From the Euler equations of incompressible fluids to gravitation through optimal transport

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INCOMPRESSIBLE FLUIDS

One can describe the motion of an incompressible fluid inside a bounded domain D in $\mathbf{R^d}$ by a time-dependent family $\mathbf{t \rightarrow M_t}$ of maps, in the Hilbert space $\mathbf{H = L^2(D, R^d)}$, valued in the subset $\mathbf{VPM(D)}$ of all Lebesgue measure-preserving maps

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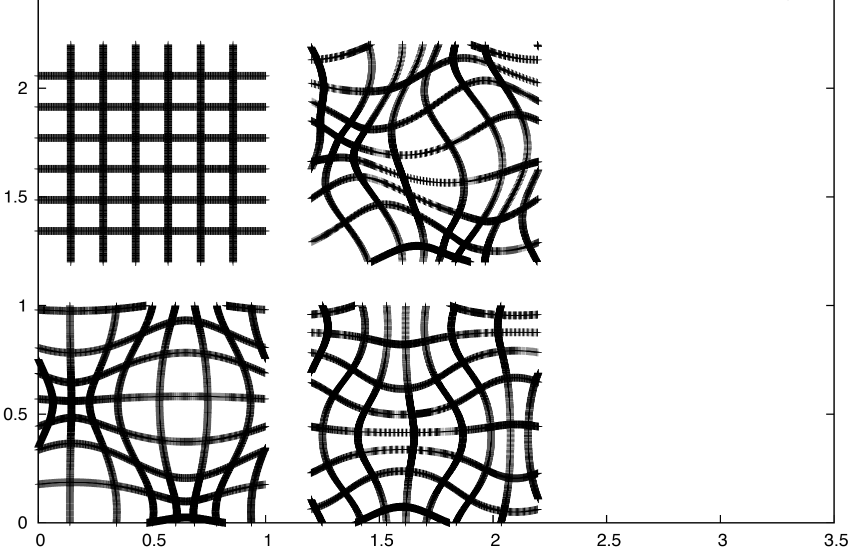
$$\boxed{\mathbf{VPM(D)} = \{\mathbf{M} \in \mathbf{H},\ \int_D \mathbf{q}(\mathbf{M}(\mathbf{x}))\mathbf{dx} = \int_D \mathbf{q}(\mathbf{x})\mathbf{dx},\ \forall \mathbf{q} \in \mathbf{C}(\mathbf{R^d})\}}$$



three maps of the periodic square: one is area preserving

THE EULER EQUATIONS OF INCOMPRESSIBLE FLUIDS

Solutions of the Euler equations, introduced in 1755, correspond to those curves $t \to M_t \in VPM(D)$ for which there exists a time dependent scalar function p_t, called 'pressure field', defined on D, such that

$$\boxed{\frac{d^2 M_t}{dt^2} + (\nabla p_t) \circ M_t = 0}$$

where ∇ is the gradient operator on R^d (with respect to the Euclidean norm $|\cdot|$).

THE PRINCIPLE OF LEAST ACTION

(easy) THEOREM Assume D to be convex. Let (M_t, p_t) a solution of the Euler equations, with $p_t(x)$ uniformly semi-concave in x. Then, for every sufficiently short interval $]t_0, t_1[$, M_t is the unique minimizer, among all curves along $VPM(D)$ that coincide with M_t at $t = t_0$, $t = t_1$, of the following **ACTION**

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In other words, such a curve is nothing but a (constant speed) minimizing geodesic along $VPM(D)$, with respect to the metric induced by $H = L^2(D, R^d)$ on $VPM(D)$.

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see Arnold 1966, Ebin-Marsden 1970, Arnold-Khesin book 1998

VOLUME-PRESERVING MAPS APPROXIMATED PAR PERMUTATIONS

Fix $D = [0, 1]^d$ and consider its dyadic decomposition by $N = 2^{nd}$ sub-cubes D_i^N, of barycenters a_i^N.

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We may approximate the set $VPM(D)$ of all volume-preserving maps by the discrete set $S = P_N(D)$ of all rigids permutations of the N sub-cubes.


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PENALIZATION OF THE EULER ACTION

Since minimizing geodesics along a discrete set such as the set of rigid permutations $S = P_N(D)$ do not make much sense, we rather consider a penalized version of the Euler action (*)

$$\boxed{\int_{t_0}^{t_1} \frac{1}{2}\left(\left|\left|\frac{dM_t}{dt}\right|\right|^2 + \frac{1}{\epsilon} Q[M_t]\right) dt}$$

$$\boxed{Q[M] = \inf_{s \in S} \frac{1}{2} ||M - s||^2}$$

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(*) For smooth sets, this is a consistent approximation to minimizing geodesics (cf. Rubin-Ungar, CPAM 1957).

