Monotone and Consistent discretization of the Monge-Ampere operator

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Joint work with Jean-David Benamou and Francis Collino.

Motivations

The Wide Stencil scheme

The MA-LBR scheme (Monge-Ampère with Lattice Basis Reduction)

Adaptivity, and the Stern-Brocot tree

Numerical results

Conclusion

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Let Ω be a bounded convex domain, let $\rho \in C^0(\overline{\Omega}, \mathbb{R}_+)$, and let $\sigma \in C^0(\partial\Omega, \mathbb{R}_+)$ be convex on any segment of $\partial\Omega$. Find u s.t.

$$\begin{cases} \det(\nabla^2 u) = \rho & \text{on } \Omega \\ u = \sigma & \text{on } \partial\Omega \\ u & \text{convex} \end{cases}$$

Motivation: isolate the difficulty associated to the Monge-Ampere operator $\det(\nabla^2 u)$ appearing in OT.

Optimal transport

Let Ω' be another convex domain, and let $\rho' \in C^0(\overline{\Omega'}, \mathbb{R}_+)$ with $\int_\Omega \rho = \int_{\Omega'} \rho'$. OT $\rho \to \rho'$ is the gradient ∇u of a convex potential

$$\begin{cases} \rho'(\nabla u) \det(\nabla^2 u) = \rho & \text{on } \Omega \\ [\nabla u](\Omega) = \Omega' & \\ u & \text{convex} \end{cases}$$

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Solution Regularity and Scheme Robustness

Each discretization of $\det(\nabla^2 u)$ take clues from a regularity theory for the Monge-Ampere PDE.

- Smooth solutions $\rightsquigarrow$ Finite Differences schemes.

  Pros: Simple implementation. Accurate when they work.

  Cons: Solver needs a good guess. Only capture smooth solutions.

- Viscosity solutions $\rightsquigarrow$ Degenerate Elliptic schemes.

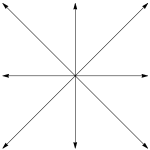
  Pros: Convergence guarantees for some discrete iterative solvers.

  Cons: Only capture viscosity solutions.

- Alexandroff solutions $\rightsquigarrow$ Power-Diagram based schemes.

  Pros: Capture the most general solutions, e.g. $u = |\cdot|$, $\rho = \pi\delta_0$.

  Cons: Complex implementation, involving a global adaptive mesh.



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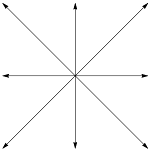

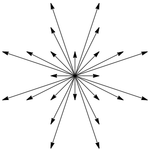

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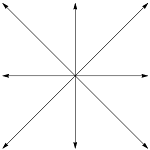

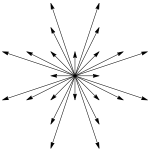

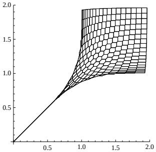

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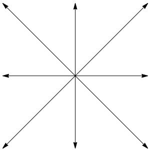
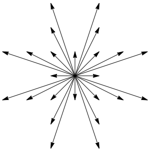
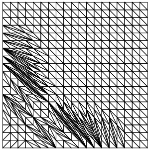

Grid based degenerate elliptic schemes

Ω: open bounded convex domain $\subset \mathbb{R}^2$.

Grid discretization. $X := \Omega \cap hR(\xi + \mathbb{Z}^2)$.

$\mathbb{U}$ denotes the collection of maps $u : X \cup \partial\Omega \to \mathbb{R}$.

Definition (Second order finite differences $\approx \langle e, \nabla^2 u(x) e\rangle$)

Let $u \in \mathbb{U}$, $x \in X$, $e \in \mathbb{Z}^2$.

- If $x \pm e \in X$ then $\Delta_e u(x) := u(x+e) - 2u(x) + u(x-e)$.
- Otherwise $\Delta_e u(x)$ involves boundary values of u, on $\partial\Omega$.

Definition (Degenerate ellipticity)

An operator $\mathcal{D} : U \to \mathbb{R}^X$ is Degenerate Elliptic with stencil $V \subset \mathbb{Z}^2$ if, for each $x \in X$, $\mathcal{D}u(x)$ is a non-decreasing locally Lipschitz function of $\Delta_e u(x)$, $e \in V$.

$$\text{Find } u : X \cup \partial\Omega \to \mathbb{R} \text{ s.t. } \begin{cases} \mathcal{D}u = \rho & \text{on } X \\ u = \sigma & \text{on } \partial\Omega. \end{cases}$$

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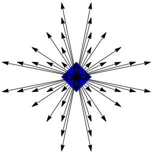

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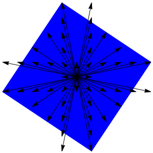

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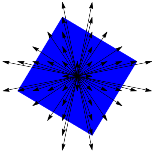

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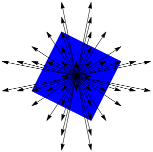

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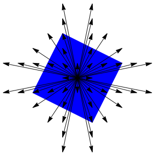

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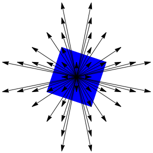

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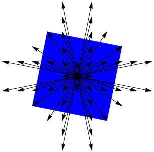

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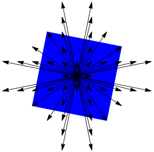

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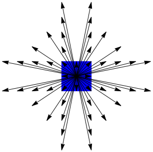

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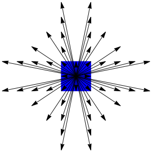

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$$u_M(x) := \frac{1}{2}\langle x, Mx\rangle, \qquad \Delta_e u_M(x) = \langle e, Me\rangle.$$

Proposition (Approximate Consistency)

For any $M \in S_2^+$, by Hadamard's inequality

$$\mathcal{D}_V u_M(x) = \min_{\substack{\{f,g\}\subset V\\ \text{orthogonal}}} \frac{\langle f, Mf\rangle}{\langle f, f\rangle}\frac{\langle g, Mg\rangle}{\langle g, g\rangle} \geq \det(M).$$

Equality holds iff $\{f, g\}$ is M-orthogonal (i.e. $\langle f, Mg\rangle = 0$).

Proof.

