Numerical Solutions of Geometric Partial Differential Equations

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Sample Equations and Schemes

Fully Nonlinear Pucci Equation

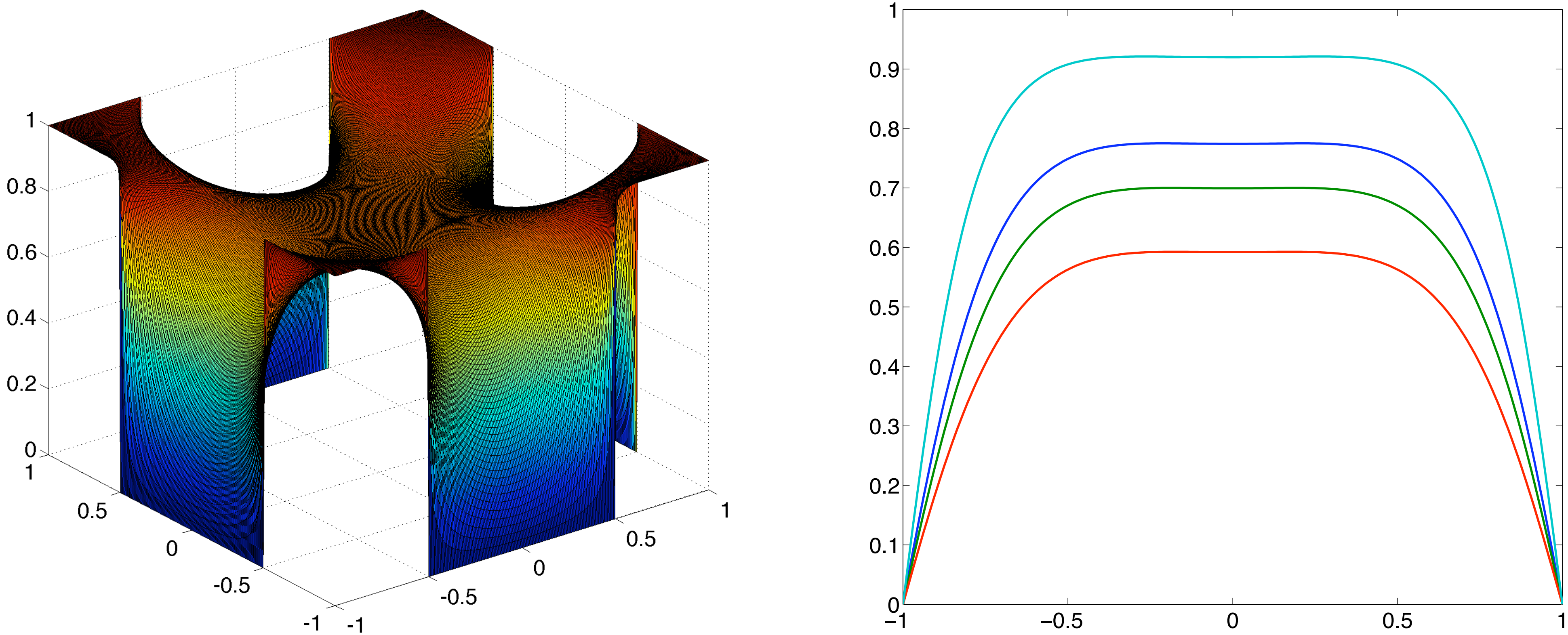


FIGURE 4. Surface plot of the Pucci solution, for $\alpha = 3$, $n = 256$. Plot of the midline of the solutions, increasing with $\alpha = 2, 2.5, 3, 5$, $n = 256$.

Mean Curvature

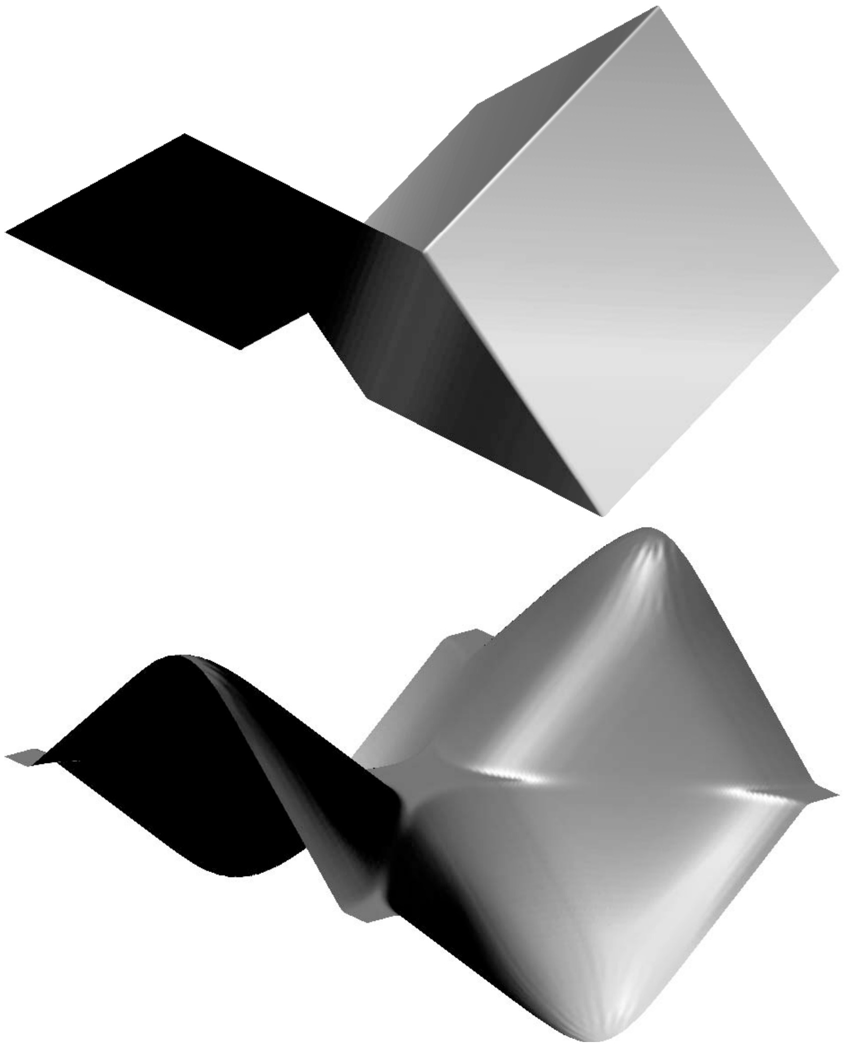
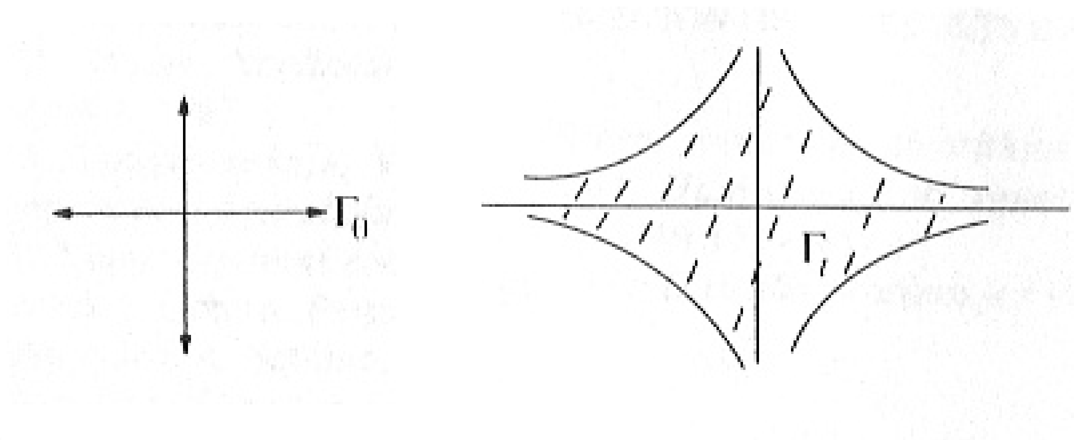


Fig. 3. Surface plot: initial data, and solution at time .03



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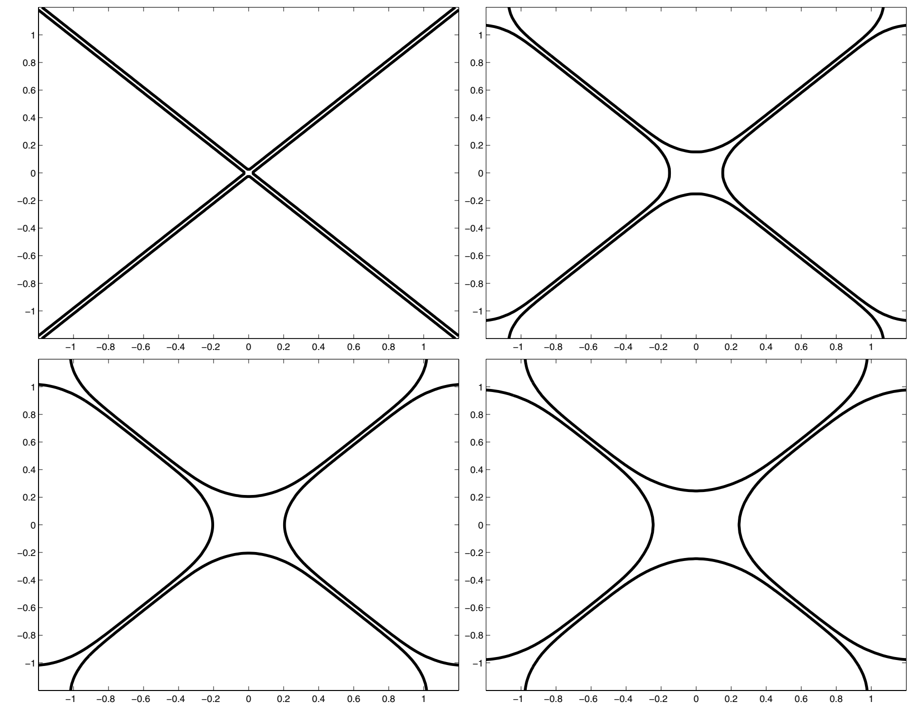
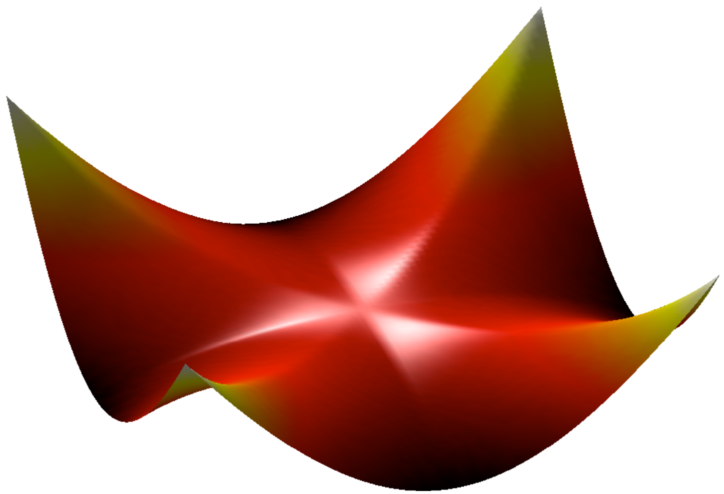
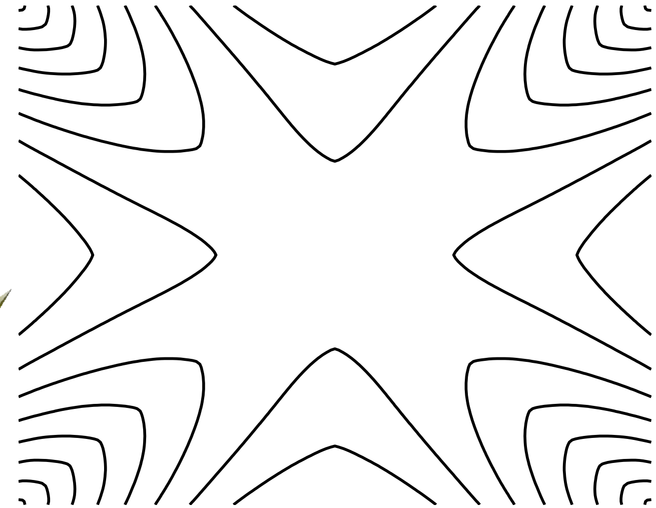