FINITE-DIMENSIONAL REDUCTION

It is consistent to limit ourself to piecewise affine maps of form

$$\boxed{M_t(x) = x - a_i^N + X_i(t), \quad x \in D_i^N, \quad i = 1, ..., N}$$

Here $X(t) \in (R^d)^N$ becomes the new, finite-dimensional, unknown.

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Here $X(t) \in (R^d)^N$ becomes the new, finite-dimensional, unknown. Accordingly, the penalized action becomes

$$\boxed{\int_{t_0}^{t_1} \frac{1}{2}\left(\left|\left|\frac{dX(t)}{dt}\right|\right|^2 + \frac{1}{\epsilon}Q[X(t)]\right)dt}$$

$$\boxed{Q[X] = \inf_{s \in S} \frac{1}{2}||X - s||^2, \quad S = \{ (a_{\sigma_1}^N, \cdots, a_{\sigma_N}^N), \quad \sigma \in \mathcal{S}_N\}}$$

where $|| \cdot ||$ now denotes the euclidean norm in $H = R^{dN}$ and $\mathcal{S}_N$ is the set of all permutations of $\{1, \cdots, N\}$.

THE RESULTING FINITE-DIMENSIONAL DYNAMICAL SYSTEM

Using the least-action principle, we end up with the following finite-dimensional dynamical system

$$\boxed{\epsilon \frac{d^2 X}{dt^2} = X - \pi[X]}$$

$$\boxed{\pi[X] = (a^N_{\sigma_1}, \cdots, a^N_{\sigma_N}), \qquad \sigma = \text{Arginf}\{\sum_{i=1}^{N} |X_i - a^N_{\sigma_i}|^2, \quad \sigma \in \mathcal{S}_N\}}$$

Here $|\cdot|$ denotes the euclidean norm and $\mathcal{S}_N$ is the set of all permutations of $\{1, \cdots, N\}$.

GRAVITATIONAL INTERPRETATION WHEN d=1

For $\mathbf{d = 1}$, we have $\mathbf{H = R^N}$ which can be interpreted as the configuration space of N points moving along the real line. We have

$$\boxed{\mathbf{S} = \{\ (\mathbf{a}^{\mathbf{N}}_{\sigma_1}, \cdots, \mathbf{a}^{\mathbf{N}}_{\sigma_{\mathbf{N}}}), \quad \sigma \in \mathcal{S}_{\mathbf{N}}\}}$$

where $\mathbf{a^N_j = j/N - 1/2}$.

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We find

$$\boxed{\epsilon \frac{d^2 X_i}{dt^2} = X_i - \frac{1}{2N} \sum_{j \neq i} \mathrm{sgn}(X_i - X_j)}$$

This describes the gravitational interaction of N parallel planes ("pancakes") with a repulsive background.

GOING BACK TO THE CONTINUOUS LIMIT

It is now tempting to go back to the continuous limit in space, while preserving the approximation parameter $\epsilon > 0$. We get a dynamical system in the Hilbert space $\mathbf{H} = \mathbf{L}^2(\mathbf{D}, \mathbf{R}^d)$

$$\boxed{\epsilon \frac{d^2 M}{dt^2} = M - \pi[M]}$$

$$\boxed{\pi[M] = \mathrm{Arginf}\{||M - s||^2, \quad s \in VPM(D)\}}$$

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(Strictly speaking, this is valid only in the non-degenerate case when this image is absolutely continuous with respect to the Lebesgue measure, otherwise, $\pi[M]$ may be multivalued.)

MONGE-AMPERE FORMULATION

According to standard optimal transport theory (Y.B. CPAM 1991)

$$\boxed{\pi[\mathbf{M}] = \mathbf{Arginf}\{||\mathbf{M} - \mathbf{s}||^2, \quad \mathbf{s} \in \mathbf{VPM(D)}\}}$$

can be written (*) in terms of Monge-Ampère equation

$$\boxed{\pi[\mathbf{M}] = (\mathbf{Id} + \epsilon\nabla\varphi) \circ \mathbf{M}, \quad \mathbf{det}(\mathbf{I} + \epsilon \mathbf{D}^2\varphi) = \int \delta(\cdot - \mathbf{M}(\mathbf{a}))\mathbf{da}}$$

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Eventually, our dynamical system reads

$$\boxed{\frac{\mathbf{d^2M}}{\mathbf{dt^2}} = -\nabla\varphi \circ \mathbf{M}, \quad \int \delta(\cdot - \mathbf{M}(\mathbf{a}))\mathbf{da} = \mathbf{det}(\mathbf{I} + \epsilon\mathbf{D}^2\varphi) \sim \mathbf{1} + \epsilon \triangle \varphi}$$

THE EARLY UNIVERSE GRAVITATIONAL MODEL

Particle trajectories $(t, a) \to X(t, a)$ are ruled by:

$$\boxed{\frac{2t}{3}\frac{d^2X}{dt^2} + \frac{dX}{dt} + \nabla\varphi(t, X(t)) = 0}$$

$$\boxed{\rho(t, x) = \int \delta(x - X(t, a))da = 1 + t \triangle \varphi(t, x)}$$

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where a denotes the initial position in R^3 and φ the gravitational potential.
This is a semi-Newtonian model.
All terms in red come from general relativity (Einstein-de Sitter "big bang" universe) .
Notice that at early times $t \downarrow 0$, "friction" takes over "inertia" (Einstein+Newton go back to Aristoteles...)