Denote $\mathbf{f} := f/\|f\|$, $\mathbf{g} := g/\|g\|$, which form an orthonormal basis of $\mathbb{R}^2$. Then, recognizing a Gram matrix

$$\det(M) = \begin{vmatrix} \langle \mathbf{f}, M\mathbf{f}\rangle & \langle \mathbf{f}, M\mathbf{g}\rangle \\ \langle \mathbf{f}, M\mathbf{g}\rangle & \langle \mathbf{g}, M\mathbf{g}\rangle \end{vmatrix} = \langle \mathbf{f}, M\mathbf{f}\rangle\langle \mathbf{g}, M\mathbf{g}\rangle - \langle \mathbf{f}, M\mathbf{g}\rangle^2 \quad \square$$

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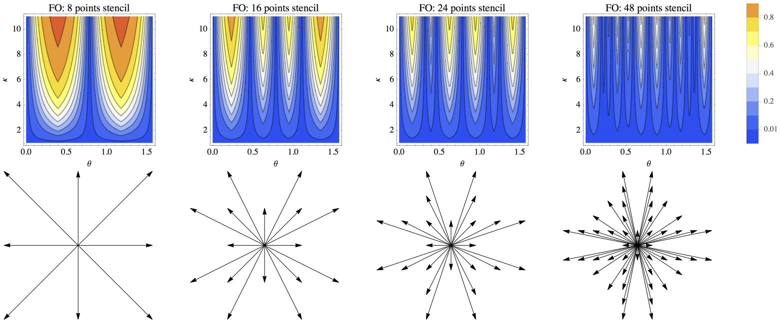

Figure : Relative consistency error $(\mathcal{D}_V(u_M) - \det(M))/\mathcal{D}_V(u_M)$, with several stencils V. Matrix $M \in S_2^+$ has condition number $\kappa^2 := \|M\|\|M^{-1}\|$ and eigenvector $(\cos\theta, \sin\theta)$.

Motivations

The Wide Stencil scheme

The MA-LBR scheme (Monge-Ampère with Lattice Basis Reduction)

Adaptivity, and the Stern-Brocot tree

Numerical results

Conclusion

Monge-Ampere with Lattice Basis Reduction

Lattice Basis Reduction is the study of preferred coordinate systems for lattices (discrete subgroups of $\mathbb{R}^d$).

Definition (Superbase of $\mathbb{Z}^2$)

A superbase is a triplet $(e, f, g) \in (\mathbb{Z}^2)^3$ such that $e + f + g = 0$ and $|\det(f, g)| = 1$. It is said M-obtuse, where $M \in S_2^+$, iff $\langle e, Mf \rangle \leq 0$, $\langle f, Mg \rangle \leq 0$, $\langle g, Me \rangle \leq 0$.

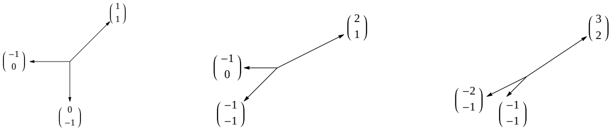

Figure : Left: An M-obtuse superbase, and the unit ball $\{\langle e, Me \rangle \leq 1\}$. Right: Likewise under change of coordinates $M^{\frac{1}{2}}$.

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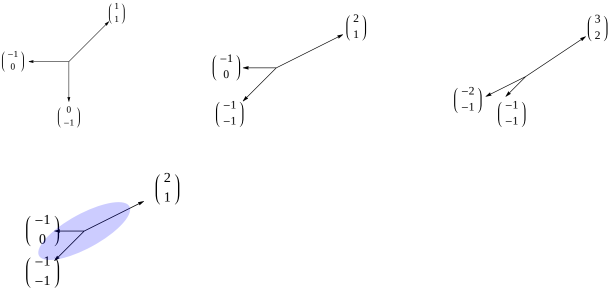

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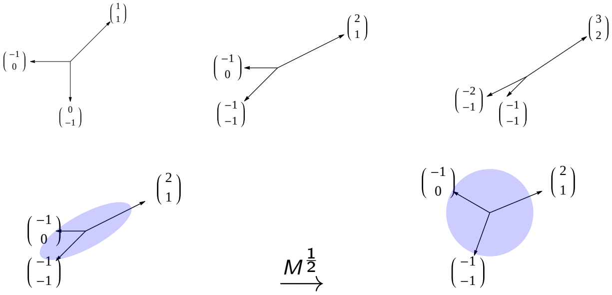

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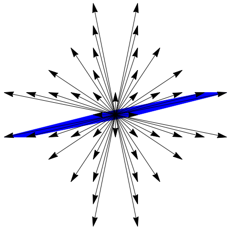
$$\mathcal{D}_V u(x) := \min_{\substack{\{e,f,g\}\subset V \\ \textit{superbase}}} h(\Delta_e^+ u(x), \Delta_f^+ u(x), \Delta_g^+ u(x)).$$

where $h : \mathbb{R}_+^3 \to \mathbb{R}_+$ is

$$h(a,b,c) = \begin{cases} bc \text{ if } a \geq b+c \text{ (and likewise permuting a,b,c)}, \\ \frac{1}{4}(2ab+2bc+2ca-a^2-b^2-c^2) \text{ otherwise.} \end{cases}$$



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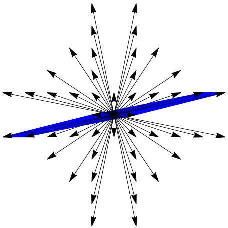

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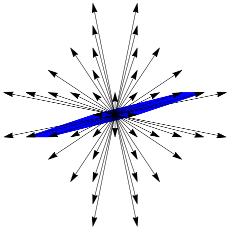

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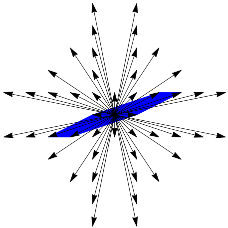

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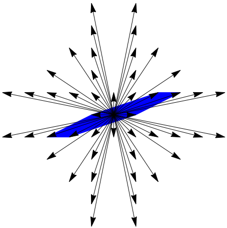

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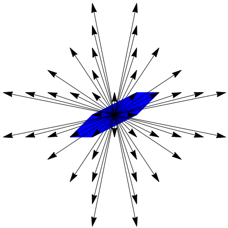

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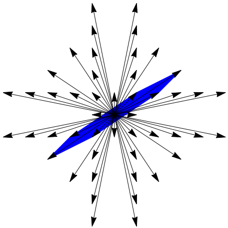

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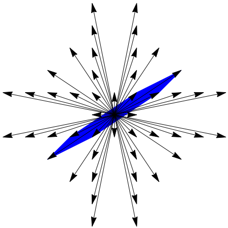

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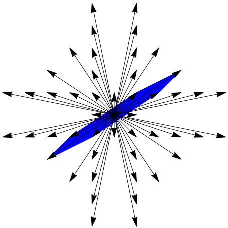

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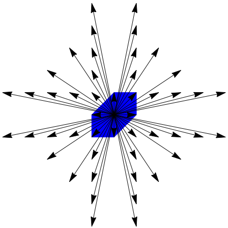


