On the optimal control of linear complementarity systems

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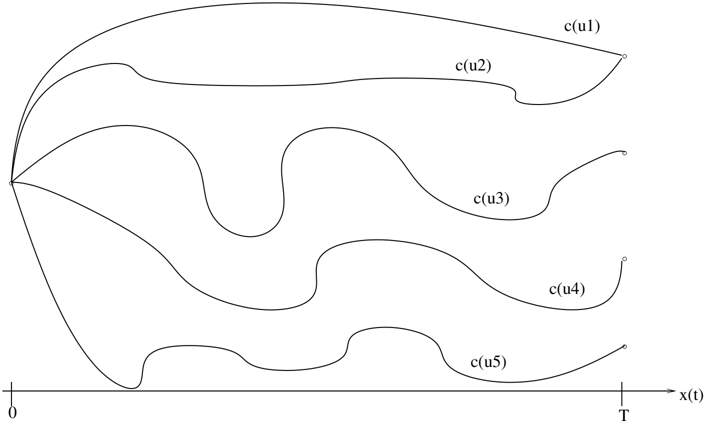
Introduction

$$
\begin{aligned}
\min\ & c(x_u, u) \\
\text{s.t. } & \dot{x}_u(t) = f(x_u(t), u(t)), \\
& (x_u(t), u(t)) \in S(t), \\
& x_u(0) = x_0, \\
& x_u(T) \text{ free.}
\end{aligned}
$$



Introduction

Problem:

$$C(u) = \int_0^T \left(x(t)^\intercal Q x(t) + u(t)^\intercal U u(t) + v(t)^\intercal V v(t)\right) dt \rightarrow \min$$

such that:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bv(t) + Fu(t) \\ 0 \leq v(t) \perp w(t) = Cx(t) + Dv(t) + Eu(t) \geq 0 \\ x(0) = x_0, \; x(T) \text{ free} \end{cases}$$

where $T > 0$, $x : [0, T] \rightarrow \mathbb{R}^n$ absolutely continuous, $v : [0, T] \rightarrow \mathbb{R}^m$, $u : [0, T] \rightarrow \mathbb{R}^{m_u}$, A, B, C, D, E, F, Q, V and U matrices of according dimensions, U supposed symmetric positive definite, Q and V positive semi-definite.

Motivation: Mechanics, Electronic Circuits, Chemical reactions

A difficult problem

$$0 \leq v(t) \perp Cx(t) + Dv(t) + Eu(t) \geq 0$$

- Existence of optimal solution not proved (classical Fillipov theory does not apply here due to lack of convexity). Cesari (2012), Theorem 9.2i and onwards
- Special cases arise when $E = 0$ and D P-matrix : switching modes are activated when the state reaches some threshold defined by the complementarity conditions. Georgescu et al. (2012), Passenberg et al. (2013)
- Since u is also involved $\implies$ mixed constraints; makes use of non-smooth analysis. Clarke and De Pinho (2010)

Direct Method

A first way to compute numerically an approximate solution: direct method.

$$\min \sum_{i=0}^{N} x_i Q x_i + u_i U u_i$$
$$\text{s.t.} \begin{cases} x_{i+1} = x_i + h(Ax_i + Bv_i + Eu_i) \\ 0 \leq v_i \perp Cx_i + Dv_i + Eu_i \geq 0 \\ x_0 \text{ fixed} \end{cases}$$

$\implies$ Mathematical Program with Equilibrium Constraints (MPEC).
Since it is a convex cost with linear constraints: there exist a solution for any fixed h.

Direct Method

2 ways to solve numerically this problem:

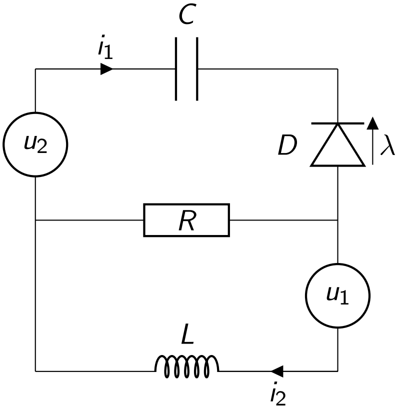
1. Cost penalization on the complementarity constraint $v^{\mathsf{T}}(Cx + Dv + Eu) = 0$: didn't work well on examples.
2. Relaxation of the complementarity: choose a sequence $\varepsilon_k \geq 0$ converging to 0 and create a sequence of optimization problems where the constraint $v^{\mathsf{T}}(Cx + Dv + Eu) = 0$ is replaced by $v^{\mathsf{T}}(Cx + Dv + Eu) \leq \varepsilon_k$: works well.

