Singular extremal of optimal control problems with L^1 cost

Joint work with A. Agrachev and I. Beschastnyi

Michele Motta (SISSA, Trieste)

5th December 2024

Séminaire McTAO, Inria Sophia Antipolis

1. Formulation of the problem
2. Motivations from applications
3. Consequences of PMP
4. Sufficient conditions for optimality

Formulation of the problem

Let M be a smooth manifold and $f_0, f_1, \dots, f_m \in \mathrm{Vec}(M)$, $T > 0$ fixed. We consider the control system on M

$$\dot{q} = f_0(q) + \sum_{i=1}^{m} u_i f_i(q), \qquad q(0) = q_0,\ q(T) = q_T, \tag{1}$$

where the control u belongs to the set

$$\mathcal{U} = \{u : [0, T] \to \mathbb{R}^m \text{ measurable, } |u(t)| \leq 1\}.$$

Formulation of the problem

Let M be a smooth manifold and $f_0, f_1, \dots, f_m \in \mathrm{Vec}(M)$, $T > 0$ fixed. We consider the control system on M

$$\dot{q} = f_0(q) + \sum_{i=1}^{m} u_i f_i(q), \qquad q(0) = q_0, \; q(T) = q_T, \tag{1}$$

where the control u belongs to the set

$$\mathcal{U} = \{u : [0, T] \to \mathbb{R}^m \text{ measurable}, \; |u(t)| \le 1\}.$$

Our cost function is

$$J(u) = \int_0^T |u(t)| dt, \; u \in \mathcal{U}. \tag{2}$$

Problem (OCP): Find the solutions of (1) minimizing (2).

1 Formulation of the problem

2 Motivations from applications

3 Consequences of PMP

4 Sufficient conditions for optimality

Motivation 2 (from Caillau et al. [2])

Aerospace engineering: minimization of fuel consumption

$$\begin{cases} \ddot{q} + \nabla V(q) = \dfrac{u(t)}{M(t)}, \\ \dot{M}(t) = -\beta |u(t)|, \end{cases} \tag{1}$$

$$M(T) \to \max \iff \int_0^T |u(t)| dt \to \min . \tag{2}$$

Motivation 2 (from Caillau et al. [2])

Aerospace engineering: minimization of fuel consumption

$$\begin{cases} \ddot{q} + \nabla V(q) = \dfrac{u(t)}{M(t)}, \\ \dot{M}(t) = -\beta |u(t)|, \end{cases} \tag{1}$$

$$M(T) \to \max \iff \int_0^T |u(t)| dt \to \min . \tag{2}$$

If $\beta = 0$, the problem is in the form seen before.

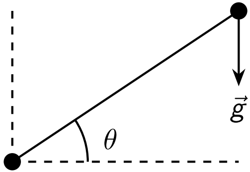
Motivation 3 (from Berret et al. [1])

Planning of arm movements: it was shown that arm's movements minimizes the *absolute work*

Planning of arm movements: it was shown that arm's movements minimizes the *absolute work*

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{\omega} = u - \vec{g}\cos\theta, \end{cases}$$

$$J(u) = \int_0^T |\omega u| dt,$$



where ω is the angular velocity and u is the net torque.

1 Formulation of the problem

2 Motivations from applications

3 Consequences of PMP

4 Sufficient conditions for optimality

Consequences of PMP 1

We apply the PMP to the OCP described before. Let

$$h_i(p,q) = \langle p, f_i(q)\rangle, \quad p \in T_q^*M, \quad q \in M, \quad i = 0, 1, \dots, m.$$

$$h_I = (h_1, \dots, h_m),$$

$$H = h_0 + \langle u, h_I\rangle - |u|.$$

Consequences of PMP 1

We apply the PMP to the OCP described before. Let

$$h_i(p,q) = \langle p, f_i(q)\rangle, \quad p \in T_q^*M, \quad q \in M, \quad i = 0, 1, \dots, m.$$

$$h_I = (h_1, \dots, h_m),$$

$$H = h_0 + \langle u, h_I\rangle - |u|.$$

By PMP, if $\tilde{u} \in \mathcal{U}$ solves the OCP, there is a Lipschitz curve λ in T^*M solving

$$\dot{\lambda}(t) = \vec{H}\big(\tilde{u}(t), \lambda(t)\big) = \vec{h}_0 + \langle \tilde{u}(t), \vec{h}_I\rangle,$$