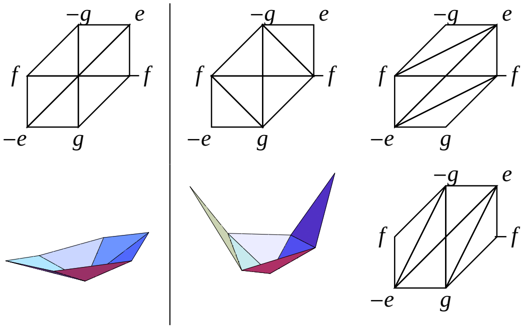
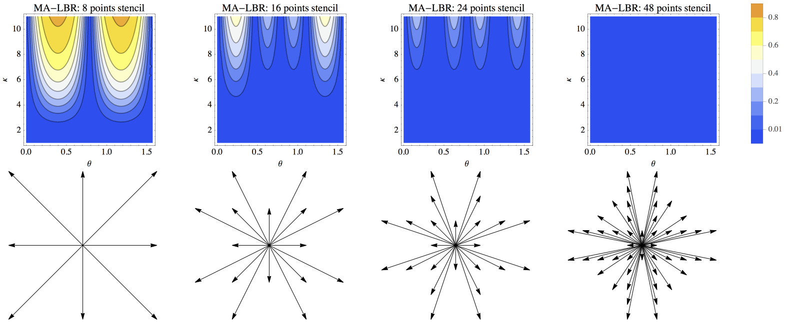


Figure : $h(a, b, c)$ can be interpreted as a subgradient measure.

Proposition (Consistency)

For any $M \in S_2^+$, $x \in X$, $\mathcal{D}_V u_M(x) \geq \det(M)$, *with equality iff* V *contains an* M *obtuse superbase.*

Relative Consistency error



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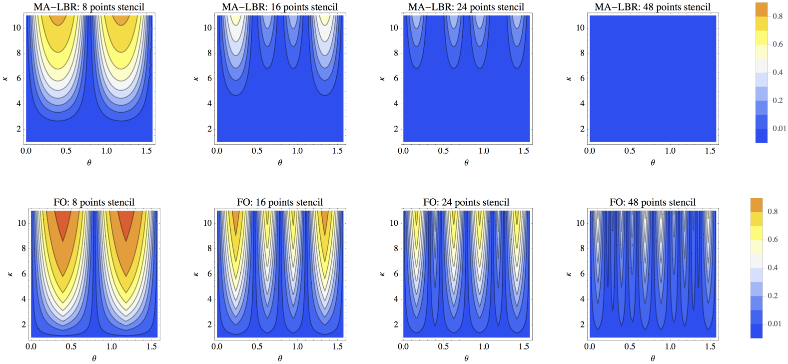

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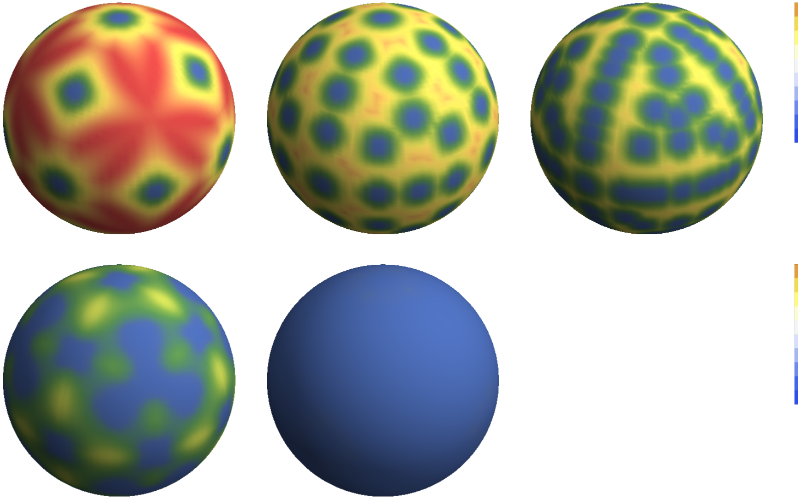

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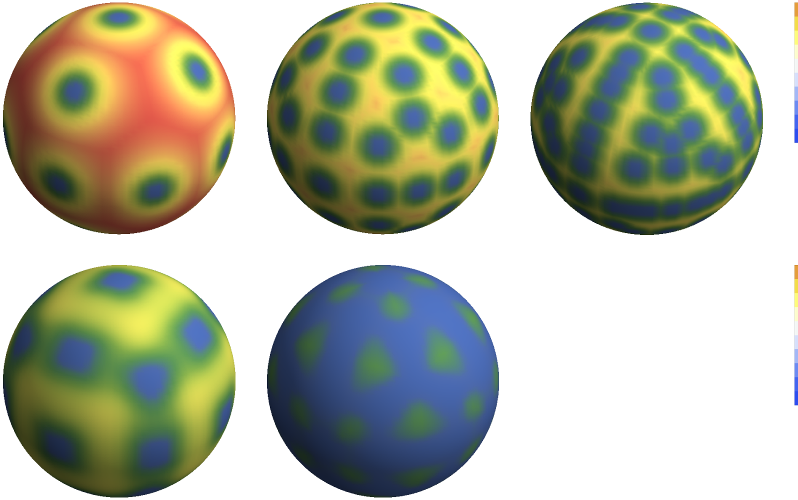

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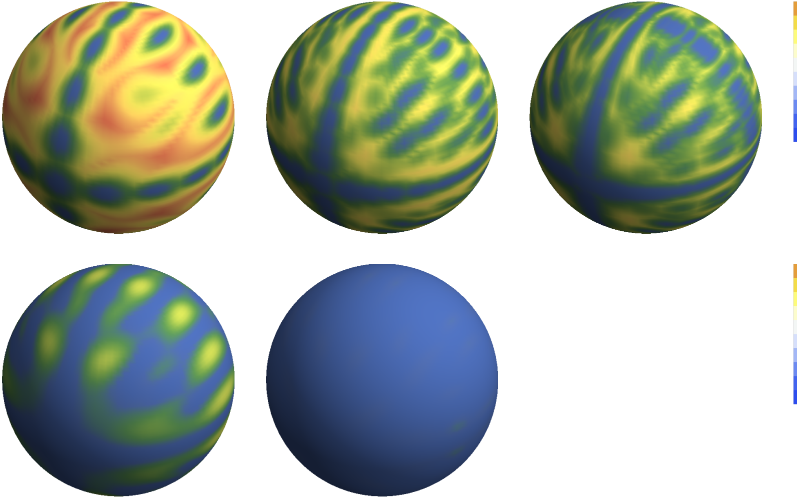