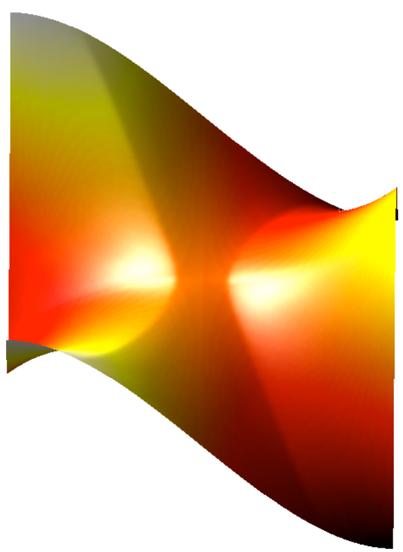
Fig. 2. Contour plots of the $-.02$, and $.02$ contours at times 0, .015, .03, .045

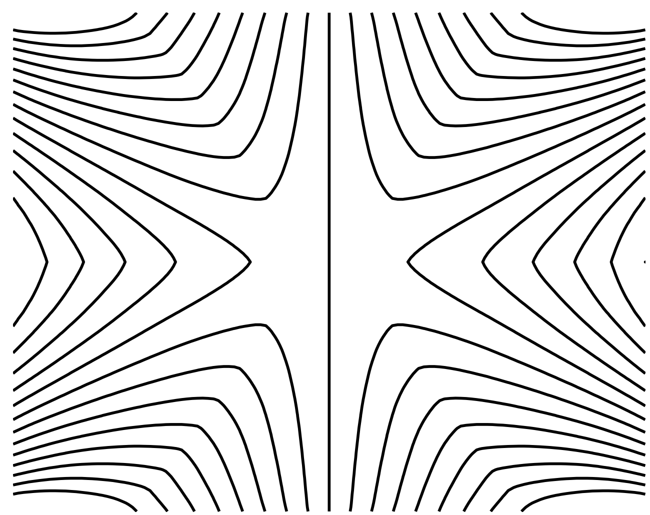
Infinity Laplacian

$$\Delta_\infty u = \frac{1}{|Du|^2} \sum_{i,j=1}^{m} u_{x_i x_j} u_{x_i} u_{x_j} = 0$$





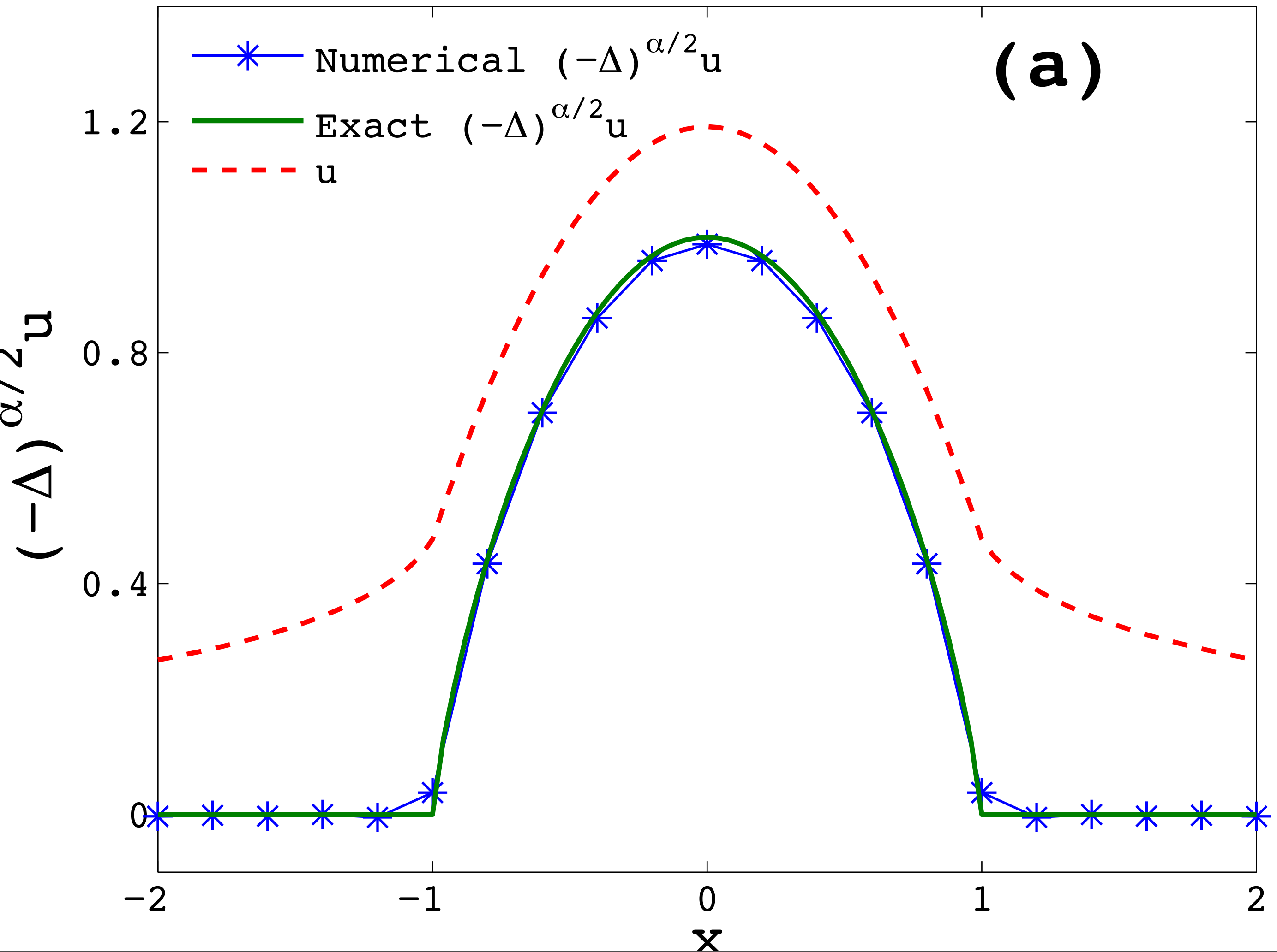




Fractional Obstacle Problem

(with) Yanghong Huang

$$
\begin{aligned}
u &\geq \varphi, && \text{in } \mathbb{R}^n, \\
(-\Delta)^{\alpha/2} u &\geq 0, && \text{in } \mathbb{R}^n, \\
(-\Delta)^{\alpha/2} u(x) &= 0, && \text{on } \{x \in \mathbb{R}^n \mid u(x) > \varphi(x)\}.
\end{aligned}
$$



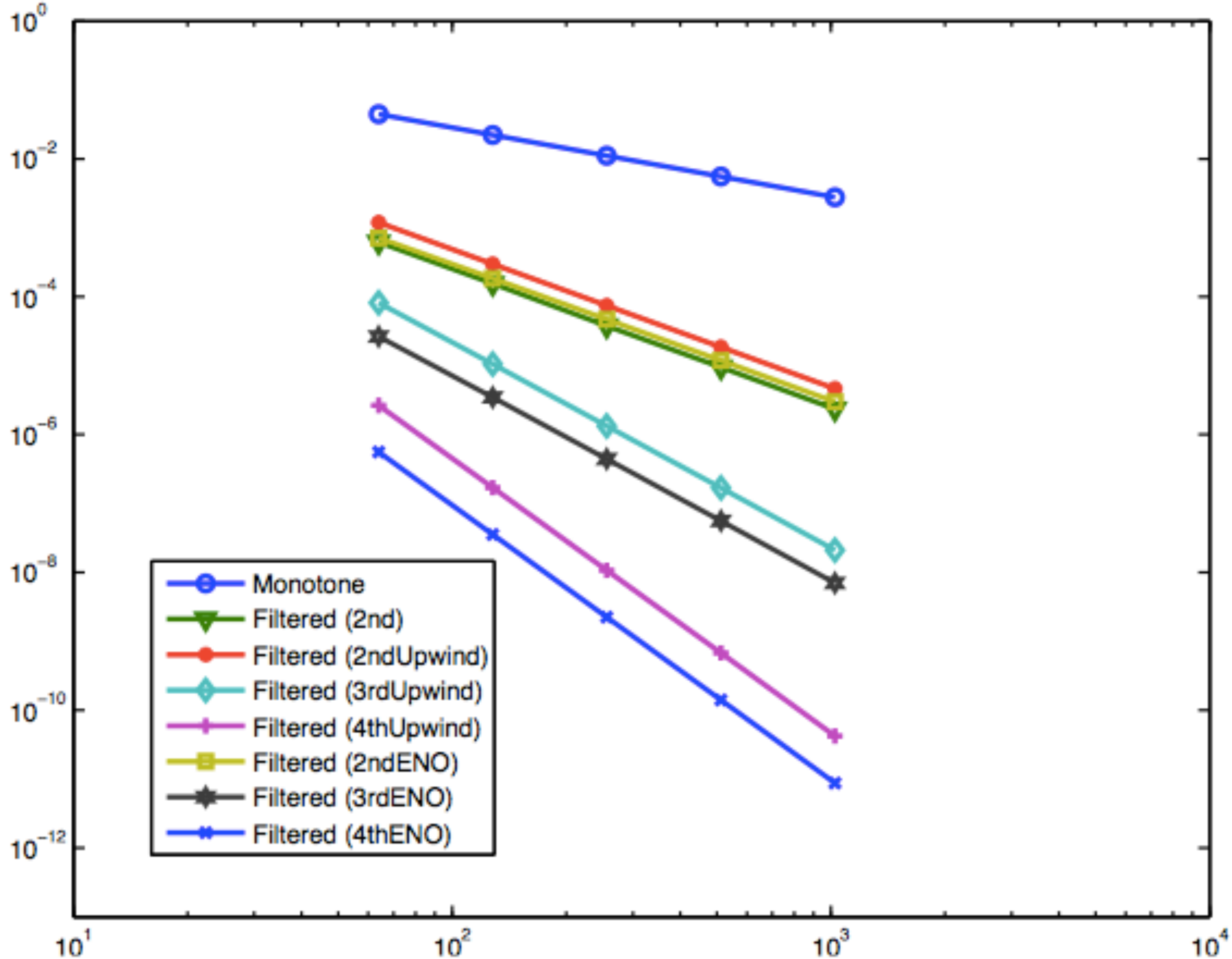
Filtered Schemes for Hamilton Jacobi with Tiago Salvador

$$\begin{cases} |\nabla u(x)| = f(x), & \text{for } x \text{ outside } \Gamma, \\ u(x) = g(x), & \text{for } x \text{ on } \Gamma. \end{cases}$$

