Inria International program
MOKALIEN Associate Team Report 2014 (1rst year)

Title: Numerical Optimal Transportation

Associate Team acronym: MOKALIEN

Principal investigator (Inria): Benamou Jean-David (Mokaplan - Rocquencourt)

Principal investigator (McGill): Oberman Adam (McGill U. - Montreal)

1 Abstract of the scientific program

The overall scientific goals is to develop numerical methods for large scale optimal transport and models based on optimal transport tools
see `https://team.inria.fr/mokaplan/files/2014/09/MOKALIEN_Proposal_2013.pdf`, section 2.

A few additional applications were suggested at our annual workshop in october
`https://team.inria.fr/mokaplan/first-meeting-in-montreal-at-u-mcgill-october-20-24-2014/`

2 Scientific progress

We list here progress made according to the proposed work program for the first year
see `https://team.inria.fr/mokaplan/files/2014/09/MOKALIEN_Proposal_2013.pdf`, section 2.3.

A1 A solver for partial transport has been implemented using Dykstra algorithm applied to an entropy regularized Kantorovich formulation, see [5].

A2 An Aleksandrov Monge-Ampère solver based on a mixed Aleksandrov/Viscosity formulation has been designed and implemented, see [6].

C1-C2 has been done but it turned out that other methods (IPFP and non-smooth convex minimization on the dual as done by Carlier, Oberman and Oudet) developed by the team were more efficient and then enabled us to treat larger cases, see [5] and [4].

C3 is still a problem of interest of the team but the analysis is harder than expected due to the difficulty of quantifying the convexity (in the usual sense) of a sum of squared Wasserstein distance.

D1 The splitting scheme for the kinetic granular media equation has been implemented by Benamou in 1d with a quadratic interaction kernel. On the other hand, a large class of Lyapunov functional has been discovered that enables to describe the large time concentrations in the phase space of solutions. The analysis of kinetic granular media equation, and the global existence of weak solutions is a challenging problem and Agueh and Carlier will continue working on it. This part of the program involves two lines that will be pursued: a theoretical one and a numerical one. Other evolutions equations where splitting could be useful from a numerical simulations point of view will be studied in a more systematic way. A typical example where this is already done is in [7] where the authors proved existence of solutions to a fractional Fokker-Planck equation using a splitting scheme combined with a Wasserstein gradient flow. In [7], the authors designed a splitting algorithm (in any dimension) and proved that it converges to a solution of this parabolic equation. It will be interesting to implement this algorithm numerically.

D2 A genuine JKO solver (implemented the exact gradient flow formulation) has been designed based on a discretization in the displacement space that preserves convexity. Simulation of crowd motion under congestion and flows in porous medium in 2D validate the approach. See [3].

E1 has been done in [1] where a new optimal method to deal with convexity constraint has been proposed. It was adapted subsequently to solve the Elliptic Monge-Ampère equation [2].

3 Additional work program items for next year

We try here to keep the classification introduced in `https://team.inria.fr/mokaplan/files/2014/09/MOKALIEN_Proposal_2013.pdf`, section 2.2.

A3 (Mérigot, Mirebeau) A geometric approach to the semi-discrete formulation of the Monge-Ampère equation. This method computes exactly the Aleksandrov solution using Laguerre Diagram. Recent works indicate that this approach can be combined with a Newton method and therefore become highly competitive.

A4 (Merigot, Mirebeau,Carlier, Nenna, Pass, Benamou, Brenier, Merigot, Mirebeau) We will investigate the applicability of geometrical methods for optimal transport to Euler fluid equations. Solutions of Euler equations can be regarded as geodesics on the space of volume preserving maps. Numerical schemes based on this interpretation rely on repeated projections onto the class of volume preserving maps, which amounts to an optimal transport problem, for which efficient numerical techniques have recently been developed. A one-dimensional experiment will be programmed. If conclusive, a two-dimensional implementation will follow.
Multi-marginal Optimal transport solver based on IPFP will be applied to the same problem.

B1 (Mérigot, Benamou, ...) We plan to use the method based on Laguerre cells for the cost function corresponding to the far-field reflector problem, that is $c = log(|x?y|)$ on the sphere. (cf ADT Mokabakour `https://team.inria.fr/mokaplan/files/2014/09/MOKABAJOUR_ADT_2014.pdf`).

B2 (Carlier, A. Blanchet, Benamou, ...) Understand and solve the non-local Monge Ampère equation corresponding to Cournot-Nash equilibria [**?**].

B3 (Carlier, Dupuis) An iterated projection strategy a la Dykstra is being studied, it already gave promising results in the Principal-agent problem of Rochet and Choné (convexity constraint) but what is probably more interesting is that it can be adapted to more general principal agent problems, namely some of those which satisfy the condition (B3) of McCann-Figalli-Kim.

D1 (Agueh, Carlier, Benamou) Global existence of weak solutions to the kinetic granular media is been investigated by Agueh and Carlier in one dimension for sufficiently regular interactions potentials. In the case where the interaction potentials are "less" regular, a blow-up in finite time of the solutions is expected, and therefore this kinetic model would be ill-posed. In this case, we plan to propose an alternative model which would best describe the physical system. We are also investigating the long-time behaviour of the solutions to the kinetic granular media in any dimension, and finally approximate these solutions numerically using a splitting scheme, as describeb in [8].