[1] S. Leyffer, G. López-Calva, and J. Nocedal. Interior methods for mathematical programs with complementarity constraints. SIAM Journal on Optimization, 17(1):52–77, 2006.

[2] C. Kanzow and A. Schwartz. A new regularization method for mathematical programs with complementarity constraints with strong convergence properties. SIAM Journal on Optimization, 23(2):770–798, 2013.

Direct Method

Denote $x_1(t) = \int_0^t i_1(s)ds + x_1(0)$ (the charge of the capacitor, in coulomb) and $x_2(t) = i_2(t)$ (the electric current, in ampere). Then the evolution of this system is described as:



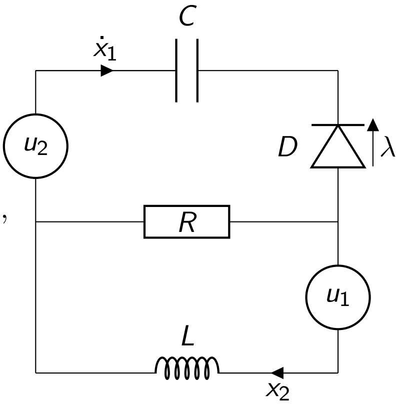
$$\begin{cases} \dot{x}_1(t) = \dfrac{-1}{RC} x_1(t) + x_2(t) - \dfrac{1}{R}\lambda(t) + \dfrac{1}{R} u_2(t), \\ \dot{x}_2(t) = \dfrac{-1}{LC} x_1(t) - \dfrac{1}{L}\lambda(t) + \dfrac{1}{L}(u_2(t) - u_1(t)), \\ 0 \leq \lambda(t) \perp \dfrac{1}{RC} x_1(t) - x_2(t) \\ \qquad\qquad + \dfrac{1}{R}\lambda(t) - \dfrac{1}{R} u_2(t) \geq 0, \end{cases}$$

The constants are chosen as $R = 10\Omega$, $C = 80\ 000\mu$F, and $L = 2$H.

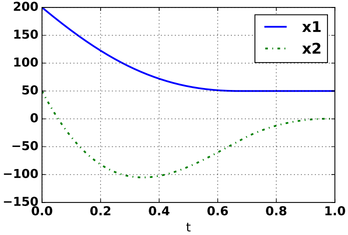
Direct Method

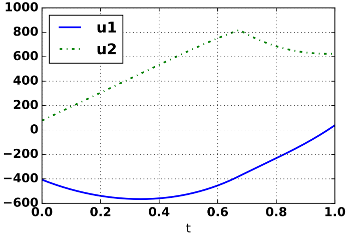
$$\min \int_0^1 [Rx_2(t)^2 + u_1(t)^2 + u_2(t)^2]dt$$

$$\text{s.t.} \begin{cases} \dot{x}_1(t) = \frac{-1}{RC}x_1(t) + x_2(t) - \frac{1}{R}\lambda(t) + \frac{1}{R}u_2(t), \\ \dot{x}_2(t) = \frac{-1}{LC}x_1(t) - \frac{1}{L}\lambda(t) + \frac{1}{L}(u_2(t) - u_1(t)), \\ 0 \leq \lambda(t) \perp \frac{1}{RC}x_1(t) - x_2(t) \\ \qquad + \frac{1}{R}\lambda(t) - \frac{1}{R}u_2(t) \geq 0, \\ x(0) = (200C, 50A) \\ x(1) = (50C, 0A) \end{cases}$$