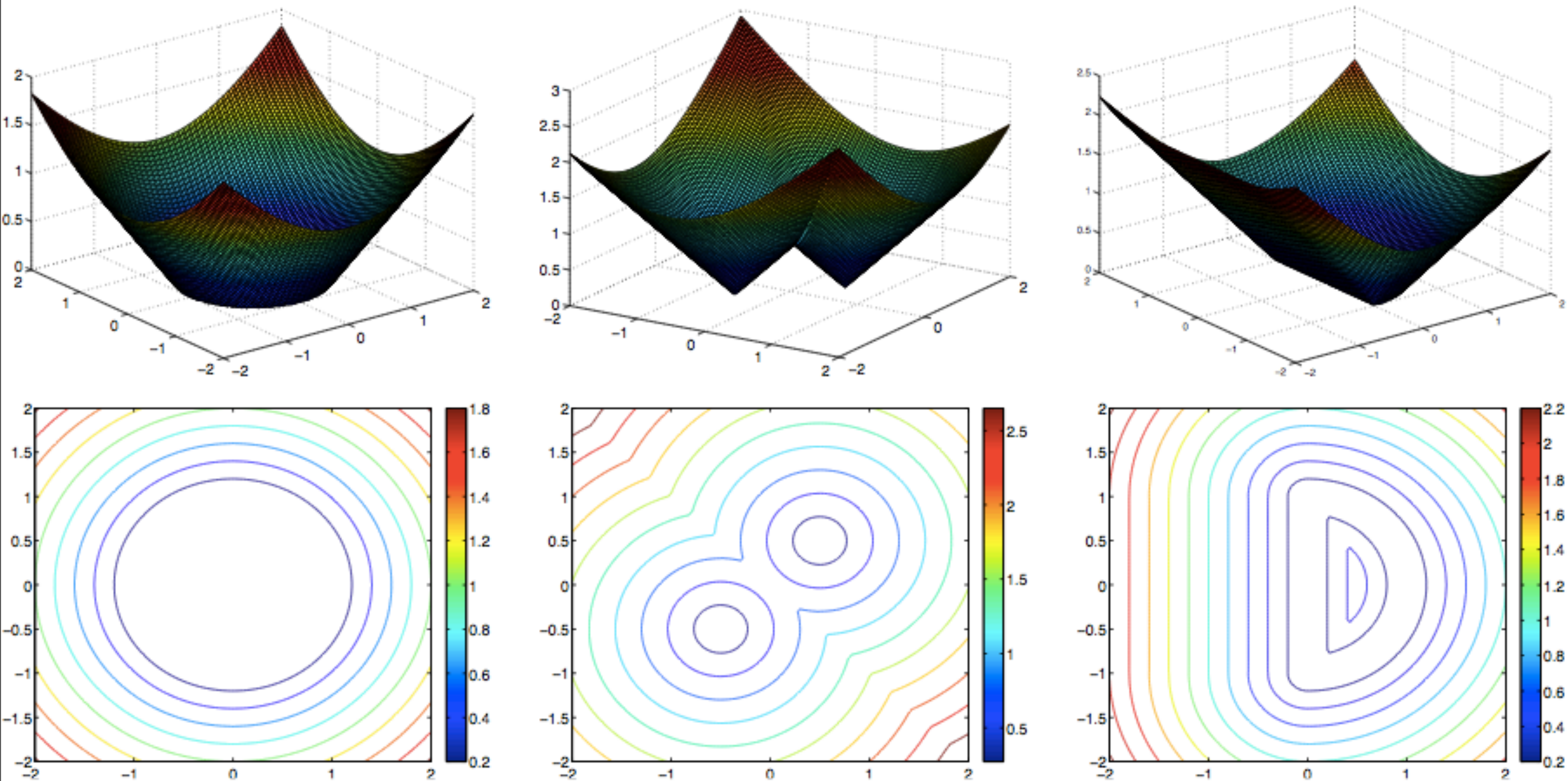
$$F^h[u] = \begin{cases} F_A^h[u], & \text{if } \left|F_A^h[u] - F_M^h[u]\right| \le \sqrt{h} \\ F_M^h[u], & \text{otherwise.} \end{cases}$$

$$F^h[u] = F_M^h[u] + \mathcal{O}(h^{1/2}).$$

Obtain High Accuracy in 1d (even if solutions not smooth)



Obtain 2nd order accuracy in 2d



General Convex Envelopes

Directionally Convex Envelopes

Rank 1 Convex Envelope: Laminate

(scalar) quasi-convex envelope: make level sets of function convex

With Yanglong Ruan

Microstructure in Laminates

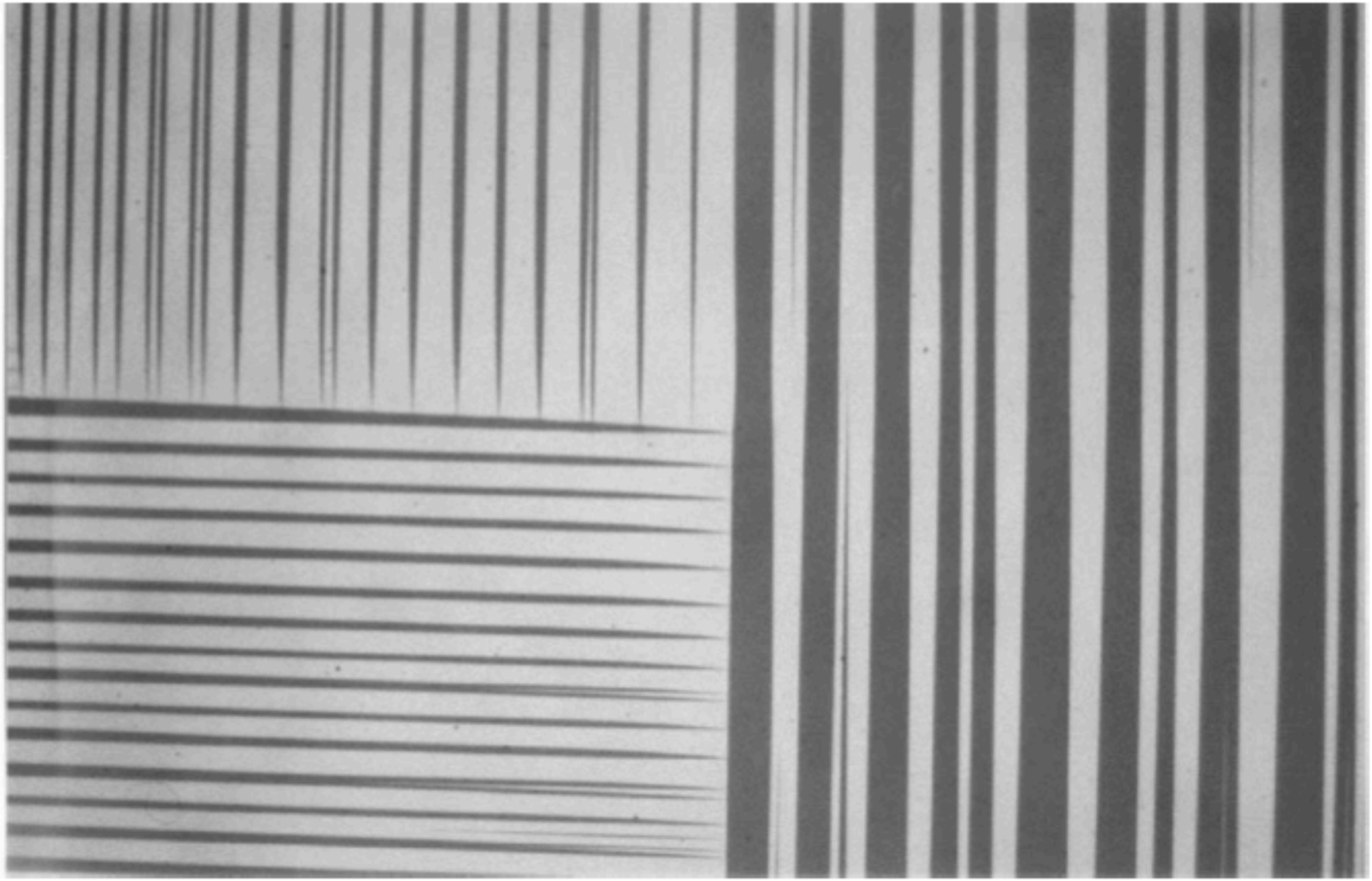
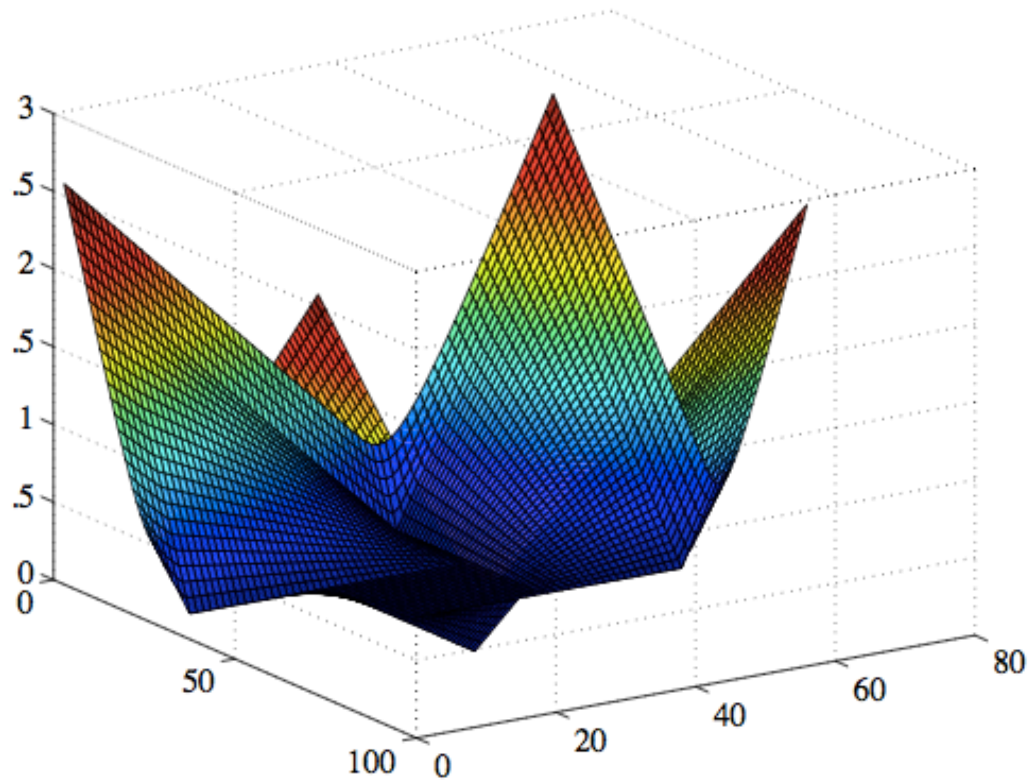