INITIAL CONSTRAINTS

Observe the degeneracy of the model at time $t = 0$

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Clusters of particles are not possible at $t = 0$ and come up later due to the concentration mechanism known as Jeans' instability.

RECONSTRUCTION OF THE EARLY UNIVERSE

Notice that the ONLY free initial values at $t = 0$ are the initial density fluctuations

$$\boxed{\rho_0'(\mathbf{x}) = \lim_{t \downarrow 0} \frac{\rho(t, \mathbf{x}) - 1}{t} = \triangle \varphi_0(\mathbf{x})}$$

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This makes plausible the EUR problem, which amounts to, following Peebles 1989, Frisch and coauthors (Nature 417) 2002, reconstructing the history of the Universe from the only observation of the HIGHLY CONCENTRATED (with essentially no Lebesgue component) density field $\boxed{\rho(T, \mathbf{x})}$ at present time T.

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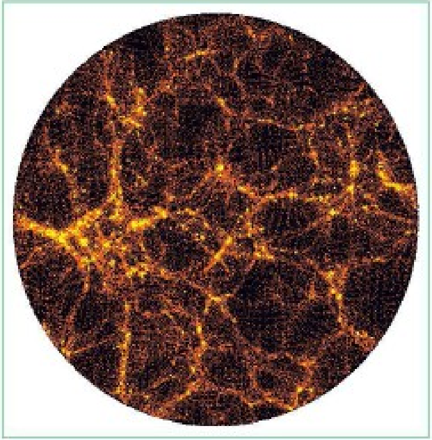
Indeed, the only initial condition to recover is just a scalar field, namely the density fluctuation field $\boxed{\rho'_0(\mathbf{x})}$ which is supposed to be a random field of very small amplitude related to the quantum theory of the VERY early universe.

The present, highly concentrated, universe



ZELDOVICH APPROXIMATION

A very simple approximate solution of the model

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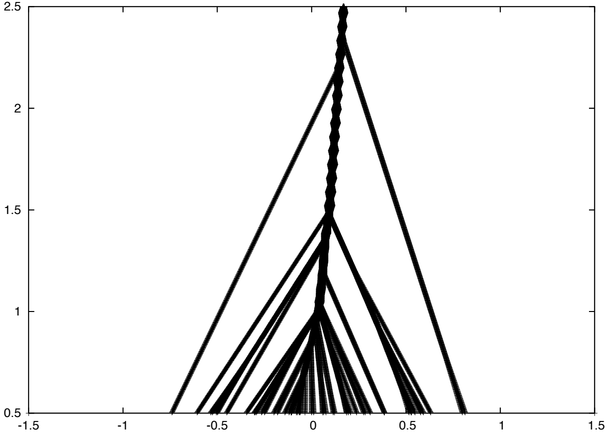
makes the reconstruction possible as a standard Monge problem with quadratic cost. (U. Frisch and coll. Nature 2002).
However, collisions cannot be taken into account this way.

Example of "Zeldovich" solutions with sticky collisions

horizontal : space /vertical : time



ZELDOVICH ANTICIPATED BY LUCRECIUS $\sim$ 99 – 55 BC

DE RERUM NATURA LIBER SECUNDUS 216 – 224

When atoms move straight down through the void by their own weight, they deflect a bit in space at a quite uncertain time and in uncertain places, just enough that you could say that their motion has changed. But if they were not in the habit of swerving, they would all fall straight down through the depths of the void, like drops of rain, and no collision would occur, nor would any blow be produced among the atoms. In that case, nature would never have produced anything.

FROM LUCRECIUS $\sim 99 - 55$ BC

DE RERUM NATURA LIBER SECUNDUS 216 – 224

Illud in his quoque te rebus cognoscere avemus, corpora cum deorsum rectum per inane feruntur ponderibus propriis, incerto tempore ferme incertisque locis spatio depellere paulum, tantum quod momen mutatum dicere possis. quod nisi declinare solerent, omnia deorsum imbris uti guttae caderent per inane profundum nec foret offensus natus nec plaga creata principiis; ita nihil umquam natura creasset.