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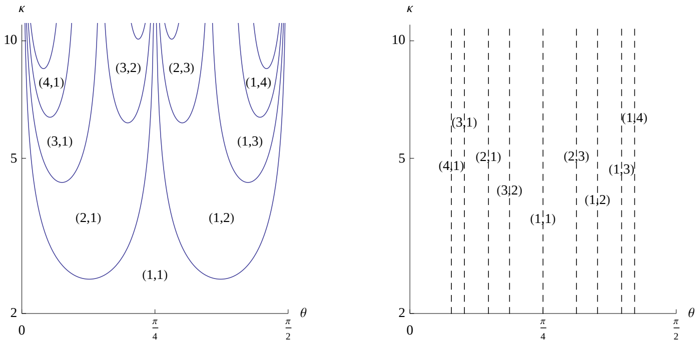
Consistency region associated to a stencil element



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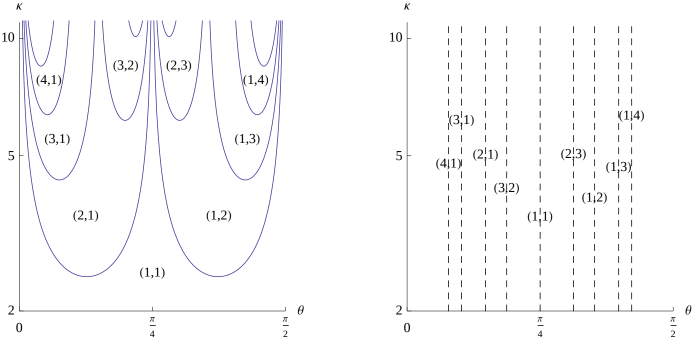

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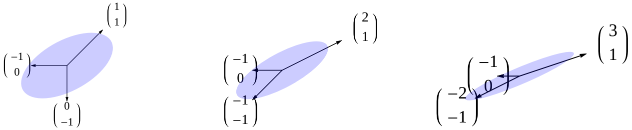

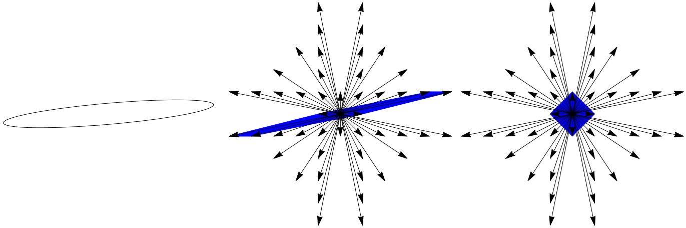

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Let $\mathcal{D} : \mathbb{U} \to \mathbb{R}^X$ be a Degenerate Elliptic operator, with finite symmetric stencil V. Let $M \in S_2^+$, $x \in X$.

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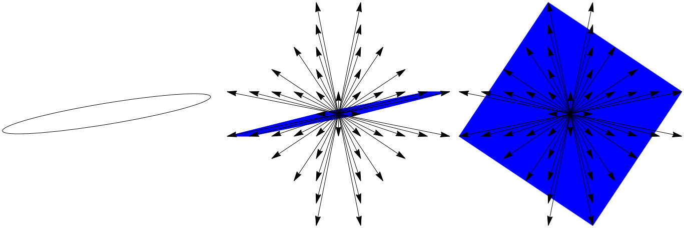

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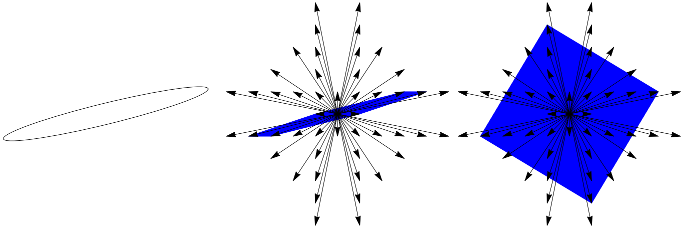

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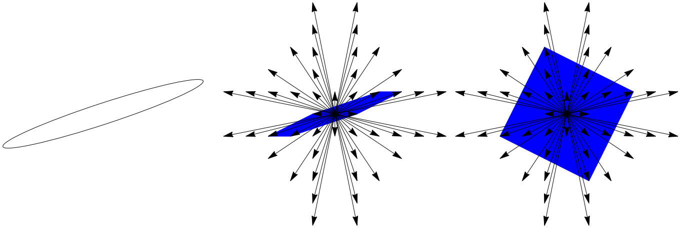

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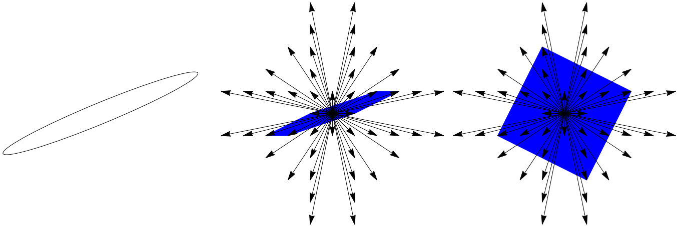

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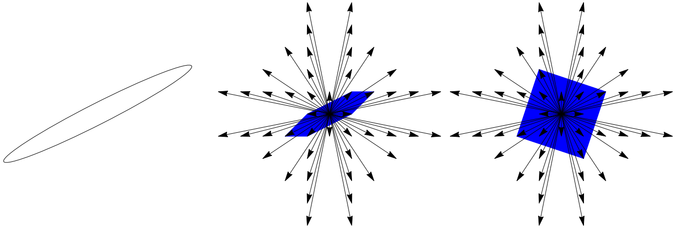

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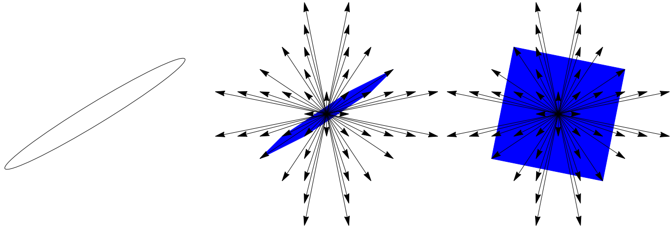

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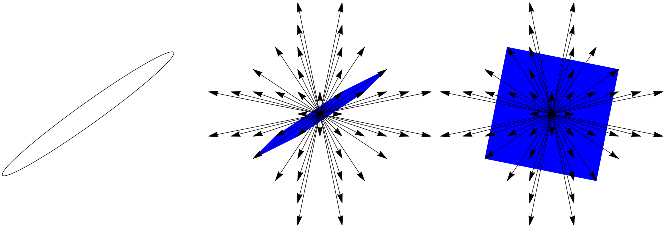

