Reachability Analysis for Systems Affine in the Time Varying Bounded Uncertainties

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Problem formulation

The problem

Initial Value Problem

$$\begin{cases} \dot{x} = f(t, x(t), u(t)) \\ x(t_0) \in \mathcal{X} \end{cases} \tag{1}$$

Affine in the uncertainties

$$f(t, x, u) = h_0(x, t) + \sum_{i=1}^{m} u_i h_i(x, t) \tag{2}$$

Immune Virus System

$$\begin{cases} \dot{x}(t) & = & \Lambda(t) - \gamma_1 x(t) - \beta x(t) v(t) \\ \dot{y}(t) & = & \beta x(t) v(t) - \gamma_2 y(t) \\ \dot{v}(t) & = & \kappa y(t) - \alpha(t) v(t) \end{cases} \tag{3}$$

The problem

Initial Value Problem

$$\begin{cases} \dot{x} = f(t, x(t), u(t)) \\ x(t_0) \in \mathcal{X} \end{cases} \tag{1}$$

Affine in the uncertainties

$$f(t, x, u) = h_0(x, t) + \sum_{i=1}^{m} u_i h_i(x, t) \tag{2}$$

Immune Virus System

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{v}(t) \end{pmatrix} = \begin{pmatrix} -\gamma_1 x(t) - \beta x(t) v(t) \\ \beta x(t) v(t) - \gamma_2 y(t) \\ \kappa y(t) \end{pmatrix} + \Lambda(t) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \alpha(t) \begin{pmatrix} 0 \\ 0 \\ -v(t) \end{pmatrix} \tag{3}$$

Problem hypotheses

Hypotheses

$$\begin{cases} \dot{x} = f(t, x(t), u(t)) = h_0(x(t), t) + \sum_{i=1}^{m} u_i(t) h_i(x(t), t) \\ x(0) \in \mathcal{X} \\ u(t) \in \mathcal{U} \end{cases} \tag{4}$$

with

- $\mathcal{X} \subset \mathbb{R}^n$ bounded
- $\mathcal{U} \subset \mathbb{R}^m$ close convex bounded
- $t \mapsto u(t)$ measurable

Carathéodory, for all $u : [0, T] \to \mathcal{U}$:

- There exists $m : [0, T] \to \mathbb{R}_+$ such that $\forall (t, x),\ \|f(t, x, u(t))\| \leq m(t)$
- There exists $k : [0, T] \to \mathbb{R}_+$ such that $\forall (t, x_1, x_2),\ \|f(t, x_1, u(t)) - f(t, x_2, u(t))\| \leq k(t) \|x_1 - x_2\|$

Difference with other tools

Other tools use at least Riemann-integrable uncertainties:

Flow*: uncertainties are assumed continuous

CORA: uncertainties are assumed Riemann-integrable

Enclosure of the solution

Inclusion using Lebesgue-integration

Lemma

Let a measurable function $u : [0, T] \to \mathcal{U}$ with a closed convex bounded set $\mathcal{U} \subset \mathbb{R}$. Let a Lesbegue-integrable function g with a decomposition in positive functions: $g = g^+ - g^-$. Then

$$\int_0^T g(s)u(s)\,ds \in \left\{ u_1 \int_0^T g^+(s)\,ds - u_2 \int_0^T g^-(s)\,ds \;\middle|\; u_1 \in \mathcal{U},\; u_2 \in \mathcal{U} \right\} \quad (5)$$

New operator

Parametric operator

$$\mathbb{P}_{(u_i)}(p) := t \mapsto x_0 + \int_0^t h_0(p(s), s)\, ds + \sum_{i=1}^{m} u_{2i-1} \int_0^t h_i^+(p(s), s)\, ds - \sum_{i=1}^{m} u_{2i} \int_0^t h_i^-(p(s), s)\, ds \quad (6)$$