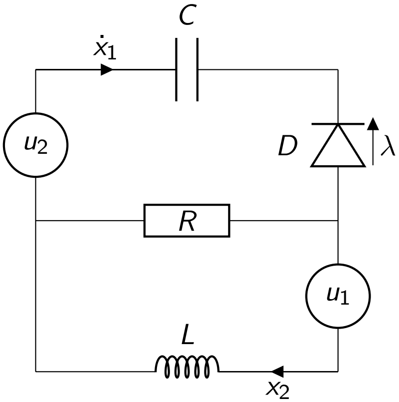


Direct Method





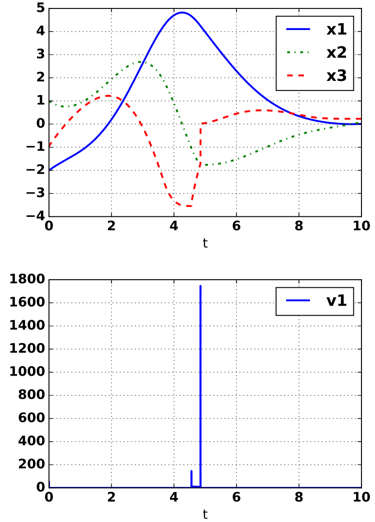
λ always naught!



Direct Method

$$\min \int_0^{10} \left(\|x(t)\|_2^2 + u(t)^2\right) dt,$$

$$\text{s.t.} \begin{cases} \dot{x}(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 0 \\ v(t) + u(t) \end{pmatrix}, \\ 0 \leq v(t) \perp \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x(t) + u(t) \geq 0, \\ x(0) = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}, \\ x(T) \text{ free}, \end{cases}$$



Resolution with Direct Method and relaxation of the complementarity. Library used: IPOPT and CasADI. $h = 10^{-3}$.

Why do we bother ?

Suppose an optimal solution exists $\Longrightarrow$ Search for necessary conditions.
Two reasons for that:

- Useful for analyzing the solution (continuity, sensitivity...)
- The direct method mostly works fine! But very slow for high precision or big systems. Possible pseudominima?

Really general necessary conditions were obtained in [1]. But as such, they are not really practical (complicated hypothesis, really general equations...).

Can it be enhanced in the case of LCS?

[1] L. Guo and J. J. Ye. Necessary optimality conditions for optimal control problems with equilibrium constraints (2016).

Weak stationarity

Define $S = \{(x,u,v) | 0 \leq v \perp Cx + Dv + Eu \geq 0\}$ and the partition of $\{1, ..., m\}$:

$$I_t^{0+}(x,u,v) = \{i \mid v_i(t) = 0 < (Cx(t) + Dv(t) + Eu(t))_i\}$$

$$I_t^{+0}(x,u,v) = \{i \mid v_i(t) > 0 = (Cx(t) + Dv(t) + Eu(t))_i\}$$

$$I_t^{00}(x,u,v) = \{i \mid v_i(t) = 0 = (Cx(t) + Dv(t) + Eu(t))_i\}$$

Theorem

Let (x^*, u^*, v^*) be a local minimizer of radius $R(\cdot)$. Suppose $\mathrm{Im}(C) \subseteq \mathrm{Im}(E)$. Then there exist an absolutely continuous function $p : [0,T] \to \mathbb{R}^n$ and measurable functions $\lambda^G : \mathbb{R} \to \mathbb{R}^m$, $\lambda^H : \mathbb{R} \to \mathbb{R}^m$ such that the following conditions hold:

1. the transversality condition: $p(T) = 0$

Weak stationarity

IPL Modeliscale

Alexandre Vieira - Bernard Brogliato - Christophe Prieur

Theorem

2. the Weierstrass condition for radius R: for almost every $t \in [t_0, t_1]$,

$$(x^*(t), u, v) \in S, \left\| \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} u^*(t) \\ v^*(t) \end{pmatrix} \right\| < R(t)$$

$$\implies \langle p(t), Ax^*(t) + Bv + Fu) \rangle - \frac{1}{2}(x^*(t)^\intercal Q x^*(t) + u^\intercal U u + v^\intercal V v)$$

$$\leq \langle p(t), Ax^*(t) + Bv^*(t) + Fu^*(t)) \rangle$$

$$- \frac{1}{2}(x^*(t)^\intercal Q x^*(t) + u^*(t)^\intercal U u^*(t) + v^*(t)^\intercal V v^*(t))$$

