Weak Solutions to a Fractional Fokker-Planck Equation via Splitting and Wasserstein Gradient Flow

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Abstract

We study a linear fractional Fokker-Planck equation that models non-local diffusion in the presence of a potential field. The non-locality is due to the appearance of the 'fractional Laplacian' in the corresponding PDE, in place of the classical Laplacian which distinguishes the case of regular diffusion. We prove existence of weak solutions by combining a splitting technique together with a Wasserstein gradient flow formulation. An explicit iterative construction is given, which we prove weakly converges to a weak solution of this PDE.

Keywords: Fractional Laplacian, Splitting, Wasserstein gradient flow

1. Introduction

The Fokker-Planck equation $\partial_t\rho = \Delta\rho + \operatorname{div}(\rho\nabla\Psi(x))$, $\rho(x,0) = \rho^0(x)$, which models diffusion in the presence of a potential field Ψ, has a well-known gradient flow formulation, thanks to the seminal paper by [5]: it is a gradient flow of the free-energy functional $F(\rho) := \int_{\mathbb{R}^d} \rho\log\rho + \rho\Psi\,\mathrm{d}x$ with respect to the quadratic Wasserstein distance W_2, on the space $\mathcal{P}_a^2(\mathbb{R}^d)$ of absolutely continuous probability measures with finite second moments. Here, the potential $\Psi : \mathbb{R}^d \to [0,\infty)$ is given, and $\rho^0(x)$ is a probability density on $\mathbb{R}^d$. The quadratic Wasserstein distance between $f, g \in \mathcal{P}_a^2(\mathbb{R}^d)$, is defined by $W_2(f,g) = \left[\inf_{T\#f=g} \int_{\mathbb{R}^d} |x - T(x)|^2 f(x)\,\mathrm{d}x\right]^{1/2}$; in fact, $W_2(f,g) = \left[\int_{\mathbb{R}^d} |x - \nabla\varphi(x)|^2 f(x)\,\mathrm{d}x\right]^{1/2}$ for a unique convex function φ satisfying $\nabla\varphi\#f = g$; for more details, we refer to [9].

If the Laplacian Δ is formally replaced by (the negative of) the non-local linear operator known as the *fractional* Laplacian, $-(-\Delta)^s$, (defined via the Fourier transform by $\mathcal{F}\left[(-\Delta)^s f\right] = |\xi|^{2s}\mathcal{F}\left[f\right]$), we obtain a 'fractional' Fokker-Planck equation,

$$\begin{cases} \rho_t = -(-\Delta)^s\rho + \operatorname{div}(\rho\nabla\Psi(x)) \quad \text{in } \mathbb{R}^d \times (0,\infty), \quad s \in (0,1) \\ \rho = \rho^0 \quad \text{on } \mathbb{R}^d \times \{t=0\}, \end{cases} \tag{1}$$

for which the gradient flow formulation is lost. However, the *transport* equation $\rho_t = \operatorname{div}(\rho\nabla\Psi(x))$, *is* a gradient flow of the potential energy $\int_{\mathbb{R}^d} \rho\Psi\,\mathrm{d}x$ w.r.t. W_2 [6] (we will call this PDE the 'Wasserstein part' of (1)). Moreover, solutions to the *fractional heat* equation $\partial_t\rho = -(-\Delta)^s\rho$ can be obtained by a different method, e.g., Fourier transform (this is the 'non-Wasserstein part' of (1)). Therefore (1) is an example of a PDE which has 'Wasserstein' and 'non-Wasserstein' parts. We ask the following question: *Suppose a PDE is comprised of a 'Wasserstein part' and a 'non-Wasserstein part'. Can we recover a solution of the PDE by combining the Wasserstein gradient flow together with solutions (obtained by another method) of the 'non-Wasserstein' part?* Our interest in the PDE (1) is a simple test for this question.

In an attempt to answer our question, we are led to the time-discrete *splitting* scheme (see e.g. [7] and the references therein). The idea of splitting is very simple. In the context of (1), we fix a finite time horizon T, a time step $\tau = T/N$ for some integer N, and divide $[0,T)$ into equal length intervals $[n\tau, (n+1)\tau)$. Starting at $t = 0$, we

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evolve the initial density ρ^0 according to the fractional heat equation for a time of duration τ, and then evolve the resulting distribution by using a Wasserstein gradient flow formulation of the transport equation for a duration τ. The resulting function after these two steps can be considered an approximation for the actual solution of (1) at time τ, and iterating these steps achieves a discrete-time approximation at times $t_n = n\tau, n \geq 1$.

By constructing a suitable time interpolation between the discrete time approximations, convergence as $\tau \to 0$ should then be investigated in the anticipation that in the limit, the approximation converges to a solution of (1).