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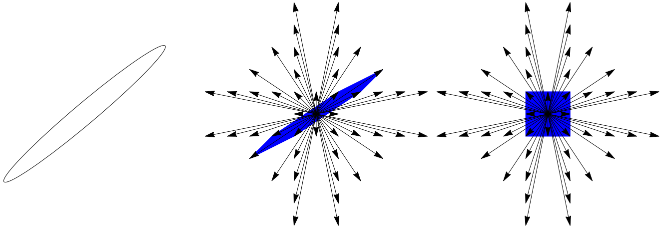

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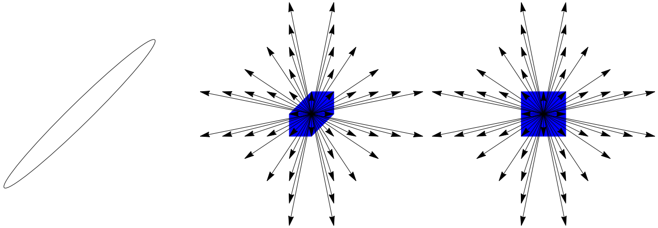

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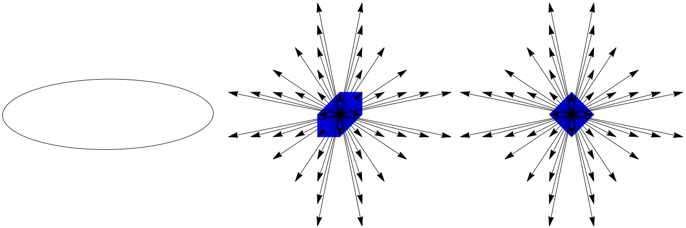

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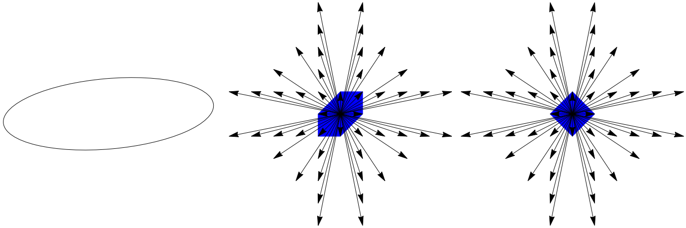

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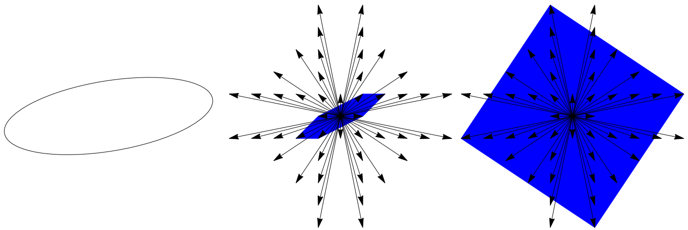

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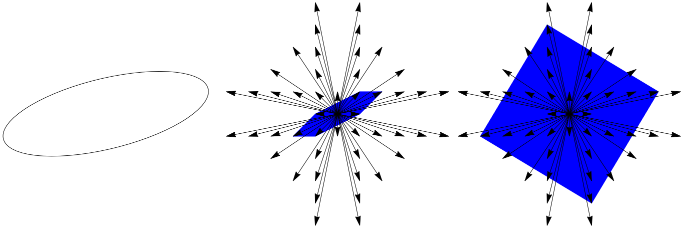

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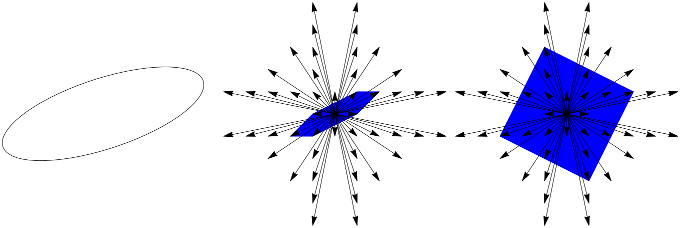

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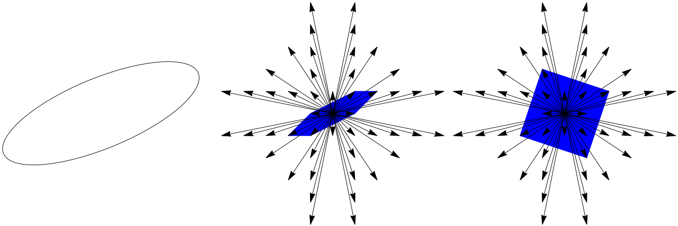

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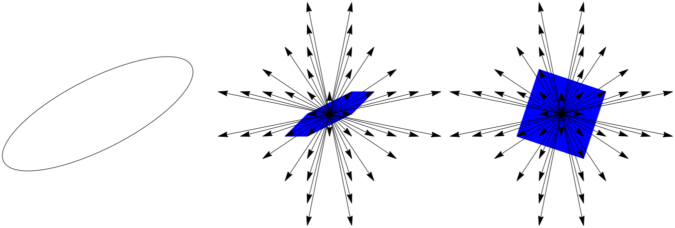

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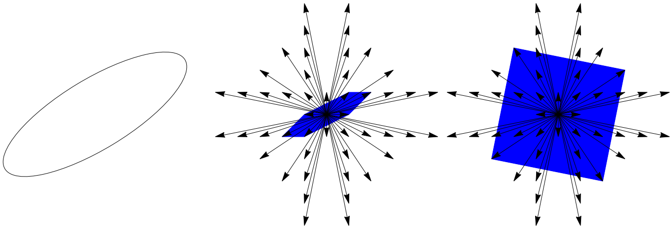

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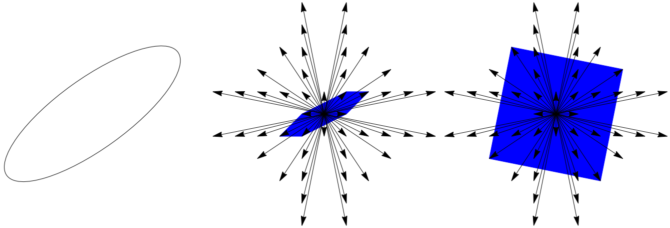

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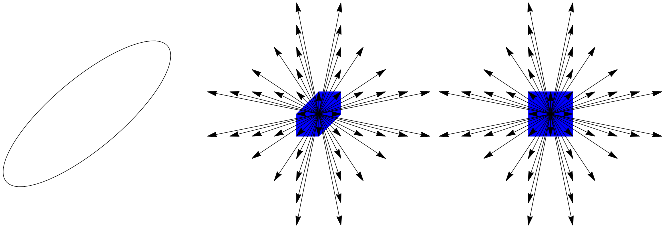

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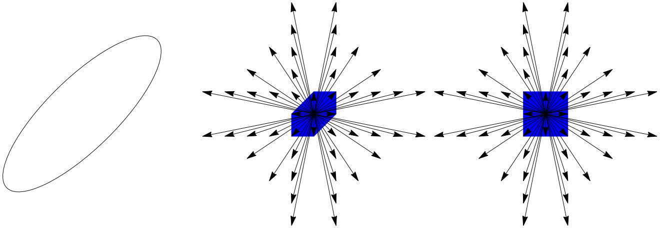

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Existence of an M-obtuse superbase

Selling's Algorithm (1857).