Weak stationarity

Theorem

3. the Euler adjoint equation: for almost every $t \in [0, T]$,

$$\begin{aligned}
\dot{p}(t) &= -A^\intercal p(t) + Qx^*(t) - C^\intercal \lambda^H(t) \\
0 &= F^\intercal p(t) - Uu^*(t) + E^\intercal \lambda^H(t) \\
0 &= B^\intercal p(t) + \lambda^G(t) + D^\intercal \lambda^H(t) \\
0 &= \lambda_i^G(t), \ \forall i \in I_t^{+0}(x^*(t), u^*(t), v^*(t)) \\
0 &= \lambda_i^H(t), \ \forall i \in I_t^{0+}(x^*(t), u^*(t), v^*(t))
\end{aligned}$$

Remark : One can also prove that these conditions are, in one sense, sufficient.

Euler equation

How can one solve the following BVP?

$$
\begin{aligned}
\dot{x} &= Ax + Bv + Fu \\
\dot{p} &= -A^\intercal p + Qx - C^\intercal \lambda^H \\
0 &= F^\intercal p - Uu + E^\intercal \lambda^H \\
0 &= B^\intercal p + \lambda^G + D^\intercal \lambda^H \\
0 &= \lambda_i^G(t),\ \forall i \in I_t^{+0}(x(t), u(t), v(t)) \\
0 &= \lambda_i^H(t),\ \forall i \in I_t^{0+}(x(t), u(t), v(t)) \\
x_0 &= x(0), \\
0 &= p(T)
\end{aligned}
$$

Euler equation

How can we solve the following BVP?

$$\begin{aligned} \dot{x} &= Ax + Bv + Fu \\ \dot{p} &= -A^\intercal p + Qx - C^\intercal \lambda^H \\ 0 &= F^\intercal p - Uu + E^\intercal \lambda^H \rightarrow \text{ isolate } u \\ 0 &= B^\intercal p + \lambda^G + D^\intercal \lambda^H \rightarrow \text{ isolate } \lambda^G \end{aligned}$$

$$\boxed{\begin{aligned} 0 &= \lambda_i^G(t),\ \forall i \in I_t^{+0}(x(t), u(t), v(t)) \\ 0 &= \lambda_i^H(t),\ \forall i \in I_t^{0+}(x(t), u(t), v(t)) \end{aligned}}$$

$$\boxed{\begin{aligned} x_0 &= x(0), \\ 0 &= p(T) \end{aligned}}$$

Strong stationarity

$$\boxed{\begin{array}{l} 0 = \lambda_i^G(t), \ \forall i \in I_t^{+0}(x(t), u(t), v(t)) \\ 0 = \lambda_i^H(t), \ \forall i \in I_t^{0+}(x(t), u(t), v(t)) \end{array}}$$

We miss a piece of information: what happens on I_t^{00} ?

Proposition

Let (x^*, u^*, v^*) be a local minimizer and suppose E invertible. Then (x^*, u^*, v^*) is strongly stationary, meaning:

$$\lambda_i^G(t) \geq 0, \lambda_i^H(t) \geq 0, \ \forall i \in I_t^{00}(x(t), u(t), v(t))$$

Strong stationarity

$$
\boxed{
\begin{aligned}
0 &= \lambda_i^G(t), && \forall i \in I_t^{+0}(x(t),u(t),v(t)) \\
0 &= \lambda_i^H(t), && \forall i \in I_t^{0+}(x(t),u(t),v(t)) \\
\lambda_i^G(t) \geq 0, \lambda_i^H(t) &\geq 0, && \forall i \in I_t^{00}(x(t),u(t),v(t))
\end{aligned}
}
$$

Almost like a linear complementarity problem!