Consequences of PMP 1

We apply the PMP to the OCP described before. Let

$$h_i(p,q) = \langle p, f_i(q)\rangle, \quad p \in T_q^*M, \quad q \in M, \quad i = 0, 1, \ldots, m.$$

$$h_I = (h_1, \ldots, h_m),$$

$$H = h_0 + \langle u, h_I\rangle - |u|.$$

By PMP, if $\tilde{u} \in \mathcal{U}$ solves the OCP, there is a Lipschitz curve λ in T^*M solving

$$\dot{\lambda}(t) = \vec{H}\big(\tilde{u}(t), \lambda(t)\big) = \vec{h}_0 + \langle \tilde{u}(t), \vec{h}_I\rangle,$$

and the control $\tilde{u}$ must satisfy

$$H\big(\tilde{u}(t), \lambda(t)\big) = \max_{|u|\leq 1} H\big(u, \lambda(t)\big) \quad \text{for a.e. } t \in [0, T].$$

Consequences of PMP 2

If we use polar coordinate for $u = rv$, $r \in [0,1]$ and $v \in S^{m-1}$, we obtain

$$H(u,\lambda) \le h_0(\lambda) + r(|h_I(\lambda)| - 1),$$

where $r = |u|$, and $h_I = (h_1, \dots, h_m)$.

Consequences of PMP 2

If we use polar coordinate for $u = rv$, $r \in [0,1]$ and $v \in S^{m-1}$, we obtain

$$H(u,\lambda) \leq h_0(\lambda) + r(|h_I(\lambda)| - 1),$$

where $r = |u|$, and $h_I = (h_1, \dots, h_m)$. So, if $|h_I| \neq 1$,

$$\begin{cases} u(t) = h_I/|h_I|, \quad \text{if } |h_I| > 1, \\ u(t) = 0, \quad \text{if } |h_I| < 1. \end{cases}$$

Consequences of PMP 2

If we use polar coordinate for $u = rv$, $r \in [0,1]$ and $v \in S^{m-1}$, we obtain

$$H(u,\lambda) \leq h_0(\lambda) + r(|h_I(\lambda)| - 1),$$

where $r = |u|$, and $h_I = (h_1, \ldots, h_m)$. So, if $|h_I| \neq 1$,

$$\begin{cases} u(t) = h_I/|h_I|, \quad \text{if } |h_I| > 1, \\ u(t) = 0, \quad \text{if } |h_I| < 1. \end{cases}$$

In this case the trajectory (p,q) is said to be *regular*. Otherwise, if $|h_I| \equiv 1$ is *singular*.

Let $h_c = \frac{1}{2}\langle h_I, h_I \rangle$. Differentiating two times in t the equation $|h_I| = 1$, one obtains

$$h_{0c}(\lambda_t) = 0,$$

$$h_{00c}(\lambda_t) - r(t) h_{cc0}(\lambda_t) = 0 \implies u_*(t) = \frac{h_{00c}}{h_{cc0}} h_I(\lambda_t),$$

where $h_{ijk} = \{h_i, \{h_j, h_k\}\}$.