Set $e_0 \leftarrow (-1,-1), \ \ e_1 \leftarrow (1,0), \ \ e_2 \leftarrow (0,1)$.

While the superbase (e_0, e_1, e_2) is not M-obtuse **do**

  Find $0 \leq i < j \leq 2$ such that $\langle e_i, Me_j \rangle > 0$,

  Set $(e_0, e_1, e_2) \leftarrow (e_i - e_j, \ e_j, \ -e_i)$.

Proposition

Selling's algorithm terminates, and the final state of (e_0, e_1, e_2) is an M-obtuse superbase.

Proof.

Introduce the energy: with $\|e\|_M := \sqrt{\langle e, Me \rangle}$

$$\mathcal{E}(e_0, e_1, e_2) := \|e_0\|_M^2 + \|e_1\|_M^2 + \|e_2\|_M^2.$$

Only finitely many superbases have their energy below a given bound. Then observe that

$$\mathcal{E}(e_i - e_j, e_j, -e_i) = \mathcal{E}(e_0, e_1, e_2) - 4\langle e_i, Me_j \rangle.$$

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Conclusion on the non-adaptive MA-LBR operator

Pros:

- More accurate than the Wide Stencil scheme, although not (much) more costly or difficult to implement.
- Consistency for all quadratic functions u_M, with condition number $\|M\|\|M^{-1}\|$ bounded by some κ_0, is achieved with a finite stencil.

Cons:

- How to a-priori choose the stencil size ?

In the following, we introduce an automatic, guaranteed and parameter free stencil construction, by reinterpreting and extending Selling's Algorithm to non-quadratic maps.

Motivations

The Wide Stencil scheme

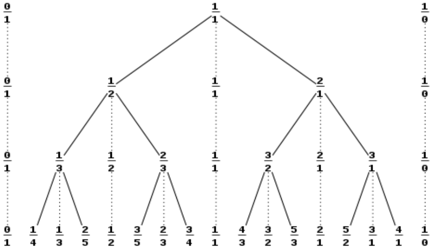
The MA-LBR scheme (Monge-Ampère with Lattice Basis Reduction)

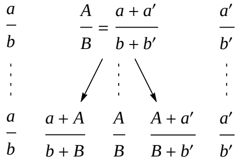
Adaptivity, and the Stern-Brocot tree

Numerical results

Conclusion

The Stern Brocot tree

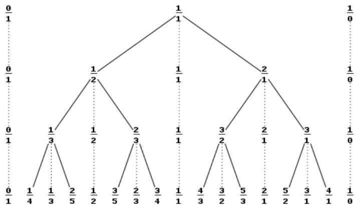




Obtain the $(n+1)$-th line by inserting $\frac{a+a'}{b+b'}$ between consecutive elements $\frac{a}{b}$ and $\frac{a'}{b'}$ of the n-th line.

- Each rational number appears exactly once, in its irreducible form.

Off topic: Fun facts on the Stern-Brocot tree



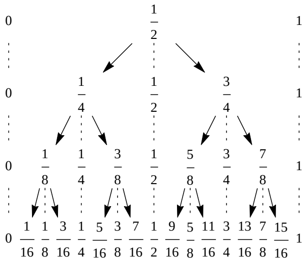


Figure : Dyadic rationals can be organized in a similar (complete infinite binary) tree. Some images from Wikipedia.

Minkowski's question mark function, $? : [0,1] \to [0,1]$

$?(x)$ is the continuous function mapping the Stern-Brocot labels in $[0,1]$ to the dyadic labels. Properties:

- $?'(x) = 0$ for almost every x. ("Slippery Devil's staircase")
- $?$ is Holder continuous, with exponent $\frac{\ln 2}{2 \ln \Phi}$, $\Phi := \frac{1+\sqrt{5}}{2}$.
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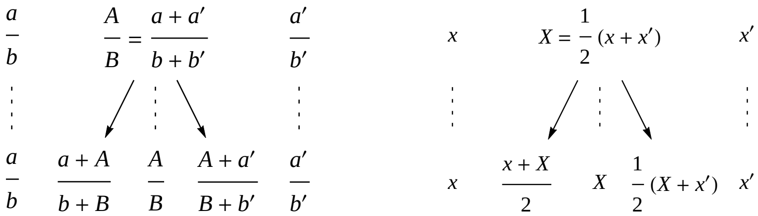

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Off topic: Fun facts on the Stern-Brocot tree

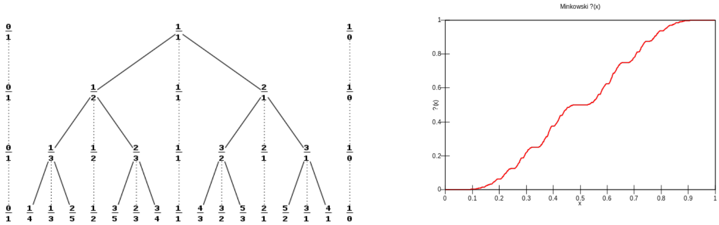


Figure : Dyadic rationals can be organized in a similar (complete infinite binary) tree. Some images from Wikipedia.

Minkowski's question mark function, $? : [0,1] \to [0,1]$

$?(x)$ is the continuous function mapping the Stern-Brocot labels in $[0,1]$ to the dyadic labels. Properties:

- $?'(x) = 0$ for almost every x. ("Slippery Devil's staircase")
- ? is Holder continuous, with exponent $\frac{\ln 2}{2 \ln \Phi}$, $\Phi := \frac{1+\sqrt{5}}{2}$.
- $?(x)$ is rational for every quadratic irrational.

Definition

For each node $\frac{a+a'}{b+b'}$ of the Stern-Brocot tree introduce

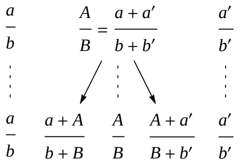
$$f = (a, b), \quad g = (a', b'), \quad e = (a + a', b + b') = f + g.$$

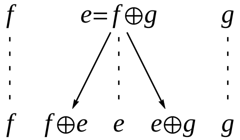
Proposition

If $e = f \oplus g$ then $(e, -f, -g)$ is a superbase of $\mathbb{Z}^2$. All superbases are of that form, up to a permutation of their elements, and a symmetry w.r.t. the origin or an axis.