Strong stationarity

Theorem

Let (x^*, u^*, v^*) be a local minimizer and suppose E invertible. Fix an arbitrary $r > 0$. Then there exist an arc p and measurable functions $\beta : [0, T] \to \mathbb{R}^m$, $\zeta : [0, T] \to \mathbb{R}$ such that, $u^*(t) = U^{-1}\left(F^\intercal p(t) + E^\intercal \beta(t) - (\zeta(t) + r)E^\intercal v^*(t)\right)$ and:

$$\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \mathcal{A}\begin{pmatrix} x \\ p \end{pmatrix} + \mathcal{B}\begin{pmatrix} \beta \\ v^* \end{pmatrix}$$

$$\begin{cases} 0 \leq \begin{pmatrix} \beta \\ v^* \end{pmatrix} \perp \mathcal{D}\begin{pmatrix} \beta \\ v^* \end{pmatrix} + \mathcal{C}\begin{pmatrix} x \\ p \end{pmatrix} \geq 0 \\ \beta \geq r v^* \\ x(0) = x_0, \;\; p(T) = 0 \end{cases}$$

How to solve a BVP LCS

$$\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \mathcal{A} \begin{pmatrix} x \\ p \end{pmatrix} + \mathcal{B} \begin{pmatrix} \beta \\ v \end{pmatrix}$$

$$\begin{cases} 0 \leq \begin{pmatrix} \beta \\ v \end{pmatrix} \perp \mathcal{D} \begin{pmatrix} \beta \\ v \end{pmatrix} + \mathcal{C} \begin{pmatrix} x \\ p \end{pmatrix} \geq 0 \\ \beta \geq rv \\ \boxed{x(0) = x_0, \ p(T) = 0} \end{cases}$$

Numerically, we usually do shooting: find the good $p(0) = p_0$ such that the computed solution $p(t; p_0)$ complies with $p(T; p_0) = 0$: nonsmooth Newton method.

- Need for an initial guess close enough
- How to compute a sensitivity matrix for $p(T; \cdot)$?

How to solve a BVP LCS

$$\boxed{\begin{aligned} &\dot{z} = \mathcal{A}z + \mathcal{B}\Lambda \\ &0 \leq \Lambda \perp \mathcal{D}\Lambda + \mathcal{C}z \geq 0 \end{aligned}}$$

Denote $\mathcal{T}_h(z)$ a linear Newton Approximation to the solution Λ of the LCP. Then, a linear Newton approximation for the solution map $z(T, \cdot)$ can be obtained by solving the DI in matrix function:

$$\dot{J}(t) \in \mathcal{A}J(t) + (\text{co } \mathcal{T}_h(z(t;\xi)))J(t), \; J(0) = I$$

... But it supposes that $\mathcal{B}$ SOL$(\mathcal{D}, \mathcal{C}z)$ is a singleton for all $z \in \mathbb{R}^{2n}$ (which we can not prove).

JS Pang, D. Stewart, Solution dependence on initial conditions in differential variational inequalities (2009)

The indirect method

The indirect method consists in solving numerically the necessary conditions (also sufficient in this case). Since it is based on a shooting method (involving a nonsmooth Newton method), the equations are solved in two steps:

1. One solves, roughly, the optimal control problem with the Direct method, in order to get a rough idea of the solution.
2. One refines the solution by solving the necessary conditions, giving the solution of the Direct method as an initial guess.

A 1D example

$$\boxed{\begin{aligned} &\dot{x} = ax + bv + fu \\ &0 \leq v \perp dv + eu \geq 0 \\ &x(0) = x_0 \end{aligned}}$$

We can show that the (strong) stationary solution in this case is given by:

$$p(t) = \left[\cosh(\sqrt{\gamma}t) - \frac{a}{\sqrt{\gamma}}\sinh(\sqrt{\gamma}t)\right] p(0) + \frac{\sinh(\sqrt{\gamma}t)}{\sqrt{\gamma}} x(0)$$

$$p(0) = -\frac{\sinh(\sqrt{\gamma}T)}{\sqrt{\gamma}\cosh(\sqrt{\gamma}T) - a\sinh(\sqrt{\gamma}T)} x(0).$$

$$u(t) = \begin{cases} fp(t) & \text{if } efp(0) \geq 0, \\ \left(f - \frac{eb}{d}\right) p(t) & \text{if } efp(0) \leq 0. \end{cases}$$