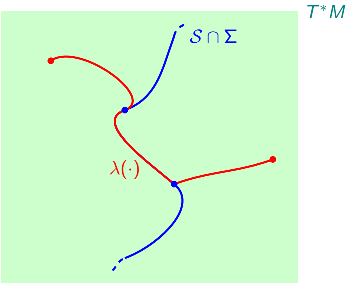
Summary on singular extremals

So, we have obtained that singular extremals satisfy

$$\dot{\lambda} = \vec{h}_0(\lambda) + \frac{h_{00c}}{h_{cc0}} \vec{h}_c(\lambda)$$

on the submanifold

$$\Sigma \cap \mathcal{S} := \{\lambda \in T^*M \mid 2h_c(\lambda) = 1\} \cap \{\lambda \in T^*M \mid h_{0c}(\lambda) = 0\}.$$



1 Formulation of the problem

2 Motivations from applications

3 Consequences of PMP

4 Sufficient conditions for optimality

Extended end-point map

The extended end-point map is

$$E_T : \mathcal{U} \to M \times \mathbb{R}, \quad E_T(u) = (q(T;u), J(u)).$$

Extended end-point map

The extended end-point map is

$$E_T : \mathcal{U} \to M \times \mathbb{R}, \quad E_T(u) = (q(T; u), J(u)).$$

The Hessian of E_T at u_* is the quadratic form $Q_T : \mathrm{Ker} D_{u_*} E_T \to \mathbb{R}$:

$$Q_T(v) = \int_0^T \frac{|w(t)|^2}{r} + \sigma_{\lambda_0}\left(Z_t v(t), \int_0^t Z_s v(s) ds\right) dt$$

where

- $v = \rho h_I + w$, with $\langle w, h_I \rangle = 0$;

Extended end-point map

The extended end-point map is

$$E_T : \mathcal{U} \to M \times \mathbb{R}, \quad E_T(u) = (q(T;u), J(u)).$$

The Hessian of E_T at u_* is the quadratic form $Q_T : \mathrm{Ker} D_{u_*} E_T \to \mathbb{R}$:

$$Q_T(v) = \int_0^T \frac{|w(t)|^2}{r} + \sigma_{\lambda_0}\left(Z_t v(t), \int_0^t Z_s v(s) ds\right) dt$$

where

- $v = \rho h_I + w$, with $\langle w, h_I \rangle = 0$;
- σ is the standard symplectic form on T^*M;

Extended end-point map

The extended end-point map is

$$E_T : \mathcal{U} \to M \times \mathbb{R}, \quad E_T(u) = (q(T;u), J(u)).$$

The Hessian of E_T at u_* is the quadratic form $Q_T : \mathrm{Ker} D_{u_*} E_T \to \mathbb{R}$:

$$Q_T(v) = \int_0^T \frac{|w(t)|^2}{r} + \sigma_{\lambda_0}\left(Z_t v(t), \int_0^t Z_s v(s) ds \right) dt$$

where

- $v = \rho h_I + w$, with $\langle w, h_I \rangle = 0$;
- σ is the standard symplectic form on T^*M;
- Θ_t is the flow of $\vec{H}(u_*(\cdot), \cdot)$;

Extended end-point map

The extended end-point map is

$$E_T : \mathcal{U} \to M \times \mathbb{R}, \quad E_T(u) = (q(T; u), J(u)).$$

The Hessian of E_T at u_* is the quadratic form $Q_T : \mathrm{Ker} D_{u_*} E_T \to \mathbb{R}$:

$$Q_T(v) = \int_0^T \frac{|w(t)|^2}{r} + \sigma_{\lambda_0}\left(Z_t v(t), \int_0^t Z_s v(s) ds \right) dt$$

where

- $v = \rho h_I + w$, with $\langle w, h_I \rangle = 0$;
- σ is the standard symplectic form on T^*M;
- Θ_t is the flow of $\vec{H}(u_*(\cdot), \cdot)$;
- $Z_t v = (\Theta_t^{-1})_* \langle v(t), \vec{h}_I \rangle$.

Second order necessary condition

Theorem

If the singular control u_* is optimal, then

$$h_{c0c}(\lambda_t) \geq 0, \quad \text{for } t \in [0, T].$$

Second order necessary condition

Theorem

If the singular control u_* is optimal, then

$$h_{c0c}(\lambda_t) \geq 0, \quad \textit{for } t \in [0, T].$$

Idea of the proof.