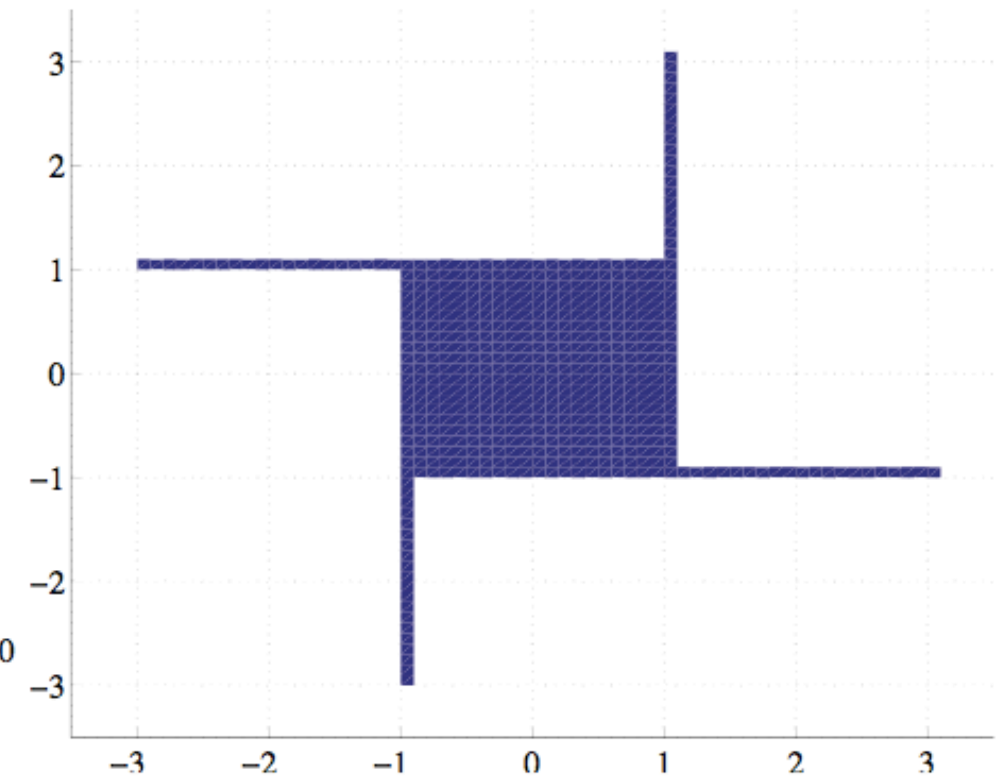
Figure 1: Microstructure in a Cu-Al-Ni single crystal; the imaged area is approximately 2 mm × 3 mm (courtesy of C. Chu and R.D. James, University of Minnesota)

Four Gradient Example

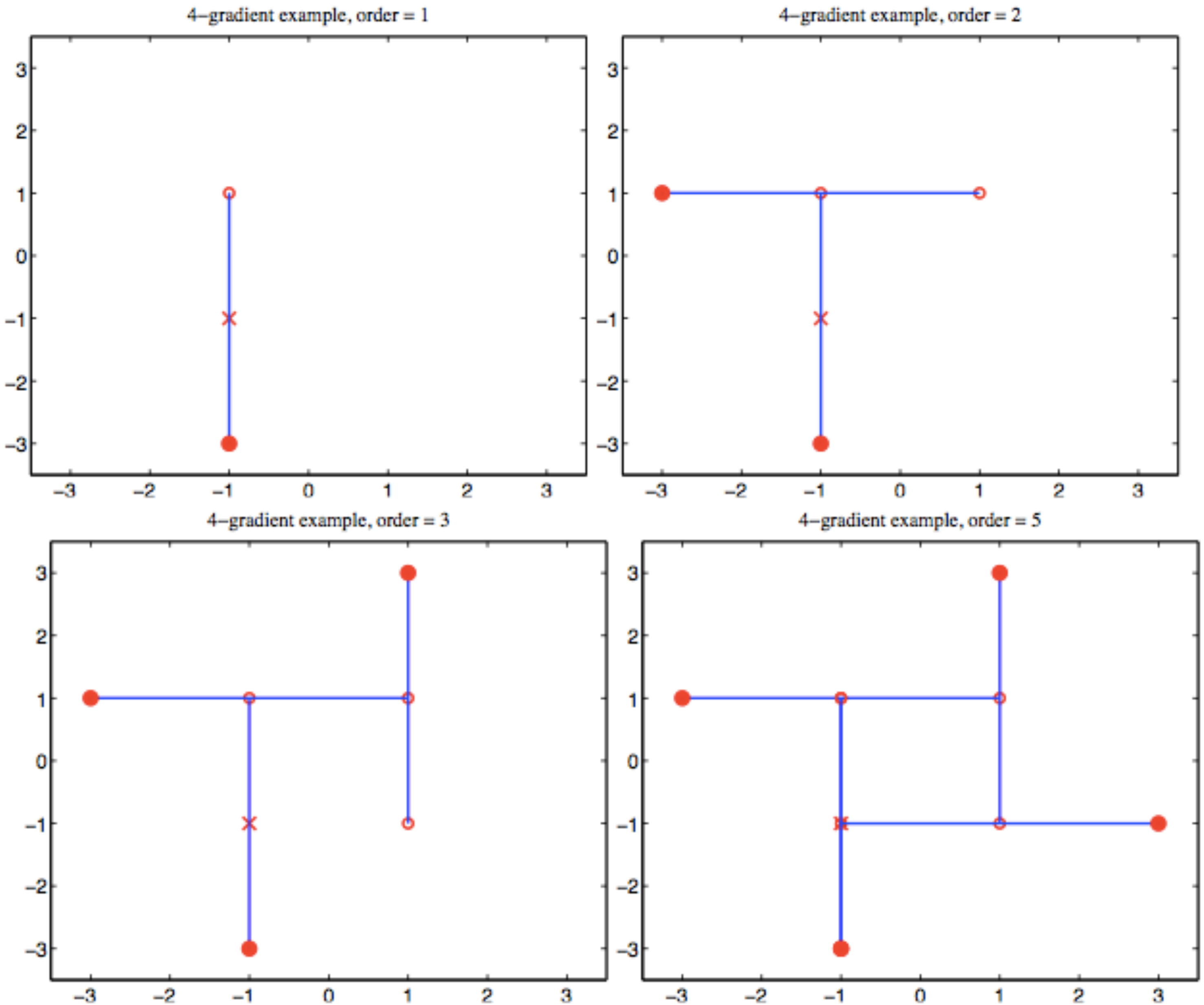
energy of 4–gradient example



envelope of 4–gradient example



Four Gradient Example



Four D (2X2) Example

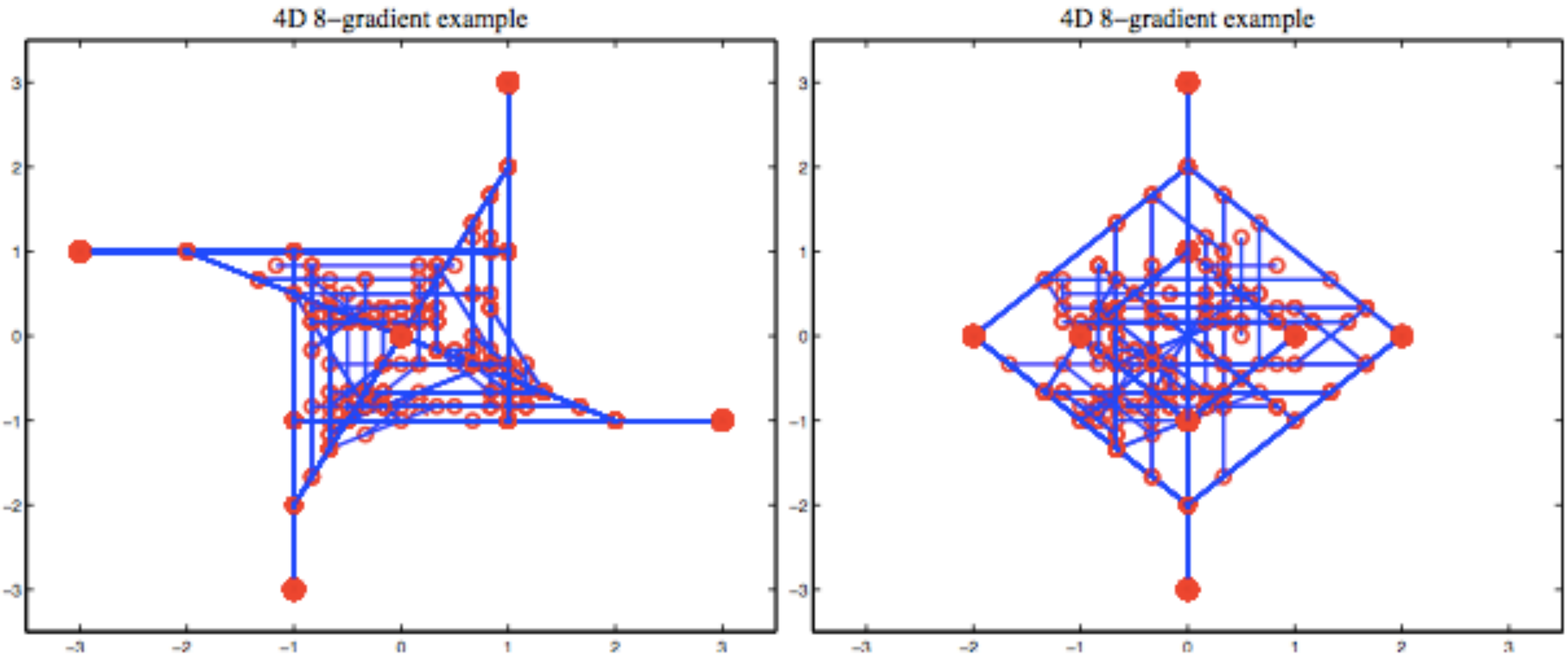


FIGURE 10. Laminate projected onto $x - w$ and $y - w$ plane. The starting point does not fall on any coordinate plane.

