, \phi]\,, \qquad n \geq 0\,. \end{cases}$$

Open questions:

- Do all the solutions blowup in finite time for super-crtical χ?
- Can we construct global-in-time solution for all χ, see [Biler, Corrias, Dolbeault]?

THE GENERALISED PARABOLIC-ELLIPTIC KS SYSTEM ($d \geq 3$, $\tau = 0$, $\alpha \geq 0$)

$$\begin{cases} \partial_t \rho = \Delta(\rho^m) - \chi \nabla \cdot (\rho \nabla \Phi) & \text{in } (0, +\infty) \times \mathbb{R}^d \\ \Delta \Phi = -\rho & \text{in } (0, +\infty) \times \mathbb{R}^d \,. \end{cases} \tag{gKS}$$

where

$$m = m_d := 2 - \frac{2}{d} \in (1, 2)\,.$$

THE FREE ENERGY

Define

$$t \mapsto \mathcal{F}[\rho] := \int_{\mathbb{R}^d} \frac{\rho^m}{m-1} + \frac{\chi\, c_d}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\rho(x)\, \rho(y)}{|x-y|^{d-2}}\, \mathrm{d}x\, \mathrm{d}y$$

Its time derivative, *the Fisher information*, for solutions to (gKS) is

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{F}[\rho(t)] = - \int_{\mathbb{R}^d} \rho \left| \nabla \left(\frac{m}{m-1} \rho^{m-1} - \phi \right) \right|^2 \mathrm{d}x \,.$$

VARIANT TO THE HARDY-LITTLEWOOD-SOBOLEV (VHLS) INEQUALITY [∼ LIEB, 1983]

For all $h \in \mathrm{L}^1(\mathbb{R}^d) \cap \mathrm{L}^m(\mathbb{R}^d)$ there exists an optimal constant C_{HLS} such that

$$\left| \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{h(x)\, h(y)}{|x-y|^{d-2}}\, \mathrm{d}x\, \mathrm{d}y \right| \leq C_{\mathrm{HLS}}\, \|h\|_1^{2/d}\, \|h\|_m^m \,.$$

Let us define the *critical chemo-sensitivity* χ_c by

$$\chi_c := \left[\frac{2}{(m-1)\, C_{\mathrm{HLS}}\, c_d} \right]^{d/2} .$$

MAIN THEOREM: CRITICAL MASS [B., CARRILLO, LAURENÇOT, 2009]

Under the assumptions

$$\rho_0 \geq 0\,, \quad \rho_0 \in L^1(\mathbb{R}^2, (1+|x|^2)\,\mathrm{d}x)\,, \quad \text{and} \quad \rho_0 \in L^m(\mathbb{R}^2)\,.$$

There exists a constant χ_c such that

- if $\chi < \chi_c$, solutions exist globally in time and there is a radially symmetric compactly supported self-similar solution,
- if $\chi = \chi_c$, solutions exist globally in time. There are infinitely many compactly supported stationary solutions,
- if $\chi > \chi_c$, there are solutions which blowup in finite time and self-similar blowing up solutions.

Define

$$\mu_\chi := \inf_{h\in\mathcal{Y}} \mathcal{F}_\chi[h] \quad \text{where} \quad \mathcal{Y} := \{h \in \mathrm{L}^1(\mathbb{R}^d) \cap \mathrm{L}^m(\mathbb{R}^d)\} \,.$$

INFIMUM OF THE FREE ENERGY

We have

$$\mu_\chi = \begin{cases} 0 & \text{if } \chi \in (0, \chi_c], \\ -\infty & \text{if } \chi > \chi_c. \end{cases}$$

Moreover, the infimum is not achieved in the case $\chi < \chi_c$ while

IDENTIFICATION OF THE MINIMISERS IN THE CASE $\chi = \chi_c$

Let ζ be the unique positive radial classical solution to

$$\Delta\zeta + \frac{m-1}{m}\,\zeta^{1/(m-1)} = 0 \quad \text{in} \quad B(0,1) \quad \text{with} \quad \zeta = 0 \quad \text{on} \quad \partial B(0,1)\,.$$