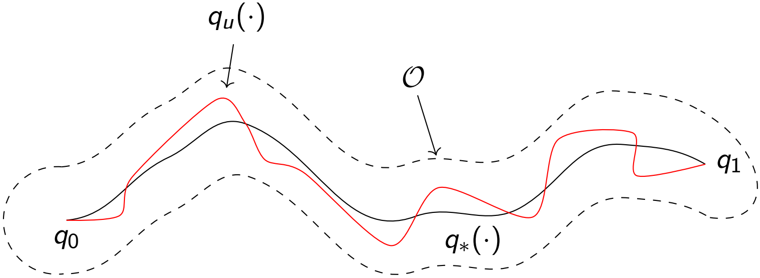
After an integration by part in the direction h_I, the second variation reads

$$Q_T(v) = \int_0^T \begin{pmatrix} w(t) \\ \phi(t) \end{pmatrix}^T \begin{pmatrix} \frac{\mathrm{Id}}{r} & \sigma\big(Z_t \cdot, Z_t h_I\big) \\ \sigma\big(Z_t \cdot, Z_t h_I\big)^T & \sigma\big(Z_t h_I, \frac{d}{dt}(Z_t h_I)\big) \end{pmatrix} \begin{pmatrix} w(t) \\ \phi(t) \end{pmatrix} dt$$
$$+ \text{l. o. t.}$$

The determinant of the invertible part is h_{c0c}/r^{m-1}. □

Locally strongly optimal trajectories

Locally strongly optimal trajectories



$$q_u(\cdot) \subset \mathcal{O},\ q_u(0) = q_0,\ q_u(T) = q_1 \implies \|u_*\|_{L^1} \leq \|u\|_{L^1}$$

Sufficient conditions: the classical result

The main strategy to prove sufficient conditions is to use the *fields of extremals*.

Sufficient conditions: the classical result

The main strategy to prove sufficient conditions is to use the *fields of extremals*. Let

$$\dot{q} = f(q,u), \quad \int_0^T L(q,u)dt \to \min,$$

$$H_M(\lambda) = \max_u \langle p, f_u(q)\rangle - L(q,u), \quad a \in C^\infty(M),$$

$$\mathcal{L}_0 = \{(q, d_q a) \mid q \in M\} \subset T^*M, \quad \mathcal{L}_t = \exp(t\vec{H}_M)(\mathcal{L}_0).$$

and $\pi : T^*M \to M$ the canonical projection.

Sufficient conditions: the classical result

The main strategy to prove sufficient conditions is to use the *fields of extremals*. Let

$$\dot{q} = f(q,u), \quad \int_0^T L(q,u)dt \to \min,$$

$$H_M(\lambda) = \max_u \langle p, f_u(q)\rangle - L(q,u), \quad a \in C^\infty(M),$$

$$\mathcal{L}_0 = \{(q, d_q a) \mid q \in M\} \subset T^*M, \quad \mathcal{L}_t = \exp(t\vec{H}_M)(\mathcal{L}_0).$$

and $\pi : T^*M \to M$ the canonical projection.

Theorem

Let $\tilde{\lambda}_t$ be a normal extremal trajectory. If $\pi_{|\mathcal{L}_t}$ is a local diffeomorphism near $\tilde{\lambda}_t$ for every t, then $\tilde{q}(t) = \pi(\lambda_t)$ is locally strongly optimal.

Sufficient conditions: proof of the classical result

Lemma (Poincaré-Cartan)

Let s be the Liouville 1-form of T^*M. Then, the 1-form

$$s - H_M dt,$$

is exact on $\mathcal{L} = \{(t, \ell) \in T^*M \mid t \in [0, T], \ell \in \mathcal{L}_t\}$.