Numerical Solution of the Infinity Laplace Equation via

solution of the absolutely minimizing Lipschitz extension problem in a discrete setting

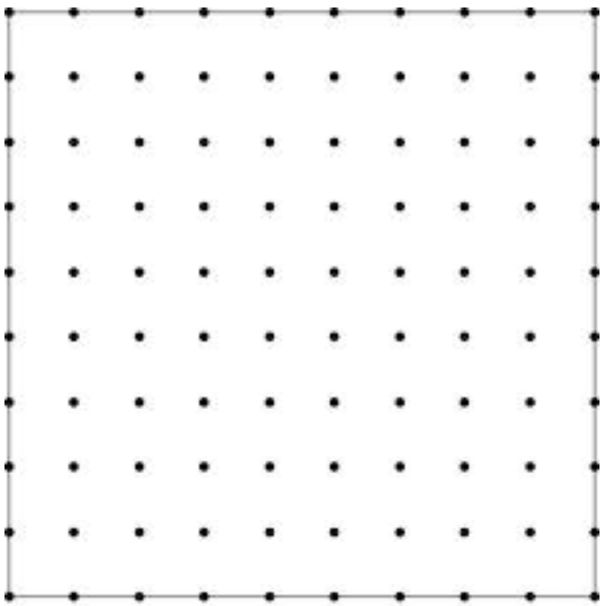
The discrete Lipschitz extension problem.

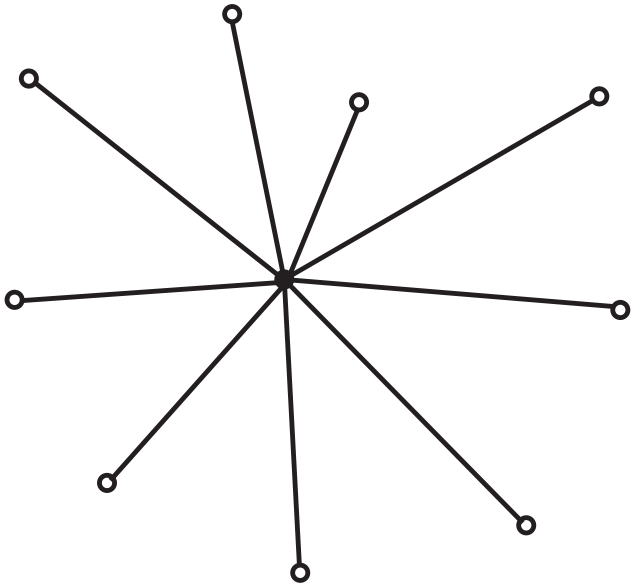
Definition. Given distinct $x_0, \dots, x_n$ in $\mathbb{R}^m$, and values $u_i = u(x_i)$, for $i = 1, \dots n$, the discrete Lipschitz constant at x_0, is

$$L(u_0) = \max_{i=1}^{n} L^i(u_0) = \max_{i=1}^{n} \frac{|u_0 - u_i|}{|x_0 - x_i|}$$

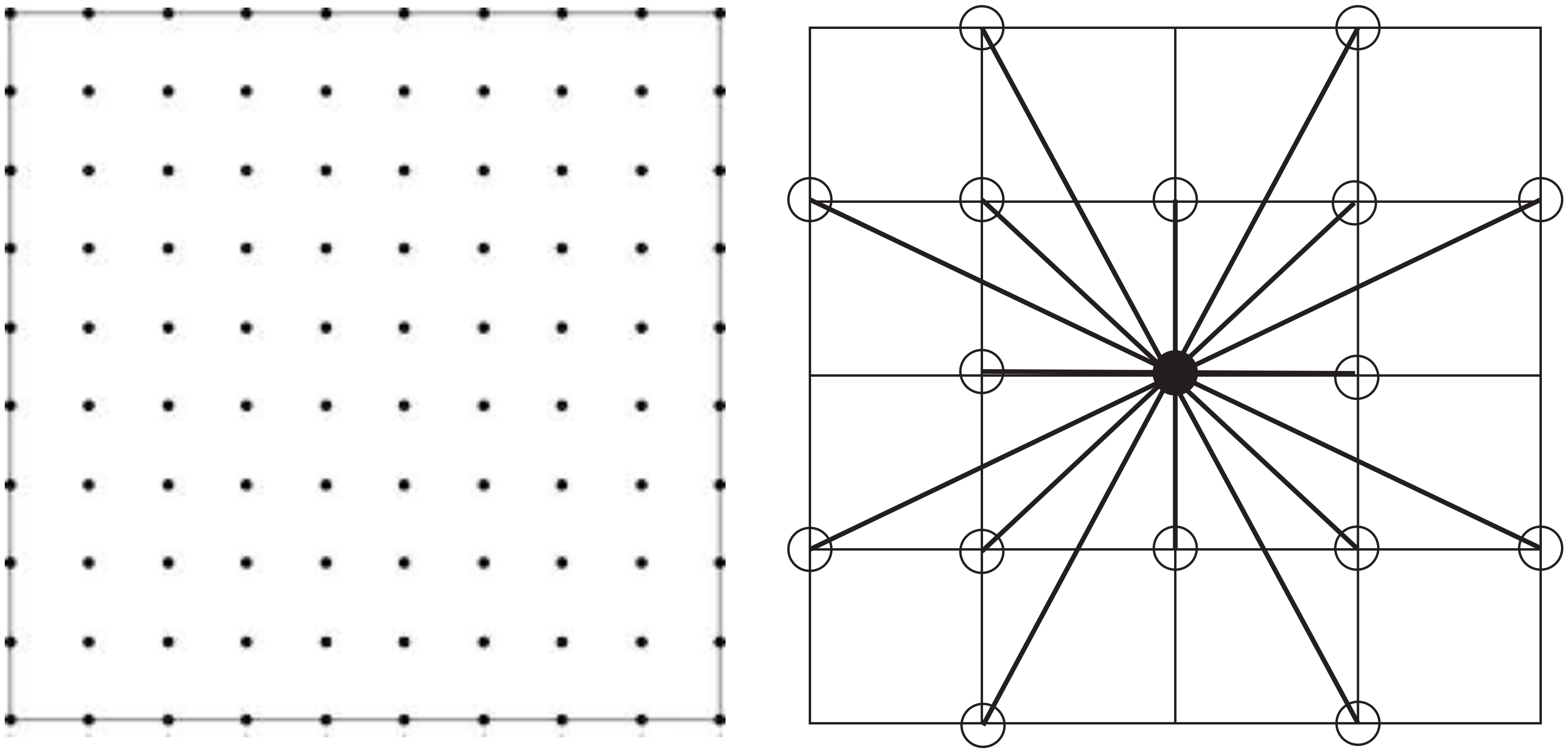
Problem. Minimize the discrete Lipschitz constant of u at x_0, (computed with respect to the points $x_1, \dots, x_n$) over the value $u_0 = u(x_0)$

$$\min_{u_0} L(u_0)$$





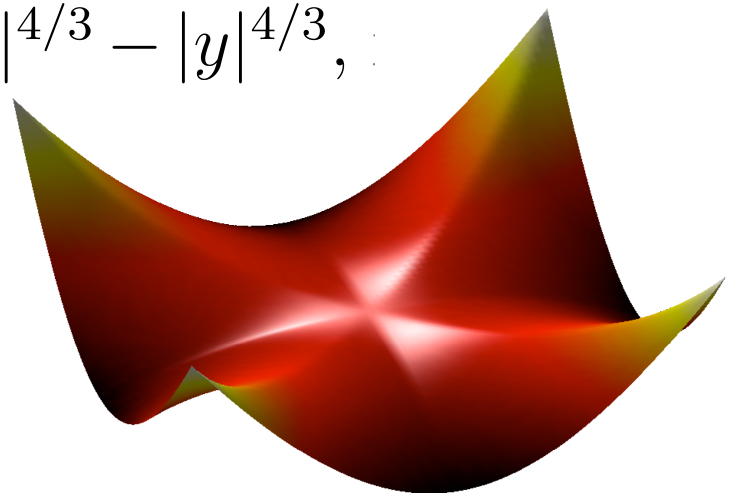
Now solve the problem at every point on a grid.

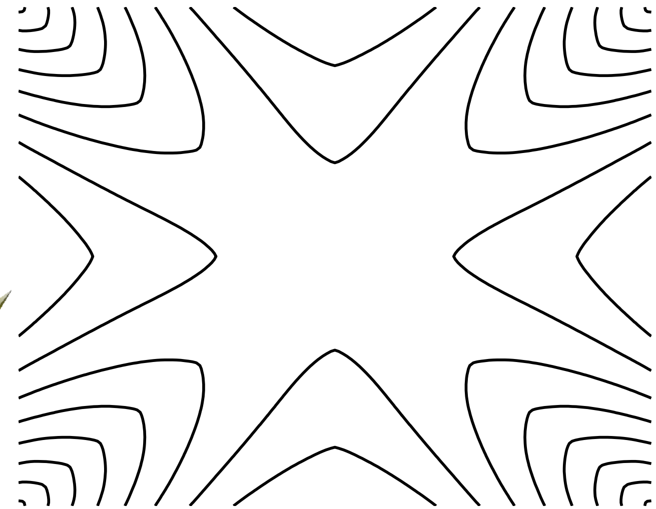


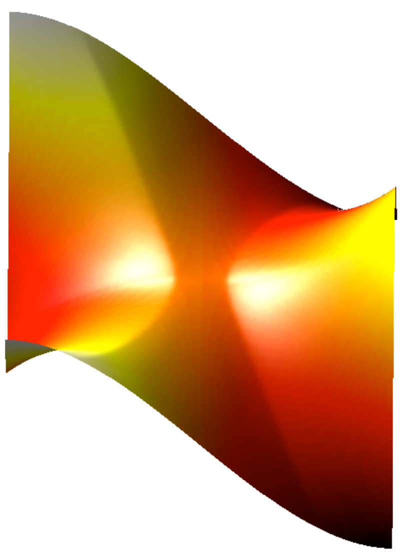
Infinity Laplacian

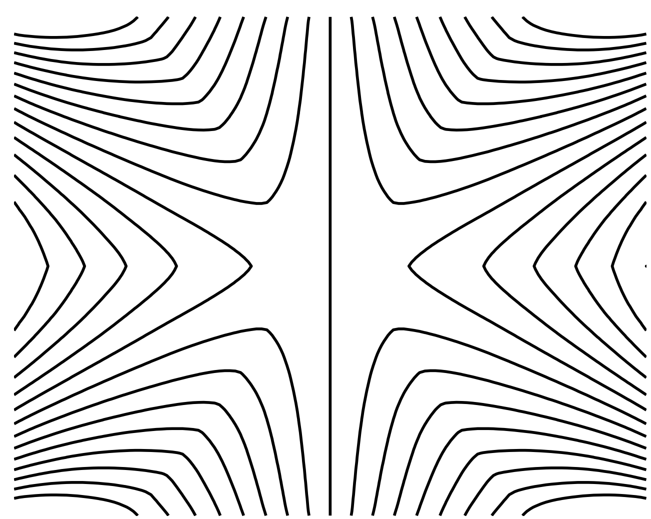
$$\Delta_\infty u = \frac{1}{|Du|^2} \sum_{i,j=1}^{m} u_{x_i x_j} u_{x_i} u_{x_j} = 0$$