$$x(t) = \dot{p}(t) + ap(t).$$

A 1D example

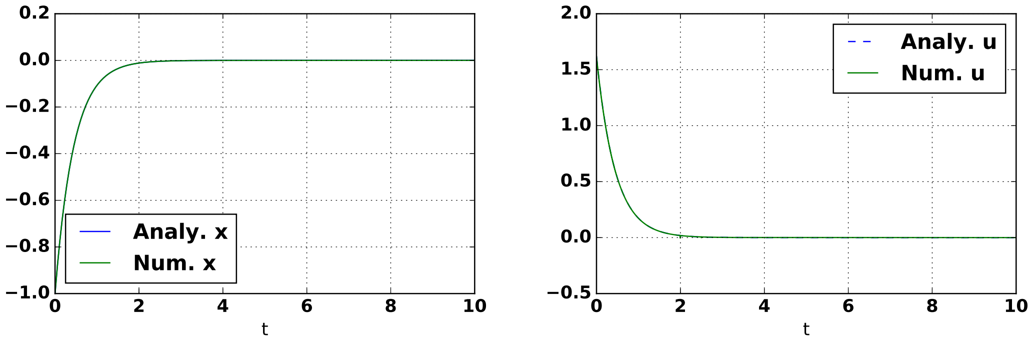


Figure: Solution via indirect method : state x and control u, on $[0, 10]$. $a = 1, b = 0.5, d = 1, e = -2, f = 3, x(0) = -1$. Initial guess with direct method and 300 nodes. Indirect method with 10 000 nodes and 20 intervals of shooting. Obtained in 54s. (In order to have this same precision with the direct method : 453s.)

Compare direct and indirect method

Let us compare the time of computation using the Direct Method and the Hybrid Approach (rough direct + refinements with indirect) in this example:

$$\min \int_0^1 \left(\|x(t)\|_2^2 + 25\|u(t)\|_2^2\right) dt,$$
$$\text{s.t.} \begin{cases} \dot{x}(t) = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} x(t) + \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} v(t) + \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} u(t), \\ 0 \leq v(t) \perp \begin{pmatrix} 3 & -1 \\ -2 & 0 \end{pmatrix} x(t) + v(t) + \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} u(t) \geq 0, \quad \text{a.e. on } [0,1] \\ x(0) = \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}, \ x(T) \text{ free}, \end{cases} \tag{1}$$

Compare direct and indirect method

h_D	Time spent (s)
10^{-2}	1.31
10^{-3}	37.50
10^{-4}	400.65
10^{-5}	∞
10^{-6}	∞

Table: Time spent With Direct Method

Parameters	Time spent (s)
$h_D = 10^{-1}$, $h_I = 10^{-2}$, $n_S = 5$	1.39
$h_D = 10^{-1}$, $h_I = 10^{-3}$, $n_S = 10$	11.26
$h_D = 10^{-2}$, $h_I = 10^{-4}$, $n_S = 20$	97.56
$h_D = 10^{-3}$, $h_I = 10^{-5}$, $n_S = 50$	1 298.62
$h_D = 10^{-4}$, $h_I = 10^{-6}$, $n_S = 100$	32 163.36

Table: Time spent with Hybrid approach

Minimal time

Let us review a different problem:

$$T^* = \min \; T(x,u,v)$$
$$\text{s.t.} \quad \begin{cases} \dot{x}(t) = Ax(t) + Bv(t) + Fu(t) \\ 0 \leq v(t) \perp Cx(t) + Dv(t) + Eu(t) \geq 0, \\ u(t) \in \mathcal{U} \\ x(0) = x_0, \; x(T^*) = x_f. \end{cases}$$

where $\mathcal{U}$ is a finite union of polyhedral compact convex sets.
Since u is now constrained: no more possibility to have strong stationarity and do the same manipulations. One still could have a weaker result, but not really useful as is.

A bang-bang property

$$T^* = \min \; T(x,u,v)$$
$$\text{s.t.} \; \begin{cases} \dot{x}(t) = Ax(t) + Bv(t) + Fu(t) \\ 0 \leq v(t) \perp Dv(t) + Eu(t) \geq 0, \\ u(t) \in \mathcal{U} \\ x(0) = x_0, \; x(T^*) = x_f. \end{cases}$$