Let u be any admissible control and q the corresponding trajectory, $\pi(\lambda_t) = q(t)$, $\gamma = (t, \lambda_t), \tilde{\gamma} = (t, \tilde{\lambda}_t)$,

$$\int_0^T L(q,u)dt = \int_0^T \langle \lambda_t, f_u(q) \rangle - H_u(\lambda_t)dt \geq \int_0^T \langle \lambda_t, f_u(q) \rangle - H_M(\lambda_t)dt$$

$$= \int_\gamma s - H_M dt = \int_{\tilde{\gamma}} s - H_M dt = \int_0^T \langle \tilde{\lambda}_t, f_u(\tilde{q}) \rangle - H_M(\tilde{\lambda}_t)dt =$$

$$= \int_0^T L(\tilde{q}, \tilde{u})dt.$$

Sufficient conditions: super-hamiltonian

Let $a \in C^\infty(M)$, $H_S(t,\cdot) \in C^\infty(T^*M)$, Φ_t the flow of H_S.

Theorem (Stefani, Zezza [3])

If $H_S(t,\cdot)$ satisfy

1. $H_S(t,\ell_t) \geq H_M(\ell_t)$, *where* $\ell_t = \Phi_t(\ell_0)$, $\ell_0 \in \mathcal{L}_0$ *;*
2. $H_S(t,\tilde{\lambda}_t) = H_M(\tilde{\lambda}_t)$, *for a.e.* $t \in [0,T]$*;*
3. $\vec{H}_S(t,\tilde{\lambda}_t) = \vec{H}_M(\tilde{\lambda}_t)$, *for a.e.* $t \in [0,T]$*;*
4. *the function* $\Psi : \mathcal{L} \to \mathbb{R} \times M$

$$\Psi(t,\ell_0) = (t,\pi(\ell_t)),$$

is a smooth diffeomorphism.

Then, q_* is locally strongly optimal.

Sufficient condition: small time

Theorem

If

$$h_{c0c}(\lambda_t) > 0, \quad \textit{for } t \in [0, T], \qquad \text{(SGLC)}$$

then for every $t \in [0, T]$ *there is some* $\tau > 0$ *such that* $q_{*|[t,t+\tau]}$ *is locally strongly optimal.*

Sufficient condition: small time

Theorem

If

$$h_{c0c}(\lambda_t) > 0, \quad \textit{for } t \in [0, T], \qquad \text{(SGLC)}$$

then for every $t \in [0, T]$ *there is some* $\tau > 0$ *such that* $q_{*|[t,t+\tau]}$ *is locally strongly optimal.*

Proof.

- Since

$$d_{(0,\ell_0)}\Psi = \begin{pmatrix} 1 & * \\ 0 & \mathrm{Id} \end{pmatrix}$$

the projection Ψ is a local diffeomorphism near $t = 0$;

Sufficient condition: small time

Theorem

If

$$h_{c0c}(\lambda_t) > 0, \quad \text{for } t \in [0, T], \qquad \text{(SGLC)}$$

then for every $t \in [0, T]$ *there is some* $\tau > 0$ *such that* $q_{*|[t,t+\tau]}$ *is locally strongly optimal.*

Proof.

- Since

$$d_{(0,\ell_0)}\Psi = \begin{pmatrix} 1 & * \\ 0 & \mathrm{Id} \end{pmatrix}$$

the projection Ψ is a local diffeomorphism near $t = 0$;

- a is constructed solving

$$\sum_{i=1}^{m} |\langle d_q a, f_i(q)\rangle|^2 = 1, \quad d_{q_0} a = p_0.$$

Sufficient condition: construction of the super-hamiltonian

Pontryagin Hamiltonian was $H = h_0 + \langle u, h_I \rangle - |u|$ and $h_c = \frac{1}{2}|h_I|^2$, so that $\vec{h}_c = \overrightarrow{|h_I|}$ on Σ.