$$f(x, y) = |x|^{4/3} - |y|^{4/3},$$









Metric induced by different stencils

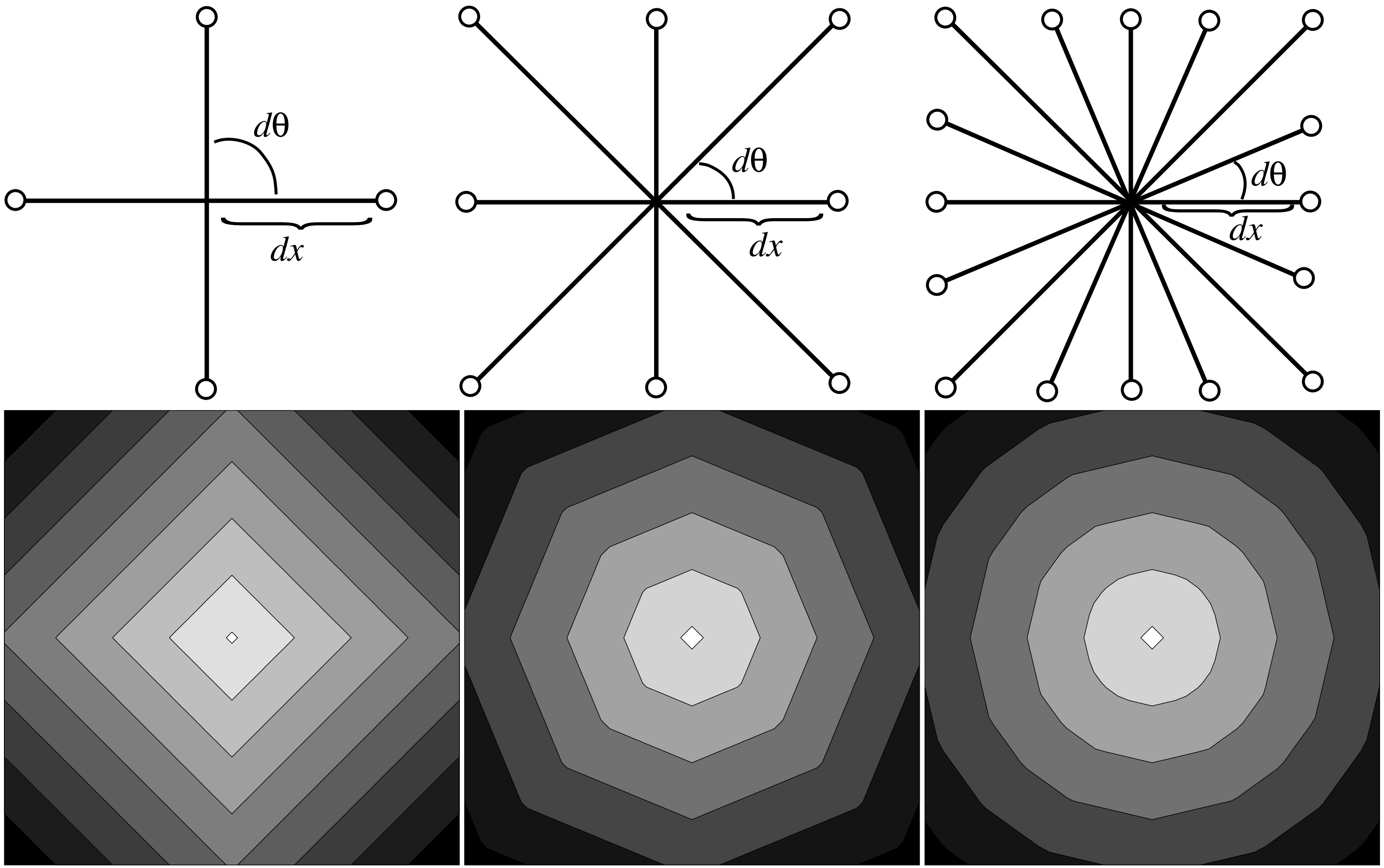


FIGURE 1. Grids for the 5, 9, and 17 point schemes, and level sets of the cones for the corresponding schemes.

Convergence of the scheme

Theorem. Let u be a C^2 function in a neighborhood of x_0. Suppose we are given neighbors $x_1, \ldots, x_n$, arranged symmetrically on a grid. Let u_* be the solution of the discrete minimal Lipschitz extension problem computed with respect to the points $x_1, \ldots, x_n$, and let i, j be the indices which maximize the relaxed discrete gradient. Then

$$-\Delta_\infty u(x_0) = \frac{1}{d_i d_j}(u(x_0) - u_*) + O(d\theta + dx)$$

Theorem (Convergence)**.** The solution of the difference scheme defined above converges (uniformly on compact sets) as $dx, d\theta \to 0$ to the solution of (IL).

Proof. Convergence to the solution of (IL) follows from consistency and degenerate ellipticity (monotonicity) of the scheme by [Barles-Souganidis]. □

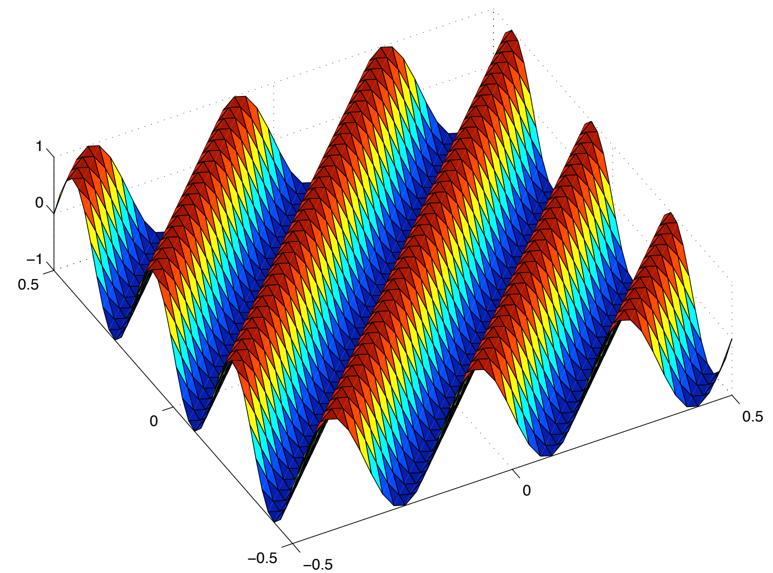
Mean Curvature

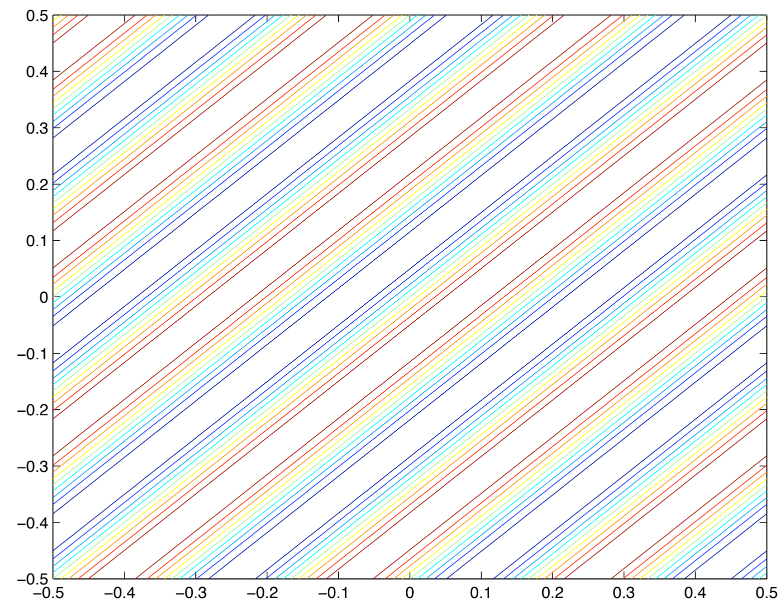
Interpretations

- Catte-Dibos-Koepfler (1985) morphological scheme for mean curvature
- Kohn-Serfaty (2005) deterministic control based approach to motion by mean curvature.
- Ryo Takei (2007) M.S. thesis: www.sfu.ca/~rrtakei

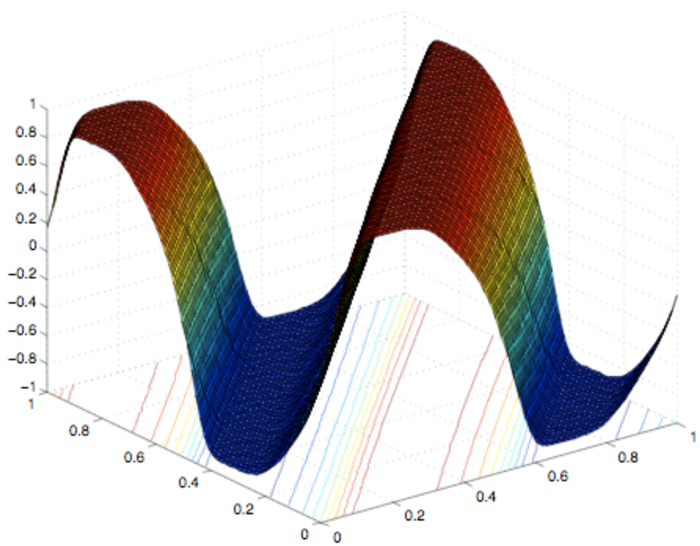
Failure of naive difference scheme

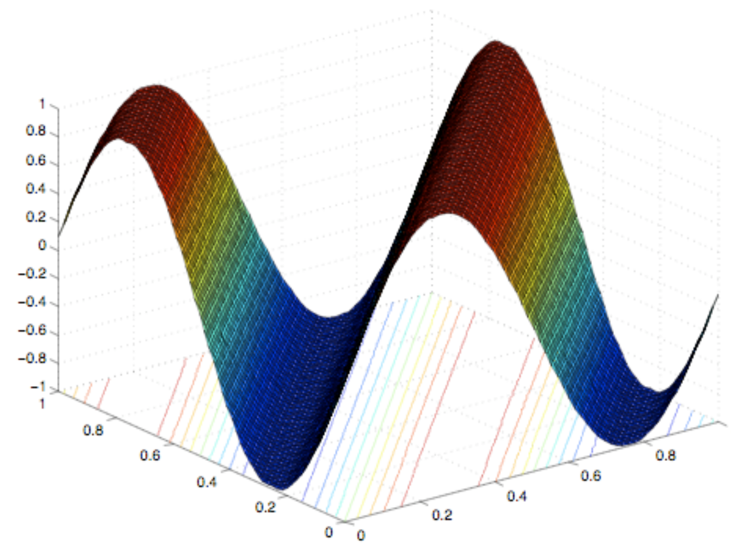
- Simply replace all the terms in the equation by a finite difference. Explicit in time.
- Use exact steady solution (with straight level sets) on periodic domain.
- Numerical solution contracts over time to a constant.
- Monotone scheme converges for this example.





Other schemes





- discretize equation written in divergence structure
- will get "capping" at local min/max of level set function.
- div(grad u/ |grad u|). When u has local max, get nonzero divergence, even if function has straight level sets.
- Expect similar behavior for FEM method.