Our goal is thus as follows: Combine a Wasserstein gradient flow formulation of the transport equation together with solutions of the fractional heat equation via splitting, and prove, in the limit as the time step τ goes to 0, that we recover a solution of the original PDE (1).

We suspect that, for the PDE (1), there might be other means of proving existence of solutions. Indeed, Duhamel's principle and a fixed point method may be combined to deduce existence of a ('mild') solution (see e.g. [3]). We therefore emphasize that the method, rather than the end result, is the main interest here. Our next goal is to apply the same type of splitting scheme to more involved PDE, e.g., a non-linear PDE containing a Wasserstein gradient flow part and a non-Wasserstein part.

Finally, we warn the readers that due to the compact format of this paper, some proofs are only sketched, and the bibliography has been shortened; for more details on the proofs and for a more exhaustive bibliography, we refer to [1] and the references therein.

1.1. Assumptions on Initial Data and Potential

In the sequel, we impose the following assumptions on the initial data and potential.

(A1) $\rho^0 \in \mathcal{P}_a^2 \cap L^p(\mathbb{R}^d)$ for some $1 < p \leq \infty$, $\int_{\mathbb{R}^d} \rho^0 \Psi \,\mathrm{d}x < \infty$.

(A2) $\Psi \in C^{1,1} \cap C^{2,1}(\mathbb{R}^d)$, $\Psi \geq 0$.

2. The Fractional Heat Equation

We start by recalling some of the properties of the fractional Laplacian. The fractional Laplacian is defined on Schwartz functions $f \in \mathcal{S}(\mathbb{R}^d)$ through the Fourier transform by $(-\Delta)^s f(x) := \mathcal{F}^{-1}\left(|\cdot|^{2s}\mathcal{F}[f]\right)(x)$, $s \in (0,1)$, where we use the Fourier definition $\mathcal{F}[f](\xi) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i\langle x,\xi\rangle} f(x)\,\mathrm{d}x$. The fact that it is a non-local linear operator is explicitly characterized by the equivalence on the space of Schwartz functions of the Fourier definition with a singular integral formulation, $(-\Delta)^s f(x) = -\frac{1}{2} C_{d,s} \int_{\mathbb{R}^d} \frac{f(x+y)+f(x-y)-2f(x)}{|y|^{d+2s}}\,\mathrm{d}y$ where $C_{d,s}$ is an explicitly known function of d and s (see, for instance, [4] for a proof). An important property that we will use in the sequel is an integration by parts for the fractional Laplacian: $\int_{\mathbb{R}^d} [(-\Delta)^s f] g \,\mathrm{d}x = \int_{\mathbb{R}^d} f[(-\Delta)^s g]\,\mathrm{d}x$ whenever these integrals are well-defined.

To obtain solutions to the fractional heat equation $\rho_t = -(-\Delta)^s \rho$, given an initial probability density $u^0(x)$ on $\mathbb{R}^d$, we apply the Fourier transform to the PDE, and deduce that

$$\rho(x,t) = \Phi_s(\cdot,t) * u^0(x)$$

is the solution, where $\Phi_s(x,t) := \frac{1}{(2\pi)^{d/2}} \mathcal{F}^{-1}\left(e^{-t|\cdot|^{2s}}\right)(x)$ is the *fractional* heat kernel. It is not difficult to deduce standard properties for Φ_s, including that $\Phi_s \in C^\infty(\mathbb{R}^d \times (0,\infty))$, $\Phi_s(t)$ is radial, and $\|\Phi_s(t)\|_{L^1(\mathbb{R}^d)} = 1$. Two properties that differentiate Φ_s from the classical Gaussian kernel are given below (the second is a consequence of the first, for which we refer to [2]).

Proposition 2.1. *For every $t > 0$,*

1. *(A two-sided estimate)*
$$C^{-1}\left(t^{-d/(2s)} \wedge \frac{t}{|x|^{d+2s}}\right) \leq \Phi_s(x,t) \leq C\left(t^{-d/(2s)} \wedge \frac{t}{|x|^{d+2s}}\right) \tag{2}$$
for all $x \in \mathbb{R}^d$, where $a \wedge b := \min\{a,b\}$ for $a, b \in \mathbb{R}$.

2. *(Infinite Second Moment) $\int_{\mathbb{R}^d} |x|^2 \Phi_s(x,t)\,\mathrm{d}x = +\infty$ for every $s \in (0,1)$.*

3. Transport as Gradient Flow

The following proposition highlights the Wasserstein gradient flow interpretation of the transport equation $\rho_t = \operatorname{div}(\rho \nabla \Psi(x))$; we refer to [6] for a proof.