We just suppose that D is a P-matrix. Denote by $\Omega = \{(u,v) \in \mathcal{U} \times \mathbb{R}^m | 0 \leq v \perp Dv + Eu \geq 0\}$, and $\mathrm{Acc}_\Omega(x_0, t)$ the accessible set at time t, starting from x_0 with controls having values in Ω.
For an index set $\alpha \subseteq \{1,...,m\}$, denote by $\mathbb{R}^m_\alpha$ the set of points q in $\mathbb{R}^m$ such that $q_\alpha \geq 0$, $q_{\overline{m}\setminus\alpha} \leq 0$, and define $E^{-1}\mathbb{R}^m_\alpha = \{\tilde{u} \in \mathbb{R}^m | E\tilde{u} \in \mathbb{R}^m_\alpha\}$ (E is not necessarily invertible)

A bang-bang property

Theorem

For a certain $\alpha \subseteq \{1, ..., m\}$, denote by $\mathcal{E}_\alpha$ the set:

$$\mathcal{E}_\alpha = \left\{ (u, v) \in \mathrm{Ext}\left(\mathcal{U} \cap E^{-1}\mathbb{R}^m_\alpha\right) \times \mathbb{R}^m \middle| \begin{array}{l} v_\alpha = 0, \;\; D_{\overline{\alpha}\bullet} v + E_{\overline{\alpha}\bullet} u = 0, \\ v \geq 0, Dv + Eu \geq 0 \end{array} \right\}$$

and by $\mathcal{E}$ the set $\mathcal{E} = \bigcup_{\alpha \subseteq \overline{m}} \mathcal{E}_\alpha$. Then, for all $t > 0$ and all $x_0 \in \mathbb{R}^n$,

$$\mathrm{Acc}_\Omega(x_0, t) = \mathrm{Acc}_\mathcal{E}(x_0, t)$$

Explanation when $E = I$ **:** Find all extreme points of each intersection of $\mathcal{U}$ with each orthant of $\mathbb{R}^m$. For each of these extreme points, find the solution v of the LCP. The optimal control can be searched as a succession of arcs taking these values.

A bang-bang property

Example

$$T^* = \min \; T(x,u,v)$$

$$\text{s.t.} \quad \begin{cases} \dot{x}(t) = ax(t) + bv(t) + fu(t), \\ 0 \leq v(t) \perp v(t) + u(t) \geq 0, \qquad \text{a.e. on } [0, T^*] \\ u(t) \in \mathcal{U} = [-1, 1] \\ (x(0), x(T^*)) = (x_0, x_f), \end{cases}$$

In this case, $\mathcal{E} = \{(-1,1),(0,0),(1,0)\}$. We can therefore search for the optimal solution with controls (u,v) in $\mathcal{E}$.
One can prove that, under complete controllability of the system, the optimal control (u^*,v^*) is constant along $[0, T^*]$ and equal to $(-1,1)$ or $(1,0)$.

A bang-bang property

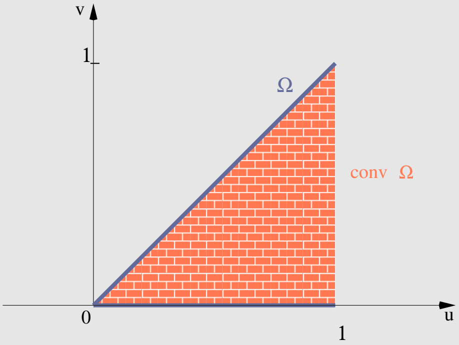
This second example, close to the previous one, suggests that the hypothesis of D P-matrix can possibly be relaxed.

Example

$$T^* = \min \ T(x,u,v)$$

$$\text{s.t.} \begin{cases} \dot{x}(t) = Ax(t) + Bv(t) + Fu(t), \\ 0 \leq v(t) \perp -v(t) + u(t) \geq 0 \\ u(t) \in \mathcal{U} = [-1,1] \\ (x(0), x(T^*)) = (x_0, x_f), \end{cases}$$

In this case, one can prove that $\mathcal{E} = \{(0,0),(1,0),(1,1)\}$ also works for covering the entire accessibility set.



Conclusion

- First stationarity and geometrical results, that we can use analytically and numerically.
- Numerical algorithms working fast, even with high precision.