AN ERSATZ: MONGE-AMPERE GRAVITATION

Following Y.B. Conflu. Math. 2011, let us substitute the Monge-Ampère equation $\boxed{\rho(\mathbf{t},\mathbf{x}) = \det(\mathbf{I} + \mathbf{t}\mathbf{D}^2\varphi(\mathbf{t},\mathbf{x}))}$ **for the Newtonian Poisson equation** $\boxed{\rho(\mathbf{t},\mathbf{x}) = 1 + \mathbf{t} \triangle \varphi(\mathbf{t},\mathbf{x})}$

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This approximation is exact for parallel pancakes, asymptotically correct both at early times and for weak fields, makes Zeldovich approximation exact.
This leads to the MONGE-AMPERE GRAVITATIONAL MODEL

$$\boxed{\frac{2t}{3}\frac{d^2X}{dt^2} + \frac{dX}{dt} + \nabla\varphi(t, X(t)) = 0}$$

$$\boxed{\rho(t,x) = \int \delta(x - X(t,a))da = \det(1 + tD^2\varphi(t,x))}$$

A LEAST ACTION PRINCIPLE FOR MAG

The configuration space is the Hilbert space H of all L^2 maps. Using again optimal transport calculus, we get the following action for the MAG model

$$\boxed{\int_{t_0}^{t_1} t^{-1/2} \left|\left| t\frac{dX}{dt} - \nabla Q[X(t)]\right|\right|^2 dt, \quad Q[X] = \inf_{s\in S} \frac{||X-s||^2}{2}}$$

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where $S \subset H$ is the subset of all volume-preserving maps.

Observe that this action is just zero for Zeldovich solutions.

A TIME-DISCRETE ACTION

We end up with the following time-discrete action

$$\boxed{\sum_n t_{n+1}^{-1/2} || t_{n+1} \frac{X_{n+1} - X_n}{t_{n+1} - t_n} - (X_n - \pi[X_n]) ||^2}$$

which is easy to handle in one space dimension, using fast sorting algorithms.
In that case, remarkably enough, the discrete action takes into account sticky collisions, as shown by the following numerical simulations.

NUMERICS

In the next slides,
we show simulations of parallel pancake gravitational interactions with sticky collisions directly based on the minimization of the fully space and time discrete version of the action.

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The calculation entirely relies on many ($\sim 10^5$) iterations of an elementary sorting algorithm.**

NUMERICS

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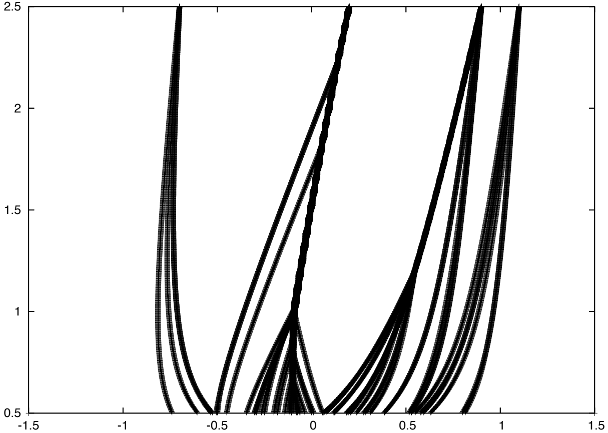
we show simulations of parallel pancake gravitational interactions with sticky collisions directly based on the minimization of the fully space and time discrete version of the action.
The calculation entirely relies on many ($\sim 10^5$) iterations of an elementary sorting algorithm.
Unfortunately, calculations would be considerably more expensive in 3D than the ones performed by Frisch and coll. (Nature 2002).

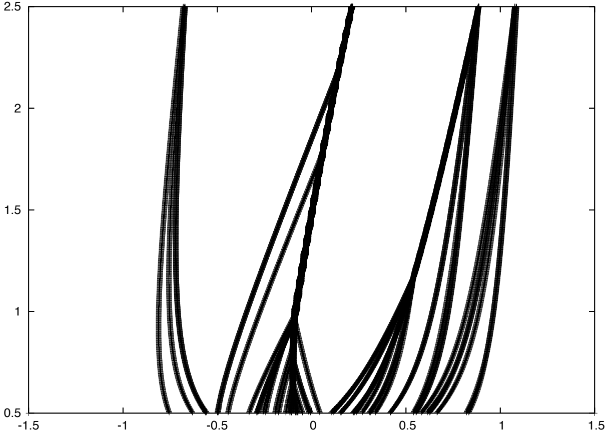
reconstructed trajectories

horizontal : 51 grid points in x /vertical : 60 grid points in t



IVP with reconstructed velocities

horizontal : 51 grid points in x /vertical : 60 grid points in t



SOME REFERENCES

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The EUR problem and the pressure-less Euler-Poisson model

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