Proposition 3.1. *Given $\rho_\tau^{n-1} \in \mathcal{P}_a^2(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \rho_\tau^{n-1} \Psi \, \mathrm{d}x < \infty$, the minimizer ρ_τ^n over all $\rho \in \mathcal{P}_a^2(\mathbb{R}^d)$ of the functional*

$$\rho \mapsto I_{\rho_\tau^{n-1}}[\rho] := \frac{1}{2\tau} W_2(\rho_\tau^{n-1}, \rho)^2 + \int_{\mathbb{R}^d} \rho \Psi \, \mathrm{d}x \tag{3}$$

is given by $\rho_\tau^n = \rho_\tau^{n-1}(x + \tau \nabla \Psi(x)) \det\left(I + \tau D^2 \Psi(x)\right)$, and, for every $\xi \in C_c^\infty(\mathbb{R}^d)$ the following 'error' estimate holds:

$$\left| \int_{\mathbb{R}^d} \frac{\rho_\tau^n - \rho_\tau^{n-1}}{\tau} \xi \, \mathrm{d}x + \int_{\mathbb{R}^d} \rho_\tau^n \nabla \Psi \cdot \nabla \xi \, \mathrm{d}x \right| \leq \frac{1}{2\tau} \left\| D^2 \xi \right\|_{L^\infty(\mathbb{R}^d)} W_2(\rho_\tau^{n-1}, \rho_\tau^n)^2. \tag{4}$$

Remark: We point out that it is possible to solve the transport equation directly by using the method of characteristics, and this will be enough to proceed with our splitting scheme. This in fact suggests that we could generalize our method to other advection fields which are not necessarily gradient and may depend on time. However, we like to think that viewing the transport equation as a gradient flow of the potential energy is a more 'natural' viewpoint of the dynamics, and this is the path we follow here; for further comments, we refer to [1, 6]. Besides, we hope that this interpretation will allow to extend our splitting scheme to other PDE whose gradient flow part cannot be solved explicitly, contrarily to the one we consider here.

4. Construction of Splitting

Having obtained discrete approximate solutions for both 'Wasserstein' and 'non-Wasserstein' parts of (1), we now apply the splitting scheme. Accordingly, we should use the solution of the fractional heat equation as the initial density for the transport gradient flow, in the following way: Evolve ρ^0 by the fractional heat equation to obtain $\Phi_s(\tau) * \rho^0$ after a time τ, and then use this as the initial density in the variational problem (3).

However, we cannot be certain that this initial density $\Phi_s(\tau) * \rho^0$ has finite potential energy, (e.g., consider the potential $\Psi = |x|^2$, along with the fact that $\Phi_s(t)$ has an infinite second moment), which is one of the assumptions in Proposition 3.1. Therefore, we introduce an additional parameter $R > 0$, and starting with $n = 0$, replace $\Phi_s(\tau) * \rho^0$ with the normalized 'approximation'

$$\bar{\rho}_{\tau,R}^1 := \frac{(\Phi_s^\tau 1_{B_R}) * \rho^0}{\left\|\Phi_s^\tau 1_{B_R}\right\|_{L^1(\mathbb{R}^d)}}, \tag{5}$$

where $\Phi_s^\tau(x) := \Phi_s(x, \tau)$. Then we denote $\rho_{\tau,R}^1$ as the minimizer of (3) with the initial density $\bar{\rho}_{\tau,R}^1$. We continue this procedure for every n (substituting ρ^0 with $\rho_{\tau,R}^{n-1}$ in (5) for $n > 1$).

The following lemma gives an estimate for the potential energy of our new approximation.

Lemma 4.1. *Define $\bar{\rho}_{\tau,R}^1$ by (5). Then*

$$\int_{\mathbb{R}^d} \bar{\rho}_{\tau,R}^1 \Psi \, \mathrm{d}x \leq \int_{\mathbb{R}^d} \rho_{\tau,R}^0 \Psi \, \mathrm{d}x + \frac{\left\|D^2 \Psi\right\|_{L^\infty(\mathbb{R}^d)}}{2} \frac{\int_{B_R} |y|^2 \Phi_s^\tau(y) \, \mathrm{d}y}{\int_{B_R} \Phi_s^\tau(y) \, \mathrm{d}y} < \infty. \tag{6}$$

Proof. By definition, $\int_{\mathbb{R}^d} \bar{\rho}_{\tau,R}^1 \Psi \, \mathrm{d}x = \frac{\int_{\mathbb{R}^d} \Psi(x) \int_{B_R} \rho_{\tau,R}^0(x-y) \Phi_s^\tau(y) \, \mathrm{d}y \, \mathrm{d}x}{\int_{B_R} \Phi_s^\tau(y) \, \mathrm{d}y}$. For the numerator, the substitution $z = x - y$ gives $\int_{B_R} \Phi_s^\tau(y) \int_{\mathbb{R}^d} \Psi(y+z) \rho_{\tau,R}^0(z) \, \mathrm{d}z \, \mathrm{d}y$. Substitution of a finite Taylor expansion $\Psi(y+z) = \Psi(z) + y \cdot \nabla \Psi(z) + \frac{1}{2} y^t D^2 \Psi(\xi_{y,z}) y$, where $\xi_{y,z}$ is some intermediate point on the line joining y and z, and using the fact that the Hessian of Ψ is bounded and $\int_{B_R} y \Phi_s^\tau(y) \, \mathrm{d}y = 0$ since Φ_s^τ is radial, we obtain (6). □

The estimate (6) is valid for $n > 1$, that is for $\bar{\rho}_{\tau,R}^n$. We thus have the following algorithm for the splitting scheme.