If V is a minimiser of $\mathcal{F}_\chi$ in $\mathcal{Y}$ there are $R > 0$ and $z \in \mathbb{R}^d$ such that

$$V(x) = \begin{cases} \dfrac{1}{R^d}\left[\zeta\left(\dfrac{x-z}{R}\right)\right]^{1/(m-1)} & \text{if } x \in B(z,R), \\ 0 & \text{if } x \in \mathbb{R}^d \setminus B(z,R). \end{cases}$$

As a direct consequence of the VHLS inequality, for any $h \in \mathcal{Y}$

$$C_{\mathrm{HLS}}\,\frac{c_d}{2}\chi_c^{2/d}\left(1-\left(\frac{\chi}{\chi_c}\right)^{2/d}\right)\|h\|_m^m \le \mathcal{F}[h]\,.$$

EXTENSION CRITERIA

For any $\eta > 0$ there exists $\tau_\eta > 0$ depending only on d, χ, and η such that, if

$$\|\rho(t^*)\|_m \le \eta$$

for some $t^* \in [0,\infty)$, then ρ is bounded in $\mathrm{L}^\infty(t^*, t^* + \tau_\eta; \mathrm{L}^m(\mathbb{R}^d))$.

In the Wasserstein metric the solution to the system (KS) is a gradient flow of the free energy:

$$\partial_t \rho = -\nabla_W \mathcal{F}[\rho(t)] \ .$$

THE JORDAN-KINDERLEHRER-OTTO (JKO) SCHEME

Given a time step τ, we define the solution by the minimising scheme:

$$\rho_\tau^{k+1} \in \operatorname{argmin}_{\rho \in \mathcal{Y}} \left[\frac{\mathcal{W}_2^2(\rho, \rho_\tau^k)}{2\tau} + \mathcal{F}[\rho] \right] \ .$$

THE EULER-LAGRANGE EQUATION

The distributional gradient of ρ_τ^{k+1} satisfies

$$-\nabla(\rho_\tau^{k+1})^m + \chi \rho_\tau^{k+1} \nabla c_\tau^{k+1} = \frac{\mathrm{id} - \nabla\varphi^k}{\tau} \rho_\tau^{k+1}$$

where $\nabla\varphi^k$ is the unique gradient of a lower semi-continuous convex function such that $\nabla\varphi^k \# \rho_\tau^{k+1} = \rho_\tau^k$.

HOW WOULD IT BLOWUP

Let $(t_k)_k$ go to ∞. If

$$\lim_{k\to\infty} \|\rho(t_k)\|_m = \infty\ .$$

then there are a sub-sequence $(t_{k_j})_j$ and a sequence $(x_j)_j$ in $\mathbb{R}^d$ such that

$$\lim_{j\to\infty} \left\| \rho(t_{k_j}, x + x_j) - \frac{1}{\lambda_{k_j}^d}\, V\left(\frac{x}{\lambda_{k_j}}\right) \right\|_{L^1} = 0\ ,$$

where $\lambda_k := \|\rho(t_k)\|_m^{-m/(d-2)}$ and V is the minimiser of $\mathcal{F}_{\chi_c}$ in $\mathcal{Y}$ with $\|V\|_m = 1$.

Concentration compactness argument: We set $v_k(x) := \lambda_k^d\, \rho(t_k, \lambda_k\, x)$ so that $\|v_k\|_m = 1$.

$$\lim_{k\to\infty} \mathcal{F}[v_k] = \lim_{k\to\infty} \|\rho(t_k)\|_m^{-m}\, \mathcal{F}[\rho(t_k)] = 0\ .$$

Consequently,

$$\lim_{k\to\infty} \chi \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{v_k(t,x)\, v_k(t,y)}{|x-y|^{d-2}} = \lim_{k\to\infty} \frac{2\chi}{c_d} \left(\frac{1}{m-1} \|v_k\|_m^m - \mathcal{F}[v_k] \right) > 0\ .$$

Proof of the main theorem: characterisation of the nature of the blowup + control of the 2-moment.