Scheme: part 1 of 2

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PDE: $u_t = |Du| div \left(\frac{Du}{|Du|} \right) = \frac{d^2 u}{d\mathbf{t}^2}, \qquad \mathbf{t} = \frac{(u_y, -u_x)}{\sqrt{u_x^2 + u_y^2}}$

Scheme: part 1 of 2

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Use this interpretation to discretize spatial operator by finite differences

$$\frac{d^2 u}{d\mathbf{t}^2} = \frac{u(x + dx\,\mathbf{t}) - 2u(x) + u(x - dx\,\mathbf{t})}{dx^2} + O(dx^2)$$

Scheme: part 1 of 2

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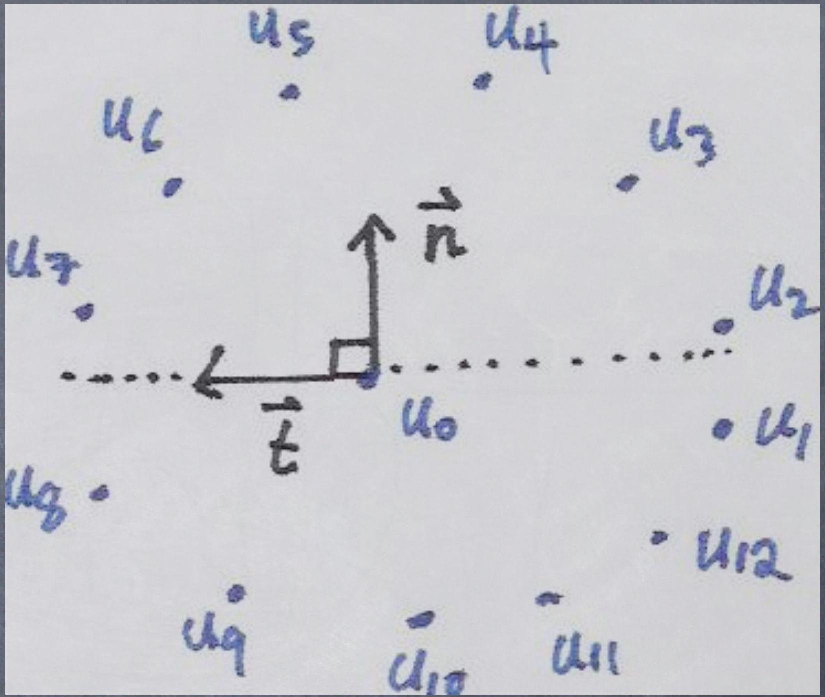
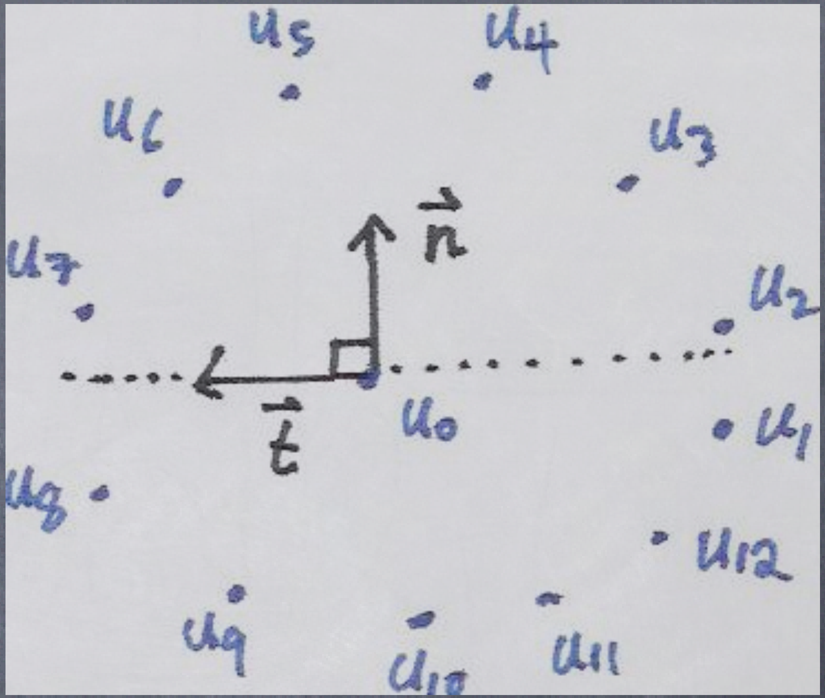
Q: How to find a monotone discretization of this operator?

Scheme: part 2 of 2



Scheme: part 2 of 2

$$u_* = \text{ median } \{u_1, u_2, \ldots, u_{12}\}$$
$$= \frac{u_2 + u_7}{2}$$



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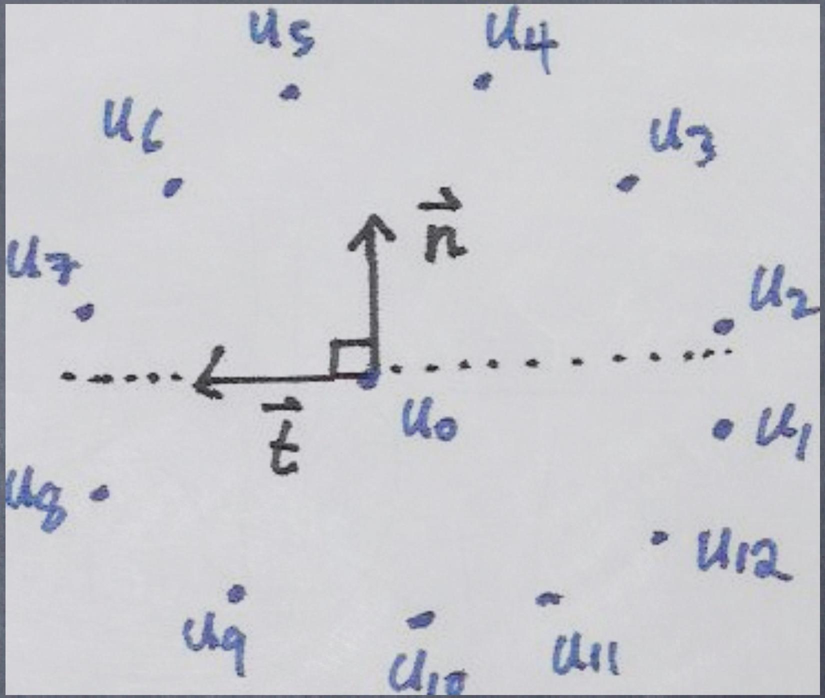

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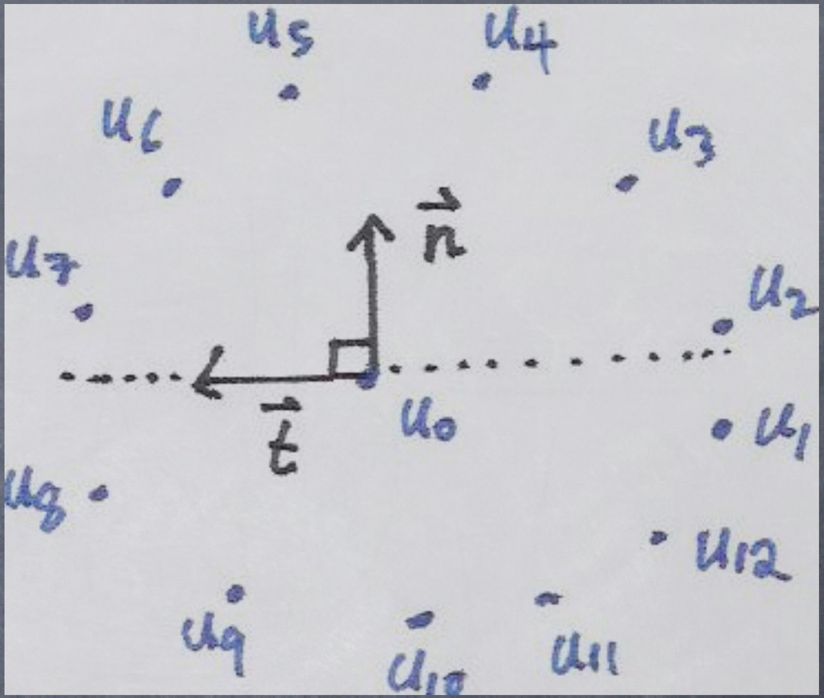

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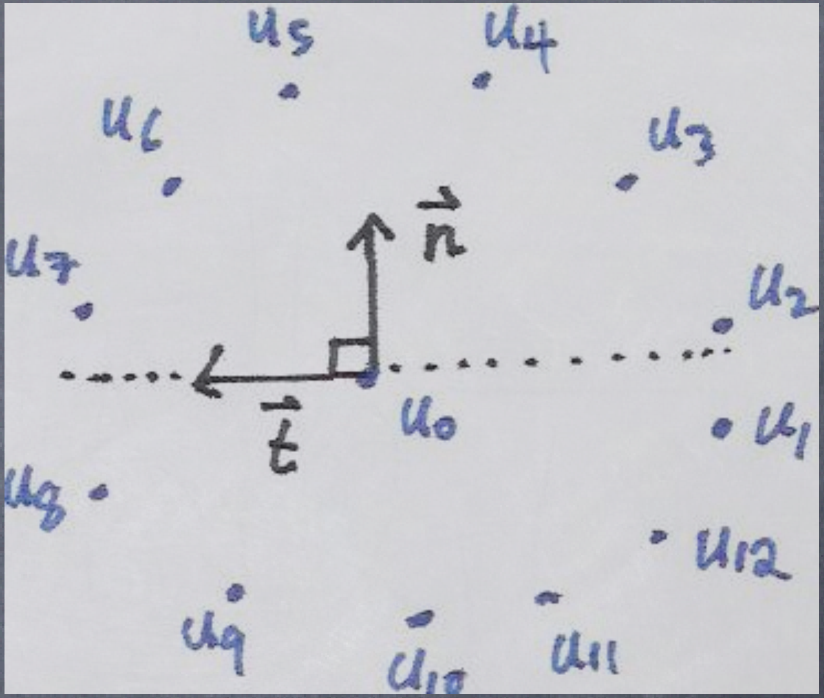
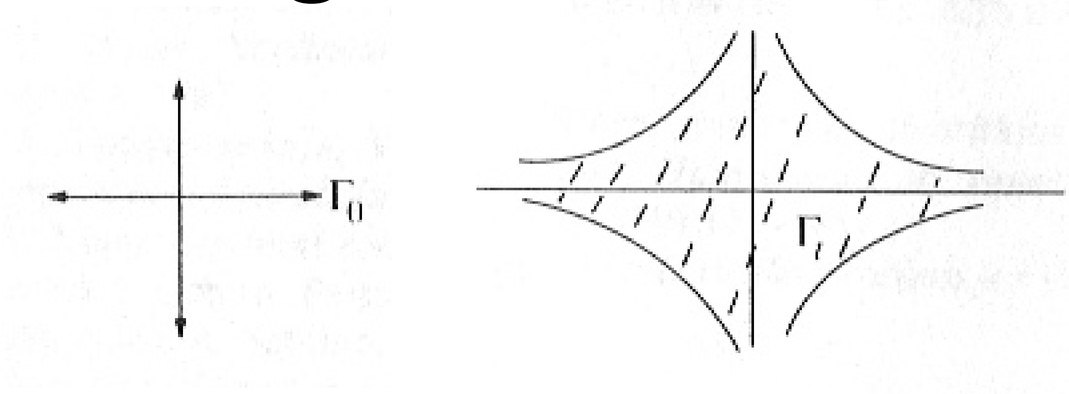


Scheme is consistent, with additional error due to directional resolution, decreased by widening stencil.