Set $\rho_{\tau,R}^0 := \rho^0$. *For* $n = 0, \dots, N-1$,

1. *Define* $\bar{\rho}_{\tau,R}^{n+1} := \frac{(\Phi_s^\tau 1_{B_R}) * \rho_{\tau,R}^n}{\|\Phi_s^\tau 1_{B_R}\|_{L^1(\mathbb{R}^d)}}$.
2. *Set* $\rho_{\tau,R}^{n+1}(x) = \bar{\rho}_{\tau,R}^{n+1}(x + \tau \nabla \Psi(x)) \det\left(I + \tau D^2 \Psi(x)\right)$.

5. Time-Dependent Approximation and Convergence

Through the scheme outlined above, we obtain a discrete-time sequence $\{\rho^k_{\tau,R}\}_{0\le k\le N}$. To study convergence, we construct the time-interpolation

$$\rho_{\tau,R}(t) := \Phi_s(t-t_n) * \rho^n_{\tau,R}, \quad t\in[t_n,t_{n+1}) \tag{7}$$

which we note is a solution of the fractional heat equation on $[t_n,t_{n+1}]$ with initial condition $\rho^n_{\tau,R}$. It is in general only right-continuous at $t=t_n$, and we define $\tilde\rho^{n+1}_{\tau,R} := \lim_{t\uparrow t_{n+1}}\rho_{\tau,R}(t)$ for later use. We need to specify an appropriate notion of convergence, and an appropriate notion of a solution to (1). We do the latter first.

Definition 5.1. *(Weak Solution) Let $T<\infty$. We say that $\rho=\rho(x,t):\mathbb{R}^d\times[0,T)\to\mathbb{R}$ is a weak solution to (1) if,*

1. *For every $\varphi\in C_c^\infty(\mathbb{R}^d\times\mathbb{R})$ with time support in $[-T,T]$,*

$$\int_{\mathbb{R}^d}\int_0^T \rho(t)\left[\partial_t\varphi(t)-(-\Delta)^s\varphi(t)-\nabla\Psi\cdot\nabla\varphi(t)\right]\,\mathrm{d}x\,\mathrm{d}t+\int_{\mathbb{R}^d}\rho^0\varphi(0)\,\mathrm{d}x=0, \tag{8}$$

2. *$\rho(x,t)\ge 0$ for a.e. $(x,t)\in\mathbb{R}^d\times(0,T)$,*

3. *$\int_{\mathbb{R}^d}\rho(t)\,\mathrm{d}x=\int_{\mathbb{R}^d}\rho^0\,\mathrm{d}x=1$ for a.e. $t\in(0,T)$.*

The following lemma tells us how 'close' the interpolation $\rho_{\tau,R}$ is to being a weak solution of (1).

Lemma 5.2. *(The Approximate Equation Satisfied by the Time Interpolation) Let $\varphi\in C_c^\infty(\mathbb{R}^d\times\mathbb{R})$ with time support in $[-T,T]$. Then*

$$\int_0^T\int_{\mathbb{R}^d}\rho_{\tau,R}(t)\left[\partial_t\varphi(t)-(-\Delta)^s\varphi(t)-\nabla\Psi\cdot\nabla\varphi(t)\right]\,\mathrm{d}x\,\mathrm{d}t+\int_{\mathbb{R}^d}\rho^0\varphi(0)\,\mathrm{d}x=\mathcal{R}(\tau,R)$$

where $\mathcal{R}(\tau,R):=E_1+E_2+E_3+E_4$, where

$$E_1:=\sum_{n=0}^{N-1}\int_{\mathbb{R}^d}\left(\tilde\rho^{n+1}_{\tau,R}-\bar\rho^{n+1}_{\tau,R}\right)\varphi(t_{n+1})\,\mathrm{d}x,\quad E_2:=\sum_{n=0}^{N-1}\left[\int_{\mathbb{R}^d}\left(\bar\rho^{n+1}_{\tau,R}-\rho^{n+1}_{\tau,R}\right)\varphi(t_{n+1})\,\mathrm{d}x-\tau\int_{\mathbb{R}^d}\rho^{n+1}_{\tau,R}\nabla\Psi\cdot\nabla\varphi(t_{n+1})\,\mathrm{d}x\right]$$

$$E_3:=-\sum_{n=1}^{N-1}\left[\int_{t_n}^{t_{n+1}}\int_{\mathbb{R}^d}\left[\rho_{\tau,R}(t)\nabla\Psi\cdot\nabla\varphi(t)-\rho^n_{\tau,R}\nabla\Psi\cdot\nabla\varphi(t_n)\right]\,\mathrm{d}x\,\mathrm{d}t\right],\quad E_4:=-\int_0^\tau\int_{\mathbb{R}^d}\Phi_s(t)*\rho^0\nabla\Psi\cdot\nabla\varphi(t)\,\mathrm{d}x\,\mathrm{d}t.$$