MOMENT ESTIMATE

For any ρ solution to (gKS)

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} |x|^2 \rho(t,x)\, \mathrm{d}x = 2\,(d-2)\, \mathcal{F}[\rho(t)]\ .$$

And, for any $\chi > \chi_c$, we can construct an intial data such that $\mathcal{F}$ is intially negative so that the solution blowsup in finite time.

The blowup time T being given, we can look for solution to (gKS) of the form

$$\rho(t,x) = \frac{1}{s(t)^d}\Psi\left(\frac{x}{s(t)}\right) \quad \text{and} \quad c(t,x) = \frac{1}{s(t)^{d-2}}\Phi\left(\frac{x}{s(t)}\right)$$

where $s(t) := [d(T-t)]^{1/d}$.

SELF-SIMILAR BLOWING-UP SOLUTIONS

There exists $\chi_2 \in (\chi_c, \infty)$ such that for any $\chi \in (\chi_c, \chi_2]$, there exists a self-similar blowing-up solution with a radially symmetric, compatly supported and non-increasing profile Ψ, satisfyng $\|\rho(t)\|_1 = \|\Psi\|_1 = 1$ for $t \in [0, T)$ and $\|\rho(t)\|_\infty \to \infty$ as $t \to T$.

Method relies on the study of an ordinary differential equation of the type:

$$\begin{cases} u''(r,a) + \dfrac{d-1}{r}\, u'(r,a) + |u(r,a)|^{p-1}\, u(r,a) - 1 = 0\,, \quad r \in [0, r_{\max}(a))\,, \\ u(0,a) = a\,, \quad u'(0,a) = 0\,, \end{cases} \tag{2}$$

with $r_{\max}(a) \in (0, \infty]$ and $p = d/(d-2)$.

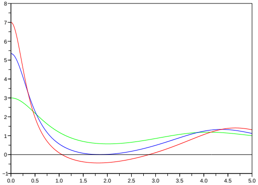


FIGURE : Behaviour of $u(., a)$ for $a > a_c$, $a = a_c$ and $a < a_c$.

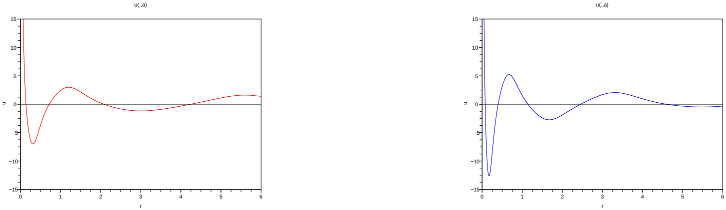


FIGURE : Positivity set of $u(., a)$ with two ($a = 50$, left) and three ($a = 90$, right) connected components ($d = 3$).

RESCALED SYSTEM

$$\begin{cases} \partial_t u &= \Delta(u^m) - \chi\nabla\cdot[u\nabla\Psi] + \nabla\cdot(xu) & \text{in } (0,\infty)\times\mathbb{R}^d\,, \\ -\Delta\Psi &= u & \text{in } (0,\infty)\times\mathbb{R}^d\,. \end{cases} \tag{rNKSd}$$

The free energy associated to (rNKSd) is given by

$$\mathcal{F}[h] + \frac{1}{2}\int_{\mathbb{R}^d} |x|^2\,|h(x)|\,\mathrm{d}x$$

We define

$$\nu_\chi := \inf\{\mathcal{F}_\chi[h] \,:\, h\in\mathcal{Z}_\chi\} \quad \text{with} \quad \mathcal{Z} = \left\{h\in\mathcal{Y} \,:\, \int_{\mathbb{R}^d} |x|^2\,|h(x)|\,\mathrm{d}x < \infty\right\}\,.$$

INFIMUM OF THE RESCALED FREE ENERGY

$$\begin{cases} \nu_\chi > 0 & \text{if } \chi < \chi_c\,, \\ \nu_{\chi_c} = 0\,, & \\ \nu_\chi = -\infty & \text{if } \chi > \chi_c\,. \end{cases}$$

THE MINIMISERS IN THE CASE $\chi < \chi_c$

There is a unique minimiser W_χ of $\mathcal{F}_\chi$ in $\mathcal{Z}$.