Proposition

Selling's algorithm explores a single branch of the Stern-Brocot tree, characterized by $\langle f, Mg\rangle < 0$.





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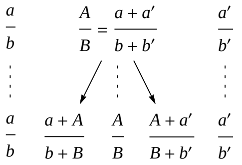

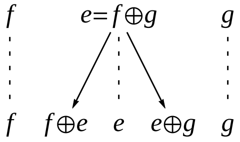

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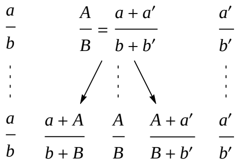

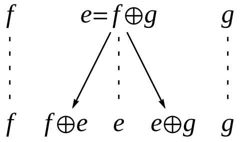

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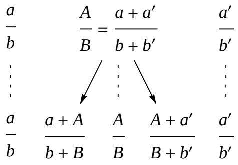

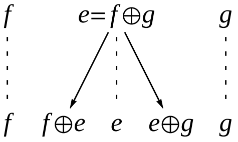

From quadratic to discrete convex functions

Definition (Hexagonal test)

Let $e = f \oplus g$, $u \in \mathbb{U}$, $x \in X$

$$H_e u(x) := \Delta_e u(x) - \Delta_f u(x) - \Delta_g u(x).$$

For a quadratic function:

$$H_e u_M(x) = \langle (f+g), M(f+g)\rangle - \langle f, Mf\rangle - \langle g, Mg\rangle = 2\langle f, Mg\rangle.$$

Test predicate is increasing along tree branches

The children of $e = f \oplus g$, are $f \oplus e$ and $e \oplus g$.

$$\langle f, Me\rangle = \langle f, Mg\rangle + \langle f, Mf\rangle$$
$$H_{f\oplus e} u(x) = H_e u(x) + \Delta_f u(x+e) + \Delta_f u(x-e).$$

(Assuming $x, x \pm e, x \pm f, x \pm g \in \Omega$)

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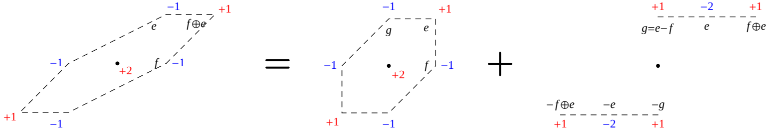
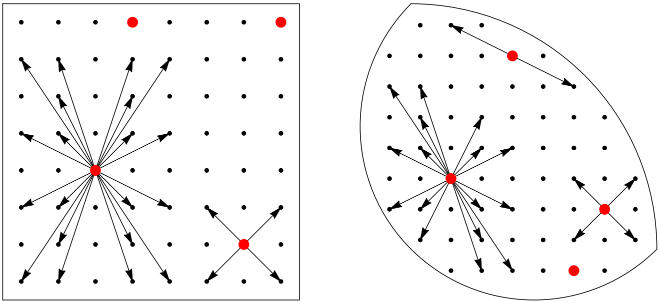


$$\mathcal{V}_\Omega(x) := \{e = f \oplus g;\ x \pm e, x \pm f, x \pm g \in \Omega\}.$$



The adaptive MA-LBR operator $\overline{\mathcal{D}}_{\mathcal{V}} u(x)$

Initialize $\mathbb{D} \leftarrow \infty$, vector $f \leftarrow (1,0)$, list $G \leftarrow [(0,1),(-1,0)]$.

While G is non-empty **do**

Denote by g the first element of G, and set $e := f + g$.

If $e \in \mathcal{V}(x)$, or $[e \in \mathcal{V}_\Omega(x)$ and $H_e u(x) < 0]$

then prepend e to G, and set

$$\mathbb{D} \leftarrow \min\{\mathbb{D},\ h(\Delta_e^+ u(x), \Delta_f^+ u(x), \Delta_g^+ u(x))\}$$

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Theorem (Adaptive pruning equals extensive sweeping)

Let $\mathcal{V}(x)$, $x \in X$, be stencils (subject to mild conditions), and let $\overline{\mathcal{V}}(x) := \mathcal{V}(x) \cup \mathcal{V}_\Omega(x)$.
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Motivations

The Wide Stencil scheme

The MA-LBR scheme (Monge-Ampère with Lattice Basis Reduction)

Adaptivity, and the Stern-Brocot tree

Numerical results

Conclusion

Test protocol

Domain $\Omega =]0,1[^2$. A strictly convex test function $U \in C^0(\overline{\Omega}, \mathbb{R})$ is recovered by solving: a discretization of

$$\begin{cases} \det(\nabla^2 u) = \det(\nabla^2 U) & \text{on } \Omega \\ u = U & \text{on } \partial\Omega \\ u & \text{convex.} \end{cases} \tag{1}$$

We use a damped Newton solver, with a strictly convex initialization satisfying the boundary conditions.

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Compared numerical schemes

- Finite Differences.

$$\mathcal{D}^{\mathrm{FD}} := \Delta_{(1,0)}\Delta_{(0,1)} - (\Delta_{(1,1)} - \Delta_{(1,-1)})^2/16.$$

- Wide-Stencil scheme of Froese and Oberman.

$$\mathcal{D}_V^{\mathrm{FO}} u(x) := \min_{\substack{\{f,g\}\subset V\\ \text{orthogonal}}} \frac{\Delta_f^+ u(x)}{\|f\|^2} \times \frac{\Delta_g^+ u(x)}{\|g\|^2}.$$

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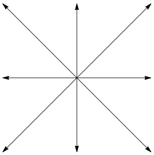

Eight point formula

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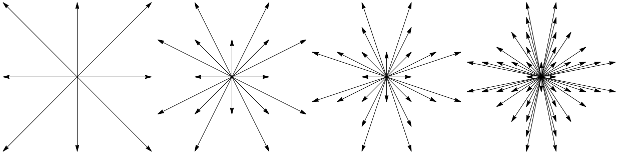

Different stencils tested

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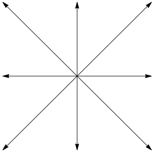
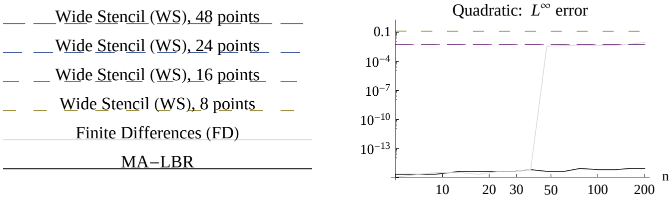


With adaptivity

Quadratic test case: $U(x) := \frac{1}{2}\langle x, Mx\rangle$, $\kappa = 10$, $\theta = \pi/3$



Smoothed cone: $U(x) := \sqrt{\delta^2 + \|x - x_0\|^2}$, with $\delta := 0.1$ and $x_0 := (1/2, 1/2)$.