Proof. Integrate $\int_{t_n}^{t_{n+1}}\int_{\mathbb{R}^d}\rho_\tau(t)\partial_t\varphi(t)\,\mathrm{d}x\,\mathrm{d}t$ by parts in t, use the fact that ρ_τ solves the fractional heat equation on (t_n,t_{n+1}), and then integrate by parts placing $(-\Delta)^s$ on $\varphi(t)$. The rest follows from addition and subtraction of like terms, recalling $\rho_{\tau,R}(t_n)=\rho^n_{\tau,R}$, and $\tilde\rho^{n+1}_{\tau,R}=\lim_{t\uparrow t_{n+1}}\rho_{\tau,R}(t)$. □

Lemma 5.3. *The remainder $\mathcal{R}(\tau,R)$ satisfies $|\mathcal{R}(\tau,R)|\le C\left(\tau+R^{-2s}+\sum_{n=0}^{N-1}W_2(\bar\rho^{n+1}_{\tau,R},\rho^{n+1}_{\tau,R})^2\right)$.*

Proof. We first estimate E_1. Write

$$|E_1|\le\sum_{n=0}^{N-1}\left|\int_{\mathbb{R}^d}\left(\tilde\rho^{n+1}_{\tau,R}-\bar\rho^{n+1}_{\tau,R}\right)\varphi(t_{n+1})\right|\le\sup_{t\in[0,T]}\|\varphi(t)\|_{L^\infty(\mathbb{R}^d)}\sum_{n=0}^{N-1}\left\|\tilde\rho^{n+1}_{\tau,R}-\bar\rho^{n+1}_{\tau,R}\right\|_{L^1(\mathbb{R}^d)}.$$

By substituting the definitions $\bar\rho^{n+1}_{\tau,R}=\rho^n_{\tau,R}*(\Phi^\tau_s 1_{B_R})/\left\|\Phi^\tau_s 1_{B_R}\right\|_{L^1(\mathbb{R}^d)}$ and $\tilde\rho^{n+1}_{\tau,R}=\rho^n_{\tau,R}*\Phi^\tau_s$ into the L^1 norm above, we obtain the estimate

$$\sum_{n=0}^{N-1}\int_{\mathbb{R}^d}\left|\tilde\rho^{n+1}_{\tau,R}-\bar\rho^{n+1}_{\tau,R}\right|\,\mathrm{d}x\le 2\sum_{n=0}^{N-1}\int_{\mathbb{R}^d\setminus B_R}\Phi^\tau_s(y)\,\mathrm{d}y\le\sum_{n=0}^{N-1}\frac{C\tau}{sR^{2s}}=\frac{CT}{sR^{2s}} \tag{9}$$

provided $R \geq \tau^{1/[2s(1+2s/d)]} \geq \tau^{1/(2s)}$ (so that using $\tau < 1$, $\Phi_s^\tau(x) \leq \frac{C\tau}{|x|^{d+2s}}\ \forall x \in \mathbb{R}^d \backslash B_R$ by (2)). Next we have, using (4),

$$|E_2| \leq \sup_{t\in[0,T]} \left\|D^2\varphi(t)\right\|_{L^\infty(\mathbb{R}^d)} \frac{1}{2}\sum_{n=0}^{N-1} W_2(\bar{\rho}_{\tau,R}^{n+1}, \rho_{\tau,R}^{n+1})^2. \tag{10}$$

We can estimate E_4 by noting that

$$|E_4| \leq \tau \sup_{t\in[0,T]} \|\nabla\Psi \cdot \nabla\varphi(t)\|_{L^\infty(\mathbb{R}^d)}. \tag{11}$$

Now, we write the integrand in E_3 as

$$\rho_{\tau,R}(t)\nabla\Psi\cdot\nabla\varphi(t) - \rho_{\tau,R}^n\nabla\Psi\cdot\nabla\varphi(t_n) = \left(\rho_{\tau,R}(t) - \rho_{\tau,R}^n\right)\nabla\Psi\cdot\nabla\varphi(t) + \rho_{\tau,R}^n\nabla\Psi\cdot\nabla\left[\varphi(t) - \varphi(t_n)\right]. \tag{12}$$