(For those interested: the whole code is on `https://gitlab.inria.fr/avieira/optLCS`)

Conclusion

What is left to be done:

- The stationarity LCS, even in this case, still is not entirely analysed.
- When the dimension of the complementarity becomes high: the numerical resolution fails.
- Get rid of some assumptions (E invertible for the quadratic cost, D P-matrix for the minimal time...).
- For minimal time problem: could we algorithmically find the extreme points $\mathcal{E}$?
- Could we extend these results with a more relaxed concept of solution (distributions)?

Bounded Slope Condition

There exists a positive measurable function k_S such that for almost every $t \in [0, T]$, the bounded slope condition holds:

$$(x, w) \in S_*^{\varepsilon,R}(t), (\alpha, \beta) \in \mathcal{N}^P_{S(t)}(x, w) \implies \|\alpha\| \leq k_S(t)\|\beta\|.$$

Strong stationarity

Define, for two scalars ζ and r:

$$\mathcal{A} = \begin{pmatrix} A & FU^{-1}F^\intercal \\ Q & -A^\intercal \end{pmatrix}$$

$$\mathcal{B} = \begin{pmatrix} FU^{-1}E^\intercal & B - (\zeta + r)FU^{-1}E^\intercal \\ -C^\intercal & (\zeta + r)C^\intercal \end{pmatrix}$$

$$\mathcal{C} = \begin{pmatrix} C & EU^{-1}F^\intercal \\ \zeta C & \zeta EU^{-1}F^\intercal - B^\intercal \end{pmatrix}$$

$$\mathcal{D} = \begin{pmatrix} EU^{-1}E^\intercal & D - (\zeta + r)EU^{-1}E^\intercal \\ \zeta EU^{-1}E^\intercal - D^\intercal & \zeta D + (\zeta + r)\left(D^\intercal - \zeta EU^{-1}E^\intercal\right) \end{pmatrix}$$

Minimal time

Theorem

Suppose:

- either $C = 0$,
- or D is a diagonal matrix with positive entries.

Let (x^*, u^*, v^*) be a local minimizer for the minimal time problem. Then (x^*, u^*, v^*) is W-stationary; i.e. there exist an arc $p : [0, T^*] \to \mathbb{R}^n$, a scalar $\lambda_0 \in \{0, 1\}$ and multipliers $\lambda^G, \lambda^H : [0, T^*] \to \mathbb{R}^m$ such that:

$$(\lambda_0, p(t)) \neq 0 \ \forall t \in [0, T^*]$$

and:

Minimal time

Theorem

$$\dot{p}(t) = -A^\intercal p(t) - C^\intercal \lambda^H(t)$$

$$0 = B^\intercal p + D^\intercal \lambda^H + \lambda^G$$

$$0 \in -F^\intercal p - E^\intercal \lambda^H + \mathcal{N}^C_{\mathcal{U}}(u^*(t))$$

$$\lambda^G_i(t) = 0, \ \forall i \in I^{+0}_t(x^*, u^*, v^*)$$

$$\lambda^H_i(t) = 0, \ \forall i \in I^{0+}_t(x^*, u^*, v^*)$$

Minimal time

Also, since the system is linear, we know there exist a second set of multipliers η^G and η^H such that:

$$0 = B^\intercal p + D^\intercal \eta^H + \eta^G$$

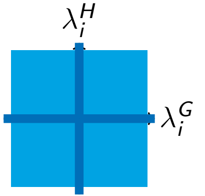
$$0 \in -F^\intercal p - E^\intercal \eta^H + \mathcal{N}^C_{\mathcal{U}}(u^*(t))$$

$$\eta_i^G(t) = 0, \ \forall i \in I_t^{+0}(x^*, u^*)$$

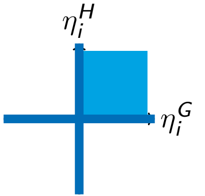
$$\eta_i^H(t) = 0, \ \forall i \in I_t^{0+}(x^*, u^*)$$

$$\eta_i^G \eta_i^H = 0 \text{ or } \eta_i^G > 0, \ \eta_i^H > 0, \ \forall i \in I_t^{00}(x^*, u^*)$$

... But they can be different from the corresponding λ^G and λ^H on a subset of $[0, T^*]$ of positive measure.



(a) W-stationarity



(b) M-stationarity