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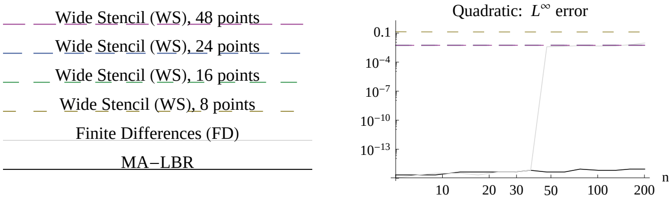

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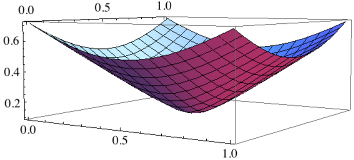

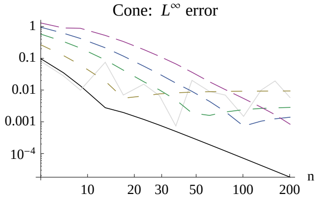

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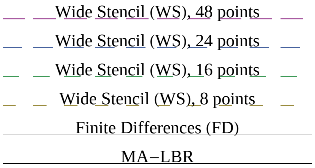

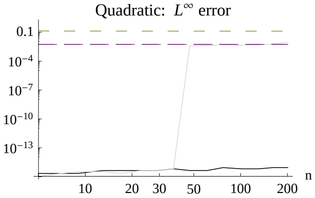

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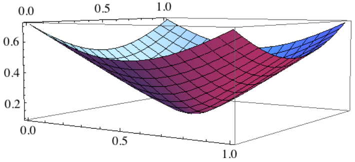

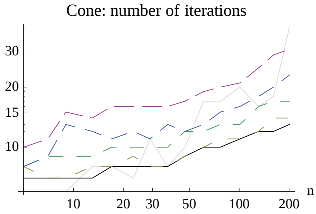

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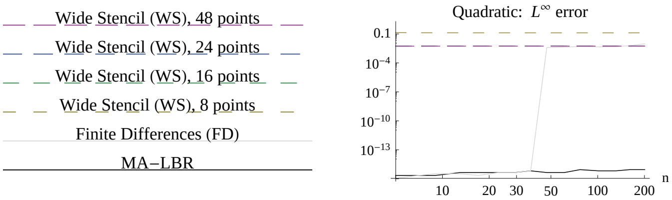
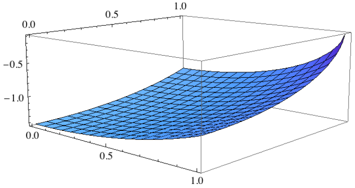
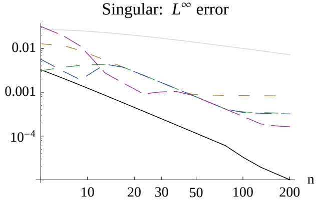


Singular: $U(x) = \sqrt{2 - \|x\|^2}$





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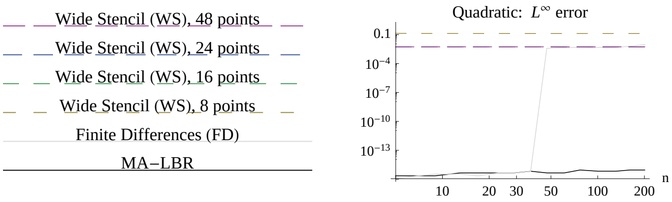

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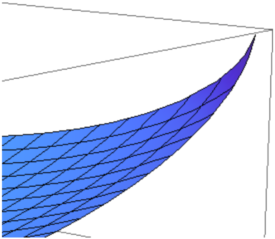

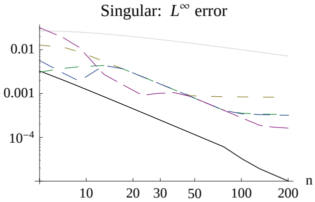

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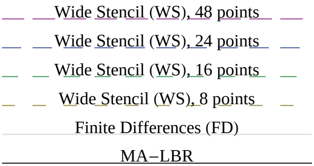

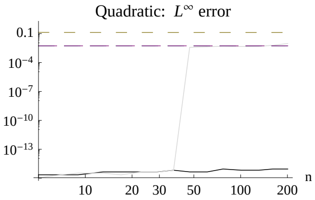

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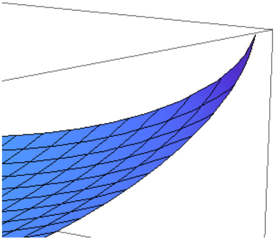

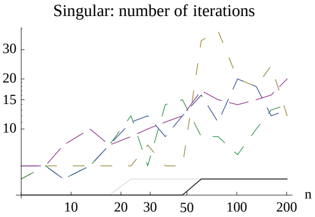

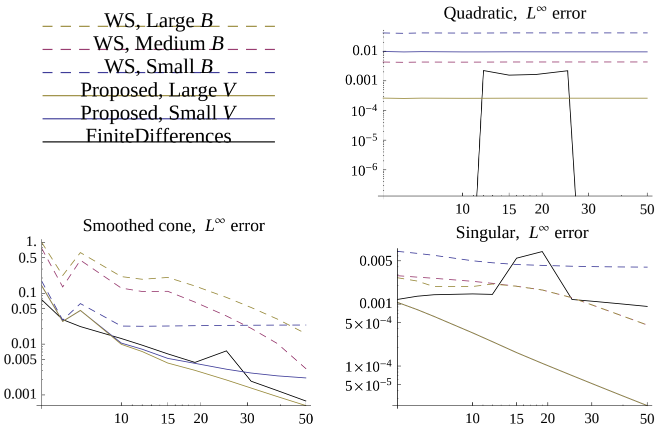

Figure : Log-Log plot of the L^∞ error, as a function of resolution n, for the three 3D test cases. Discretization set $X \subset \Omega$ has n^3 points.

Motivations

The Wide Stencil scheme

The MA-LBR scheme (Monge-Ampère with Lattice Basis Reduction)

Adaptivity, and the Stern-Brocot tree

Numerical results

Conclusion

Anisotropic PDEs discretizations on Cartesian grids are tied to:

- Lattice classification: obtuse superbases.
- Arithmetic: the Stern-Brocot tree.

Monge-Ampere schemes developed with these tools are:

- Consistent ($\neq$ approximately consistent Wide Stencil).
- Degenerate Elliptic ($\neq$ naïve finite differences).
- Cheap thanks to adaptivity.

Future work / Questions left open

- Alexandroff solutions of MA require other approaches, e.g. "Geometric" schemes.
- No Stern-Brocot tree in 3D.

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