D3 (Benamou, Carlier, Merigot, Oudet, Blanchet) Using the JKO approach to solve the Keller-Segel problem. Two difficulties need to be tackled. First, dealing with a non-quadratic interaction energy makes each Newton iteration very expensive (i.e. quadratic in the number of points). Second, the interaction energy appearing in the Keller-Segel problem is neither convex nor displacement-convex, and nor will the be the discrete optimisation problem solved at each timestep. This poses difficult theoretical and practical issues

E2 (Carlier, Mérigot,Benamou, Santambrogio) New variational models of crowd motion under congestion will be solved numerically using the ALG2 algorithm. density constraints (as in the models of Maury), velocity constraints, or constraints mixing the two as in usual traffic flow modeling (the maximal feasible velocity is constrained by the density)

E3 (Merigot, Mirebeau) We will investigate a new discretization of constraint of convexity, replacing the linear constraints used in present approaches by the non-linear constraint that the Lebesgue measure of the sub-gradient of the solution is positive at each discretization point. The Brunn-Minkowski inequality gives hope that this constraint is well behaved and compatible with interior point methods.

F corresponds to other topics.

F1 (Oberman, Mirebeau) We will apply some monotone numerical schemes with improved consistency, recently developed by Mirebeau, to Stochastic control, Hamilton-Jacobi equations, and some Homogenization problems in these areas.

F2 (Oberman, Froese, Tiago Salvador) Completing discretization of the 2-hessian equation in 3 dimensions, related to Monge-Ampère.

F3 (Oudet, Santambrogio) We will also investigate the numerical applications of some recent approximation results, in terms of $\Gamma-$convergence of elliptic functionals to singular energies, to the solution of some network problems, linked to optimal transport questions, such as the Steiner minimal connection problem, the branched transport problem, or the average distance problem under length constraints or penalization. This is based on same works by Oudet, Santambrogio, Lemenant and Bonnivard. New approximations, in the same spirit, will also be considered.

F4 (Yanglong, Finlay, Abbasi, Oberman) Applications of Wasserstein distance to parameter identification.

4 Record of activities

See `https://team.inria.fr/mokaplan/first-meeting-in-montreal-at-u-mcgill-october-20-24-2014/` for our first workshop in Montreal.

- Oberman visited INRIA in June.
- Carlier and Mérigot visited McGill in October.
- Carlier is spending 6 month in U. Victoria with M. Agueh.

Planned activites

The P.I. are coorganizing a workshop at BIRS (Banff, `https://www.birs.ca/events/2015/5-day-workshops/15w5067`. Most of Mokalien members will attend.

Carlier and Benamou will visit McGill in Ferbruary 2015.

A Mokalien Workshop will be organised at INRIA-Paris Rocquencourt most likely in October 2015.

5 Production

References

[1] Jean-Marie Mirebeau. Anisotropic and Hierarchical cones of Discrete Convex functions. https://hal.inria.fr/hal-00943096

[2] Jean-David Benamou, Francis Collino, Jean-Marie Mirebeau. Monotone and Consistent discretization of the Monge-Ampere operator https://hal.inria.fr/hal-01067540

[3] Jean-David Benamou, Guillaume Carlier, Quentin Mérigot, Edouard Oudet. Numerical Discretization of functionals involving the Monge-Ampre operator. Discretization of functionals involving the Monge-Ampre operator

[4] Guillaume Carlier, Adam Oberman, Edouard Oudet. Numerical methods for matching for teams and Wasserstein barycenters. 2 https://hal.inria.fr/hal-00987292

[5] Jean-David Benamou, Guillaume Carlier, Marco Cuturi, Luca Nenna, Gabriel Peyré Iterative Bregman Projections for Regularized Transportation Problems *in preparation*

[6] Jean-David Benamou, Brittany D. Froese A viscosity framework for computing Pogorelov solutions of the Monge-Ampere equation https://hal.inria.fr/hal-01053454

[7] Martial Agueh, Local existence of weak solutions to kinetic models of granular media. http://www.math.uvic.ca/ agueh/Agueh_Granular_May2014.pdf

[8] Martial Agueh, Malcolm Bowles Weak Solutions to a Fractional Fokker-Planck Equation via Splitting and Wasserstein Gradient Flow http://www.math.uvic.ca/ agueh/Agueh_Bowles_AML_2014.pdf

6 Non- Public Information

Juste une remarque : La notification tardive de la selection de l'équipe ainsi le retard dans l'abondement correspondant du budget de l'quipe (octobre pour une date limite de dépense en décembre !) ne facilite pas l'organisation du travail.

7 Changes on the team

The Mokaplan "Action Exploratoire" is now a joint INRIA-Paris Dauphine-CNRS team. with two additional full time researchers

- Gabriel Peyré (DR CNRS).
- François-Xavier Vialard (MCF).

and two new external collaborators

- Filippo Santambrogio, Prof U. Paris-Sud.
- Jean-Marie Mirebeau, CR CNRS Ceremade, U. Paris Dauphine.

On the Canadian side, contact has been made with two additional researchers

- Jean-Christophe Nave, Prof. McGill U. - Montreal.
- Rustum Choski, Prof. McGill U. - Montreal.

They work on numerical methods and calculus of variation problems. As such their research is potentially open for collaboration with our group on numerical Optimal Transportation. They agreed to participate to the next Mokalien Workshop in France.

8 Budget requested for the coming year

As requested in the initial proposal : 20K Euro to organise a one week workshop gathering the Canadians and French researchers (two editions already). In 2015, this WS will take place in France, most likely at INRIA.