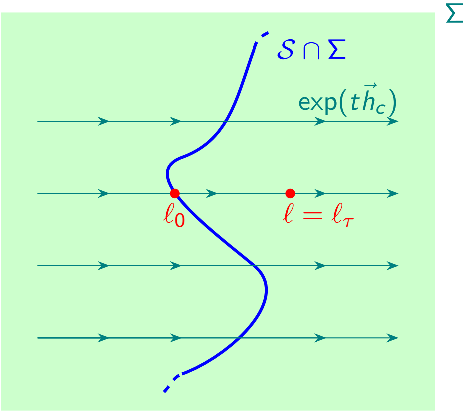
Let $\Sigma = \{\ell \in T^*M \mid |h_I(\ell)| = 1\}$, $\mathcal{S} = \{\ell \in T^*M \mid h_{0c}(\ell) = 0\}$.

We solve the equation

$$\{h_S, h_c\} = 0 \quad \text{on} \quad T^*M,$$
$$h_S = h_0 \quad \text{in} \quad \mathcal{S}.$$

and define

$$H_S = h_S + r(t)(|h_I| - 1)$$



Sufficient condition: end of the proof

- $H_S(t,\ell) \geq H(u_*(t),\ell) = h_0(\ell)$ follows by

$$h_S(\ell) - h_0(\ell) = h_S(\ell_0) - \left(h_0(\ell_0) + \tau h_{c0}(\ell_0) + \frac{\tau^2}{2} h_{cc0}(\ell_0) + o(\tau^2)\right)$$
$$= \frac{\tau^2}{2} h_{c0c}(\ell_0) + o(\tau^2);$$

Sufficient condition: end of the proof

- $H_S(t,\ell) \geq H(u_*(t),\ell) = h_0(\ell)$ follows by

$$h_S(\ell) - h_0(\ell) = h_S(\ell_0) - \Big(h_0(\ell_0) + \tau h_{c0}(\ell_0) + \frac{\tau^2}{2} h_{cc0}(\ell_0) + o(\tau^2)\Big)$$
$$= \frac{\tau^2}{2} h_{c0c}(\ell_0) + o(\tau^2);$$

- $H_S(t,\lambda_t) = H(u_*(t),\lambda_t)$, follows from of $\lambda(t) \in \mathcal{S} \cap \Sigma$ and $h_S(\ell) = h_0(\ell)$ for $\ell \in \mathcal{S}$;

Sufficient condition: end of the proof

- $H_S(t,\ell) \geq H(u_*(t),\ell) = h_0(\ell)$ follows by

$$h_S(\ell) - h_0(\ell) = h_S(\ell_0) - \big(h_0(\ell_0) + \tau h_{c0}(\ell_0) + \frac{\tau^2}{2} h_{cc0}(\ell_0) + o(\tau^2)\big)$$
$$= \frac{\tau^2}{2} h_{c0c}(\ell_0) + o(\tau^2);$$

- $H_S(t,\lambda_t) = H(u_*(t),\lambda_t)$, follows from of $\lambda(t) \in \mathcal{S} \cap \Sigma$ and $h_S(\ell) = h_0(\ell)$ for $\ell \in \mathcal{S}$;
- $\vec{H}_S(t,\lambda_t) = \vec{H}(u_*(t),\lambda_t)$, follows since $d_\ell h_0 = d_\ell h_S$ for $\ell \in \mathcal{S} \cap \Sigma$.

Sufficient condition: end of the proof

- $H_S(t,\ell) \geq H(u_*(t),\ell) = h_0(\ell)$ follows by

$$h_S(\ell) - h_0(\ell) = h_S(\ell_0) - \left(h_0(\ell_0) + \tau h_{c0}(\ell_0) + \frac{\tau^2}{2} h_{cc0}(\ell_0) + o(\tau^2)\right)$$
$$= \frac{\tau^2}{2} h_{c0c}(\ell_0) + o(\tau^2);$$

- $H_S(t,\lambda_t) = H(u_*(t),\lambda_t)$, follows from of $\lambda(t) \in \mathcal{S} \cap \Sigma$ and $h_S(\ell) = h_0(\ell)$ for $\ell \in \mathcal{S}$;
- $\vec{H}_S(t,\lambda_t) = \vec{H}(u_*(t),\lambda_t)$, follows since $d_\ell h_0 = d_\ell h_S$ for $\ell \in \mathcal{S} \cap \Sigma$.