Fattening



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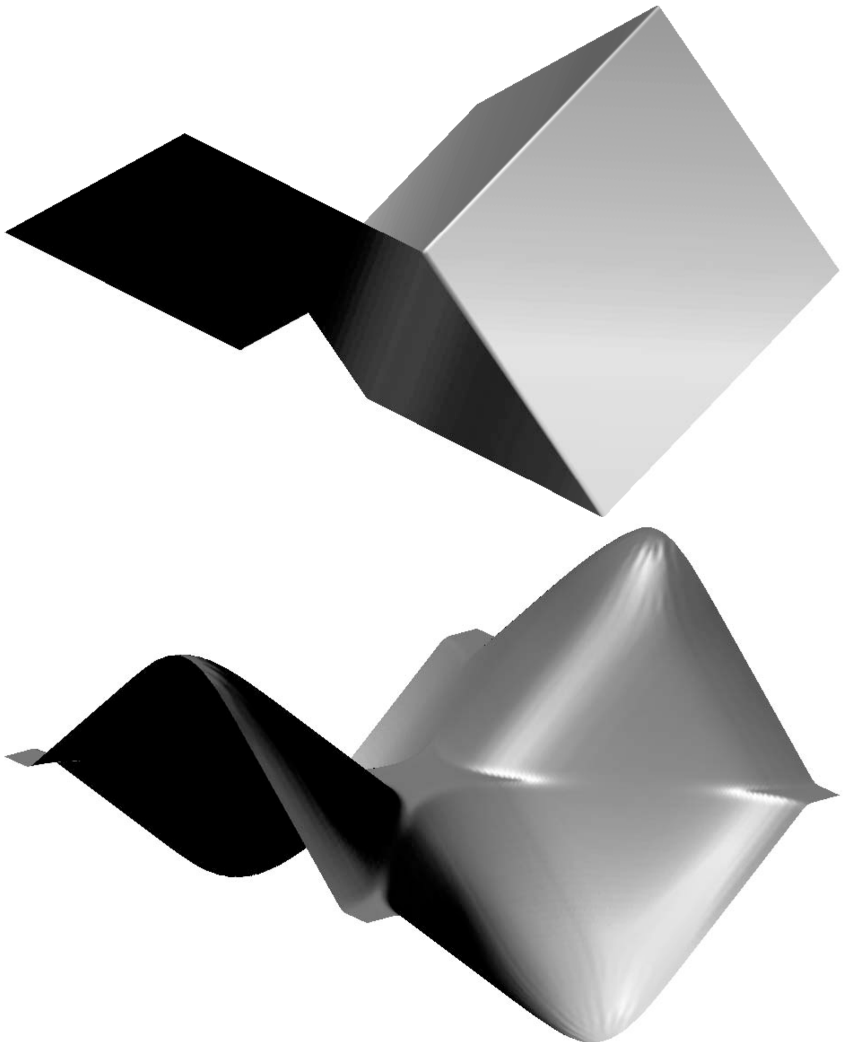
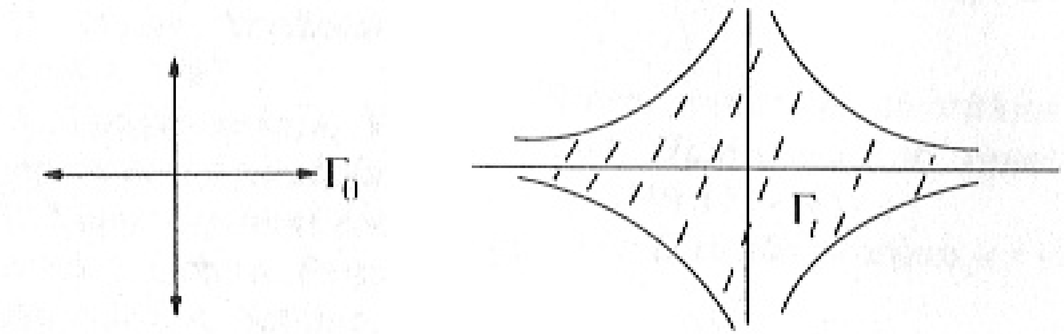


Fig. 3. Surface plot: initial data, and solution at time .03



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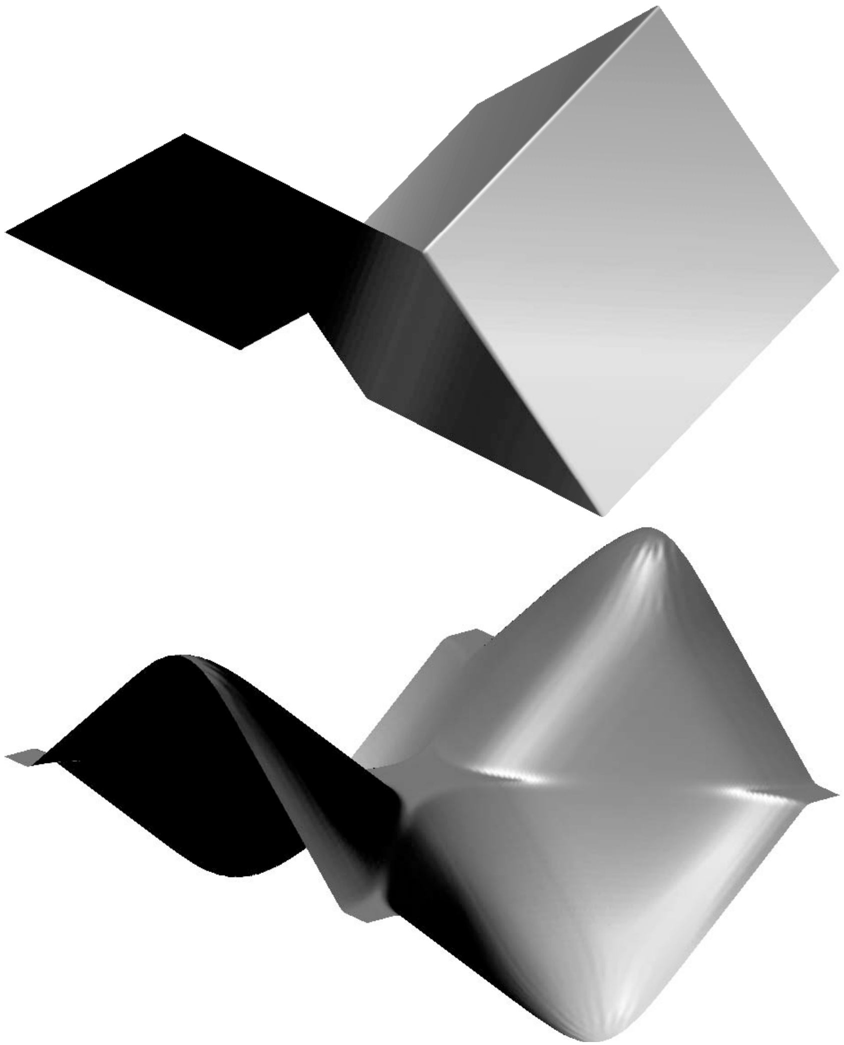

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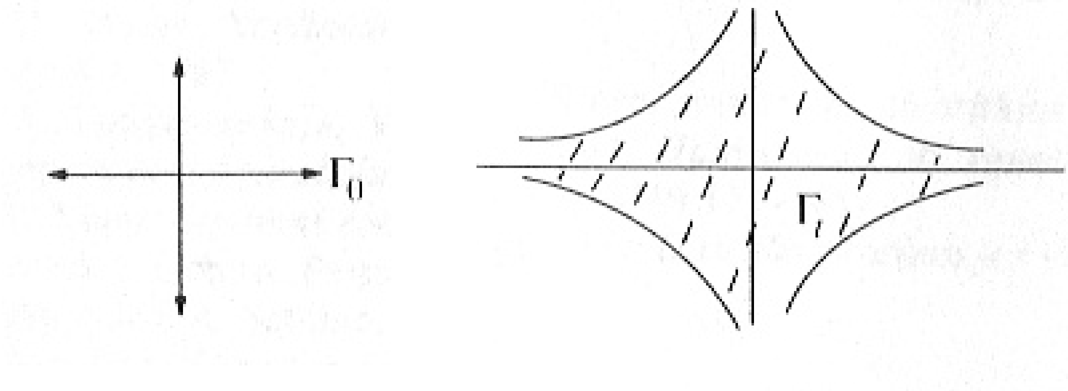

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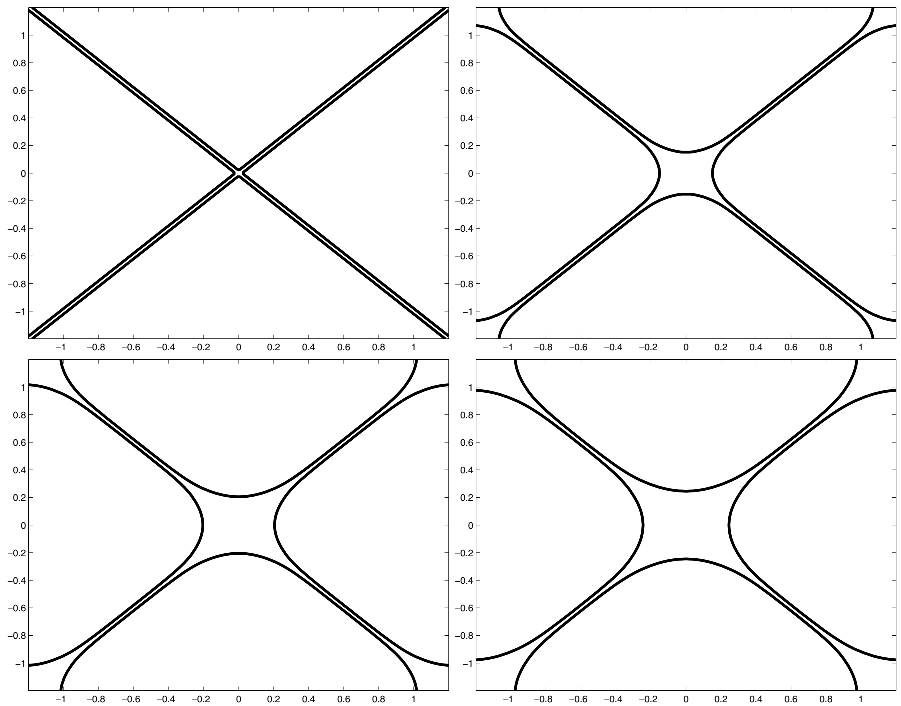

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