Global operator

$$\mathbb{P}(p) := t \mapsto \left\{ \mathbb{P}_{(u_i)}(p)(t) \;\middle|\; \forall i \in [\![1, 2m]\!],\ u_i \in \mathcal{U} \right\} \quad (7)$$

Fixed-point theorem

Theorem

Let x_0 an initial state and let $\mathcal{U}$ a closed bounded convex set of possible values of the inputs. If there exists a set of functions φ such that for all $t \in [0, T]$:

$$\mathbb{P}\left(\varphi\right)\left(t\right) \subset \varphi(t) \tag{8}$$

then for all $t \in [0, T]$, $\varphi(t)$ is an over-approximation of the reachable set at time t with the initial state x_0.

Application

Algorithm to compute an over-approximation

Algorithm

We use Taylor Models as sets representations and we replace all uncertainties by a centered Taylor Model: $u(t) \in [a, b]$ becomes $\mathrm{TM}\left(\frac{a+b}{2}, \left[\frac{a-b}{2}, \frac{b-a}{2}\right]\right)$.

1. Compute a raw enclosure of the solution
2. Decompose the functions $h_{i \geq 1}(x(t), t)$ as difference of positive functions using the raw enclosure
3. Compute the polynomial expansion up to the expected order
4. Find a valid remainder

Decomposition

Affine decomposition

Assume for all $t \in [0, T]$, $h_i(x(t), t) \in [a, b]$ and $a < 0 < b$. We define

$$\begin{cases} h_i^+(x(t), t) = \frac{b}{b-a} h_i(x(t), t) - \frac{ab}{b-a} \\ h_i^-(x(t), t) = \frac{a}{b-a} h_i(x(t), t) - \frac{ab}{b-a} \end{cases} \tag{9}$$

and we have $h_i(x(t), t) = h_i^+(x(t), t) - h_i^-(x(t), t)$.

Optimality

This decomposition minimizes $\left\| h_i^+ \right\|_1 + \left\| h_i^- \right\|_1$.

Examples

Example 1

Simple dynamics

$$\dot{x}(t) = (0.1 - t)u(t) \tag{10}$$

with $x(0) = 0$ and $\forall t \in [0, 0.2],\ u(t) \in [-1, 1]$

Case: u constant
If u is constant ($u(t) = u(0) \in [-1, 1]$), then $x(0.2) = 0$ and $x(t) \in [-0.005, 0.005]$.

Exact reachable set
The exact reachable set at time $t = 0.2$ is $x(0.2) \in [-0.01, 0.01]$.

Example 1: Decomposition

Decomposition

For all $t \in [0, 0.2]$, $h_1(x(t), t) = (0.1 - t) \in [-0.1, 0.1]$.
We deduce $h_1(x, t) = h_1^+(x, t) - h_1^-(x, t)$ with

$$\left\{ \begin{array}{l} h_1^+(x, t) = 0.1 - 0.5t \\ h_1^-(x, t) = 0.5t \end{array} \right. \tag{11}$$

Equivalent dynamics

The dynamics becomes

$$\dot{x}(t) = (0.1 - 0.5t)\, u(t) - (0.5t)\, u(t) \tag{12}$$

Example 1: Over-approximation

We replace all occurrences of $u(t)$ by $\mathrm{TM}\left(0,\ [-1,1]\right)$.
We start with an expansion to the order 0 in time:
$\varphi_0(x_0,t) = \mathrm{TM}\left(x_0,\ [0]\right)$.
Then, to expected an higher order expansion, we iterate
$\varphi_{n+1} = \mathbb{P}\left(\varphi_n\right)$:

$$
\begin{aligned}
\varphi_1(x_0,t) = {} & \mathrm{TM}\left(x_0,\ [0]\right) + \mathrm{TM}\left(0,\ [-1,1]\right) \int_0^t h_1^+\left(\varphi_0(x_0,s),s\right) ds \\
& \quad - \mathrm{TM}\left(0,\ [-1,1]\right) \int_0^t h_1^-\left(\varphi_0(x_0,s),s\right) ds \\
= {} & \mathrm{TM}\left(x_0,\ [0]\right) + \mathrm{TM}\left(0,\ [-1,1]\right) \cdot \mathrm{TM}\left(0.1t - 0.25t^2,\ [0]\right) \\
& \quad - \mathrm{TM}\left(0,\ [-1,1]\right) \cdot \mathrm{TM}\left(0.25t^2,\ [0]\right) \\
= {} & \mathrm{TM}\left(x_0,\ [0]\right) + \mathrm{TM}\left(0,\ [-0.01,0.01]\right) \\
& \quad - \mathrm{TM}\left(0,\ [-0.01,0.01]\right) \\
= {} & \mathrm{TM}\left(x_0,\ [-0.02,0.02]\right)
\end{aligned}
$$

Example 1: Over-approximation

Result

For all $t \in [0, 0.2]$, we have $x(t) \in [-0.02, 0.02]$.

(Remind) Exact reachable set

The exact reachable set at time $t = 0.2$ is $x(0.2) \in [-0.01, 0.01]$.

Decreasing exponential

Dynamics

$$\dot{x}(t) = -u(t)x(t) \tag{13}$$

with $x(0) \in [1, 1.1]$ and $\forall t, \; u(t) \in [1, 2]$.

Decreasing exponential

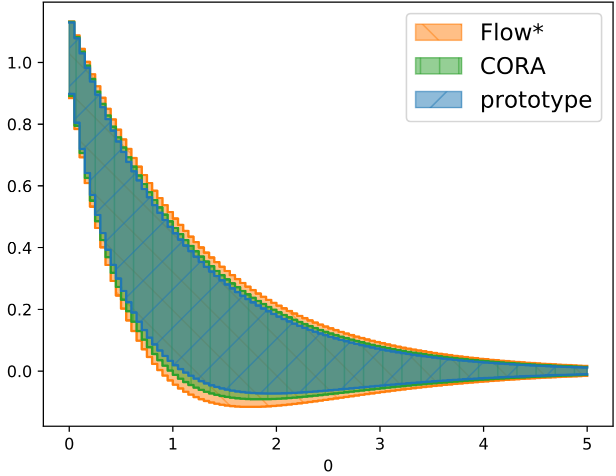

Figure: Over-approximations with fixed time-step equals to 0.05

Nonlinear perturbation

Dynamics

$$\begin{cases} \dot{x}(t) = -x(t) + x(t)y(t)u(t) \\ \dot{y}(t) = -y(t) \end{cases} \tag{14}$$

with $x(0) = 1$, $y(0) = 2$ and $u(t) \in [-1, 1]$.

Nonlinear perturbation

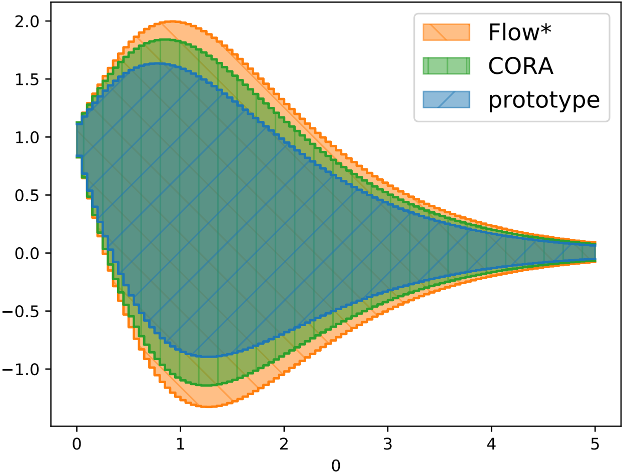

Figure: Over-approximations with fixed time-step equals to 0.05

Conclusion

Summary

- able to handle measurable bounded uncertainties
- promising results on simple examples

Futur work

- try different sets representations
- optimize the prototype

Thank you for your attention