Since $\rho_{\tau,R}(t) - \rho_{\tau,R}^n = -\int_{t_n}^t (-\Delta)^s\rho_{\tau,R}(u)\,\mathrm{d}u$, an integration by parts with $(-\Delta)^s$ gives

$$\int_{t_n}^{t_{n+1}}\int_{\mathbb{R}^d}\left(\rho_{\tau,R}(t) - \rho_{\tau,R}^n\right)\nabla\Psi\cdot\nabla\varphi(t)\,\mathrm{d}x\,\mathrm{d}t = -\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}\int_{\mathbb{R}^d}\rho_{\tau,R}(u)(-\Delta)^s\left[\nabla\Psi\cdot\nabla\varphi(t)\right]\,\mathrm{d}x\,\mathrm{d}u\,\mathrm{d}t.$$

Since $\nabla\Psi\cdot\nabla\varphi(t) \in C_c^{1,1}(\mathbb{R}^d)$, we have that its fractional Laplacian is bounded in $\mathbb{R}^d$ (see e.g. [8]). Therefore

$$\begin{aligned}\sum_{n=0}^{N-1}\left|\int_{t_n}^{t_{n+1}}\int_{\mathbb{R}^d}\left(\tilde{\rho}_{\tau,R}^{n+1} - \rho_{\tau,R}(t)\right)\nabla\Psi\cdot\nabla\varphi(t)\,\mathrm{d}x\,\mathrm{d}t\right| &\leq \sup_{t\in[0,T]}\left\|(-\Delta)^s\left[\nabla\Psi\cdot\nabla\varphi(t)\right]\right\|_{L^\infty(\mathbb{R}^d)}\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}\int_{\mathbb{R}^d}\rho_{\tau,R}(u)\,\mathrm{d}x\,\mathrm{d}u\,\mathrm{d}t\\ &\leq \sup_{t\in[0,T]}\left\|(-\Delta)^s\left[\nabla\Psi\cdot\nabla\varphi(t)\right]\right\|_{L^\infty(\mathbb{R}^d)}\frac{1}{2}T\tau. \end{aligned}\tag{13}$$

For the second term in (12), we Taylor expand $\varphi(t) - \varphi(t_n) = \partial_t\varphi(t_\lambda)(t - t_n)$ for some $t_\lambda \in (t_n, t_{n+1})$. Then

$$\begin{aligned}\sum_{n=0}^{N-1}\left|\int_{t_n}^{t_{n+1}}\int_{\mathbb{R}^d}\rho_{\tau,R}^n\nabla\Psi\cdot\nabla\left[\varphi(t) - \varphi(t_n)\right]\,\mathrm{d}x\,\mathrm{d}t\right| &\leq \sup_{t\in[0,T]}\|\nabla\Psi\cdot\nabla\partial_t\varphi(t)\|_{L^\infty(\mathbb{R}^d)}\left\|\rho_{\tau,R}^n\right\|_{L^1(\mathbb{R}^d)}\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}}(t - t_n)\,\mathrm{d}t\\ &\leq \sup_{t\in[0,T]}\|\nabla\Psi\cdot\nabla\partial_t\varphi(t)\|_{L^\infty(\mathbb{R}^d)}\frac{1}{2}T\tau.\end{aligned}\tag{14}$$

Combining (9), (10), (11), (13) and (14), we obtain the desired estimate for $\mathcal{R}(\tau, R)$. ☐

Lemma 5.4. $\sum_{n=1}^N W_2(\bar{\rho}_{\tau,R}^n, \rho_{\tau,R}^n)^2 \leq C\left[\tau\int_{\mathbb{R}^d}\Psi\rho^0\,\mathrm{d}x + T\left\|D^2\Psi\right\|_{L^\infty(\mathbb{R}^d)}\left(\tau^{1/s} + \tau R^{2-2s}\right)\right]$.

Proof. By optimality of $\rho_{\tau,R}^n$ in Problem (3), (formulated with $\bar{\rho}_{\tau,R}^n$ in place of ρ_τ^{n-1}), the following holds:

$$\frac{1}{2\tau}W_2(\bar{\rho}_{\tau,R}^n, \rho_{\tau,R}^n)^2 \leq \int_{\mathbb{R}^d}\Psi\bar{\rho}_{\tau,R}^n\,\mathrm{d}x - \int_{\mathbb{R}^d}\Psi\rho_{\tau,R}^n\,\mathrm{d}x. \tag{15}$$