In addition, W_χ is non-negative radially symmetric and non-increasing and there is a unique $\varrho_\chi > 0$ such that

$$\begin{cases} W_\chi^{m-1} := \xi_\chi & \text{in } B(0, \varrho_\chi) \\ W_\chi(x) = 0 & \text{in } \mathbb{R}^d \setminus B(0, \varrho_\chi)\,, \end{cases}$$

where

$$\begin{cases} \Delta\xi_\chi + \dfrac{m-1}{m}\left(\xi_\chi^{1/(m-1)} + d\right) = 0 & \text{in } B(0, \varrho_\chi) \\ \xi_\chi = 0 & \text{on } \partial B(0, \varrho_\chi)\,. \end{cases}$$

Open questions:

- Do these self-similar solutions attract all the solutions in the sub-critical case?
- What happens in the critical case?
- Can we contruct global-in-time solutions with supercritical χ?

Plan

THE GENERALISED PARABOLIC-PARABOLIC KS SYSTEM ($d \geq 3$, $\tau > 0$, $\alpha \geq 0$)

$$\begin{cases} \partial_t \rho = \Delta \rho^m - \chi \nabla \cdot [\rho \nabla \phi] \,, \\ \tau \partial_t \phi = \Delta \phi + \rho - \alpha \phi \,, \end{cases} \qquad (t,x) \in (0,\infty) \times \mathbb{R}^d \,, \tag{3}$$

THE FREE ENERGY

$$\mathcal{F}[\rho,\phi] := \frac{1}{\chi} \int_{\mathbb{R}^d} \frac{\rho^m}{m-1} \, \mathrm{d}x - \int_{\mathbb{R}^d} \rho \, \phi \, \mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}^d} |\nabla \phi|^2 \, \mathrm{d}x + \frac{\alpha}{2} \int_{\mathbb{R}^d} |\phi|^2 \, \mathrm{d}x \,.$$

GLOBAL EXISTENCE [B. & LAURENÇOT, 2012]

Let $\tau > 0$, $\alpha \geq 0$ and $d \geq 3$. Consider $\rho_0 \in \mathcal{P}_2(\mathbb{R}^d)$ be such that $\rho_0 \in \mathrm{L}^m(\mathbb{R}^d)$ and $\phi_0 \in \mathrm{H}^1(\mathbb{R}^d)$. If $\chi < \chi_c$ then there is a weak global-in-time solution (ρ, ϕ) to the parabolic-parabolic Keller-Segel system (3) and, for all $t > 0$

- $\rho(t)$ is bounded in $\mathcal{P}_2(\mathbb{R}^d)$ for all $t \geq 0$,
- ρ is bounded in $\mathrm{L}^\infty(0,t;\mathrm{L}^m(\mathbb{R}^d))$ and $\rho^{m/2}$ in $\mathrm{L}^2(0,t;\mathrm{H}^1(\mathbb{R}^d))$,
- ϕ is bounded in $\mathrm{L}^\infty(0,t;\mathrm{H}^1(\mathbb{R}^d)) \cap \mathrm{L}^2(0,t;\mathrm{H}^2(\mathbb{R}^d)) \cap \mathrm{W}^{1,2}(0,\mathrm{t};\mathrm{L}^2(\mathbb{R}^\mathrm{d}))$, $\phi(0) = \phi_0$.

Merci pour votre attention