Sufficient condition: end of the proof

- $H_S(t,\ell) \geq H(u_*(t),\ell) = h_0(\ell)$ follows by

$$h_S(\ell) - h_0(\ell) = h_S(\ell_0) - \left(h_0(\ell_0) + \tau h_{c0}(\ell_0) + \frac{\tau^2}{2} h_{cc0}(\ell_0) + o(\tau^2)\right)$$
$$= \frac{\tau^2}{2} h_{c0c}(\ell_0) + o(\tau^2);$$

- $H_S(t,\lambda_t) = H(u_*(t),\lambda_t)$, follows from of $\lambda(t) \in \mathcal{S} \cap \Sigma$ and $h_S(\ell) = h_0(\ell)$ for $\ell \in \mathcal{S}$;
- $\vec{H}_S(t,\lambda_t) = \vec{H}(u_*(t),\lambda_t)$, follows since $d_\ell h_0 = d_\ell h_S$ for $\ell \in \mathcal{S} \cap \Sigma$.

□

Conjugate times

Definition

The time $t_1 > 0$ is called a **conjugate time** if there is $\bar{v} \in \mathrm{Ker} D_{u_*} E_{t_1}$ such that

$$Q_{t_1}(\bar{v}, v) = 0, \quad \forall v \in \mathrm{Ker} D_{u_*} E_{t_1}$$

Conjugate times

Definition

The time $t_1 > 0$ is called a **conjugate time** if there is $\bar{v} \in \mathrm{Ker} D_{u_*} E_{t_1}$ such that

$$Q_{t_1}(\bar{v}, v) = 0, \quad \forall v \in \mathrm{Ker} D_{u_*} E_{t_1}$$

Equivalently, t_1 is a conjugate time if there is a non-constant solution to the boundary value problem on $T_{\lambda_0}(T^*M)$:

$$\dot{\eta}(t) = -\mathcal{Z}_t I_t^{-1} \sigma\Big(\mathcal{Z}_t \cdot \,, \eta(t)\Big) \quad \text{for a.e. } t \in [0, t_1],$$

$$\eta(0) \in \Big(T^*_{q_0} M + \mathbb{R} Z_I(0)\Big) \cap \big(\mathbb{R} Z_I(0)\big)^{\angle}, \quad \eta(t_1) \in T^*_{q_0} M,$$

where $Z_I(t) = Z_t h_I$ and $\mathcal{Z}_t v = Z_t w - \phi(t) \dot{Z}_I(t)$, I_t is the invertible part of $\mathrm{Hess}_{u_*} E_{t_1}$.

Theorem

If (SGLC) *holds and there are no conjugate times in* $[0, T]$, *then* q_* *is locally strongly optimal on* $[0, T]$.

Theorem

If (SGLC) *holds and there are no conjugate times in* $[0, T]$*, then* q_* *is locally strongly optimal on* $[0, T]$.

Idea of the Proof.

Again, we want to apply the argument of the fields of extremals.

In the classical regular case, you can use the absence of conjugate times to show that the projection $\Psi : \mathcal{L} \to \mathbb{R} \times M$ is a local diffeomorphism up to time T.

In this singular case, we replace the flow of the maximazed Hamiltonian with the flow of H_S. □

References

[1] Bastien Berret et al. "The Inactivation Principle: Mathematical Solutions Minimizing the Absolute Work and Biological Implications for the Planning of Arm Movements". In: *PLOS Computational Biology* (2008).

[2] Z. Chen, J.-B. Caillau, and Y. Chitour. "L^1-Minimization for Mechanical Systems". In: *SIAM Journal on Control and Optimization* (2016).

[3] Gianna Stefani and Pierluigi Zezza. "17. A Hamiltonian approach to sufficiency in optimal control with minimal regularity conditions: Part I". In: *In Imaging and Geometric Control*. Ed. by Maitine Bergounioux et al. Berlin, Boston: De Gruyter, 2017.

Thank you for your attention!