We appeal to estimate (6) (with $\bar{\rho}_{\tau,R}^n$ in place of $\bar{\rho}_{\tau,R}^1$ and $\rho_{\tau,R}^{n-1}$ in place of $\rho_{\tau,R}^0$), for a bound on the potential energy of $\bar{\rho}_{\tau,R}^n$. To estimate the term in (6) involving Φ_s^τ, we use (2) to find

$$\frac{\int_{B_R}|y|^2\Phi_s^\tau(y)\,\mathrm{d}y}{\int_{B_R}\Phi_s^\tau(y)\,\mathrm{d}y} \leq C\frac{\int_0^{\tau^{\frac{1}{2s}}} r^2\tau^{-\frac{d}{2s}}r^{d-1}\,\mathrm{d}r + \int_{\tau^{\frac{1}{2s}}}^R r^2\frac{\tau}{r^{d+2s}}r^{d-1}\,\mathrm{d}r}{\int_0^{\tau^{\frac{1}{2s}}}\tau^{-\frac{d}{2s}}r^{d-1}\,\mathrm{d}r + \int_{\tau^{\frac{1}{2s}}}^R\frac{\tau}{r^{d+2s}}r^{d-1}\,\mathrm{d}r} \leq C\left[\tau^{1/s} + \tau R^{2-2s}\right], \tag{16}$$

whence upon by substitution of (16) back into (6), and the resulting inequality into (15), summing over n, and recalling $N = \frac{T}{\tau}$ and $\Psi \geq 0$, gives us the desired inequality. ☐

Corollary 5.5. *Set $R = \tau^{-1/2}$. Then $\{\rho_{\tau,R}\}$ is a sequence in τ, which we denote by $\{\rho_\tau\}$, $\left|\mathcal{R}(\tau, \tau^{-1/2})\right| \leq C\left(\tau + \tau^s + \tau^{1/s}\right)$, and therefore $\lim_{\tau\to0}\int_0^T\int_{\mathbb{R}^d}\rho_\tau(t)\left[\partial_t\varphi(t) - (-\Delta)^s\varphi(t) - \nabla\Psi\cdot\nabla\varphi(t)\right]\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^d}\rho^0\varphi(0)\,\mathrm{d}x = 0$ for all $\varphi \in C_c^\infty(\mathbb{R}^d\times\mathbb{R})$ with time support in $[-T, T]$.*

Proof. By Lemmas 5.3 and 5.4, $|\mathcal{R}(\tau, R)| \leq C\left(\tau + R^{-2s} + \tau^{1/s} + \tau R^{2-2s}\right) = C\left(\tau + \tau^s + \tau^{1/s}\right) \to 0$ as $\tau \to 0$. ☐

Lemma 5.6. *Let $\tau > 0$ be small enough so that $\det(I + \tau D^2\Psi(x)) \leq 1 + \alpha\tau$, for some fixed $\alpha > \|D^2\Psi\|_{L^\infty(\mathbb{R}^d)}$. If $\rho^0 \in L^p(\mathbb{R}^d)$ for $1 < p \leq \infty$, then $\|\rho_\tau(t)\|_{L^p(\mathbb{R}^d)}$ is uniformly bounded for $t \in (0, T)$.*

Proof. Suppose $1 < p < \infty$. We have $\left\|\rho_\tau^n\right\|^p_{L^p(\mathbb{R}^d)} \leq (1 + \tau\alpha)^{p-1}\left\|\bar{\rho}_\tau^n\right\|^p_{L^p(\mathbb{R}^d)} \leq (1 + \tau\alpha)^{p-1}\left\|\rho_\tau^{n-1}\right\|^p_{L^p(\mathbb{R}^d)}$ by applying the definition of $\rho^n_{\tau,R}$ as the transport of $\bar{\rho}^n_\tau$, and then using Young's convolution inequality. Thus $\left\|\rho_\tau^n\right\|^p_{L^p(\mathbb{R}^d)} \leq (1 + \tau\alpha)^{n(p-1)}\left\|\rho^0\right\|^p_{L^p(\mathbb{R}^d)} \leq e^{\alpha(p-1)T}\left\|\rho^0\right\|^p_{L^p(\mathbb{R}^d)}$. Then for $1 < p < \infty$ and $t \in (t_n, t_{n+1})$, $\|\rho_\tau(t)\|_{L^p(\mathbb{R}^d)} \leq \left\|\rho_\tau^n\right\|_{L^p(\mathbb{R}^d)} \leq e^{\alpha T\left(\frac{p-1}{p}\right)}\left\|\rho^0\right\|_{L^p(\mathbb{R}^d)} < \infty$, where Young's convolution inequality was used in the first inequality. Finally, let $p \to \infty$ in the previous estimate to conclude that $\|\rho_\tau(t)\|_{L^\infty(\mathbb{R}^d)} \leq e^{\alpha T}\left\|\rho^0\right\|_{L^\infty(\mathbb{R}^d)} < \infty$ for all $t \in (0, T)$. ☐

Lemma 5.7. *There exists a non-relabelled subsequence $\{\rho_\tau\}_{\tau\downarrow 0}$ and a $\rho \in L^1 \cap L^p(\mathbb{R}^d \times (0, T))$ such that $\rho_\tau \rightharpoonup \rho$ in $L^p(\mathbb{R}^d \times (0, T))$ (or $\rightharpoonup^*$ if $p = \infty$), and ρ satisfies (8). Moreover, $\rho \geq 0$ a.e. $(x, t) \in \mathbb{R}^d \times (0, T)$.*

Proof. By Lemma 5.6, we deduce the existence of a $\rho \in L^p(\mathbb{R}^d \times (0, T))$ such that $\rho_\tau \rightharpoonup \rho$ in $L^p(\mathbb{R}^d \times (0, T))$ for a non-relabelled subsequence. Then by appealing to Corollary 5.5, we have that (8) is satisfied. It follows by definition of ρ_τ and the weak convergence to ρ, that ρ also belongs to $L^1(\mathbb{R}^d \times (0, T))$. Moreover since $\rho^0 \geq 0$, then $\rho_\tau(t) \geq 0$ by construction, and the weak convergence also implies the nonnegativity of $\rho(x, t)$ for a.e. $(x, t) \in \mathbb{R}^d \times (0, T)$. ☐

Lemma 5.8. *Suppose there exists $\rho \in L^1(\mathbb{R}^d \times (0, T))$ satisfying (8). Then $\int_{\mathbb{R}^d} \rho(t)\, \mathrm{d}x = \int_{\mathbb{R}^d} \rho^0\, \mathrm{d}x = 1$ for a.e. $t \in (0, T)$.*

Proof. Let $\eta : [0, \infty) \to [0, \infty)$ be C^∞ and satisfies $\eta(r) = 1$ for $r \leq 1$ and $\eta(r) = 0$ for $r > 2$, and let $\eta_R \in C_c^\infty(\mathbb{R}^d)$ be defined by $\eta_R(x) = \eta\left(\frac{|x|}{R}\right)$ for $R > 0$. Then for all $\theta \in C_c^\infty(-T, T)$, we choose $\varphi(t) = \theta(t)\eta_R$ in (8) to obtain

$$\int_0^T \theta'(t)\left[\int_{\mathbb{R}^d}\left(\rho(t) - \rho^0\right)\eta_R\, \mathrm{d}x\right]\mathrm{d}t = \int_0^T \theta(t)\int_{\mathbb{R}^d}\rho(t)\left[(-\Delta)^s\eta_R + \nabla\Psi\cdot\nabla\eta_R\right]\mathrm{d}x\,\mathrm{d}t. \tag{17}$$

It is not difficult to see that $\lim_{R\to\infty}(-\Delta)^s\eta_R(x) = 0$ and $\lim_{R\to\infty}\nabla\Psi(x)\cdot\nabla\eta_R(x) = 0$ pointwise on $\mathbb{R}^d$ and that by the dominated convergence theorem we may pass these limits inside the integrals in the above display. Then we obtain in the limit $R \to \infty$,

$$\int_0^T \theta'(t)\int_{\mathbb{R}^d}\rho(x, t)\,\mathrm{d}x\,\mathrm{d}t + \theta(0)\int_{\mathbb{R}^d}\rho^0(x)\,\mathrm{d}x = 0, \tag{18}$$

for every such $\theta \in C_c^\infty(-T, T)$. In particular, for every $\gamma \in C_c^\infty(0, T)$ we have $\int_0^T \gamma'(t)\int_{\mathbb{R}^d}\rho(x, t)\,\mathrm{d}x\,\mathrm{d}t = 0$, from which we deduce that $\int_{\mathbb{R}^d}\rho(x, t)\,\mathrm{d}x = C$, a.e $t \in (0, T)$. Substitution of this equality back in (18) implies $C = \int_{\mathbb{R}^d}\rho^0(x)\,\mathrm{d}x = 1$. ☐

5.1. The Main Result

Combining the results obtained in the previous sections, we now can show that the constructed splitting scheme weakly converges in L^p to a weak solution of (1).

Theorem 5.9. *Assume ρ^0 and Ψ satisfy the above assumptions (A1) and (A2). Define the sequence $\{\rho_\tau\}_{\tau\downarrow 0}$ according to (7) with $R = \tau^{-1/2}$. Then there exists a non-relabeled subsequence $\{\rho_\tau\}_{\tau\downarrow 0}$ and a $\rho \in L^1 \cap L^p(\mathbb{R}^d \times (0, T))$, such that $\rho_\tau \rightharpoonup \rho$, weakly in $L^p(\mathbb{R}^d \times (0, T))$ -(or $\rightharpoonup^*$ if $p = \infty$)-, where ρ is a weak solution of (1) according to Definition 5.1.*

Proof. By Lemma 5.7 and Lemma 5.8, we deduce ρ is a weak solution to (1), in the sense of Definition 5.1. ☐

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