Explicit semi-algebraic description of the orbit space of Weyl group actions

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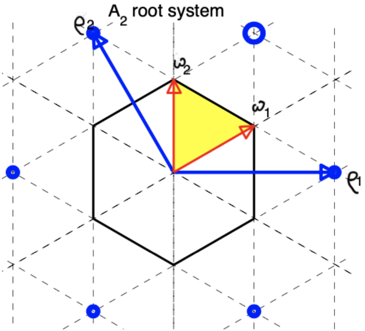
Lattice $\Omega = \mathbb{Z}\,\omega_1 \oplus \ldots \oplus \mathbb{Z}\,\omega_n$

Trigonometric polynomial: $f : \mathbb{R}^n \to \mathbb{R}$

where $f(u) = \sum_{\omega \in \Omega} a_\omega \, e^{2\pi i \langle \omega, u \rangle}$

$a_{-\omega} = \overline{a_\omega} \in \mathbb{C}$

f in $\Omega^\perp$-periodic



Compute (numerically)

$$\min_{u \in \mathbb{R}^n} f(u)$$

under the asumption that f is invariant
w.r.t a reflection group $\mathfrak{G}$ of rank n

Univariate case

$\mathfrak{G} = \{+1, -1\}$ acts on $\mathbb{R}$ and preserves $\Omega = \mathbb{Z}$

$$\begin{array}{rcl} \mathfrak{G} \times \mathbb{R} & \rightarrow & \mathbb{R} \\ (\epsilon, u) & \mapsto & \epsilon\, u \end{array}$$

Invariant trigonometric polynomials : $a_{-k} = a_k \in \mathbb{R}$

$$f(u) = \sum_{k\in\mathbb{N}} a_k \left(e^{2\pi i k u} + e^{-2\pi i k u}\right) = \sum_{k\in\mathbb{N}} 2a_k \cos(2\pi k u)$$

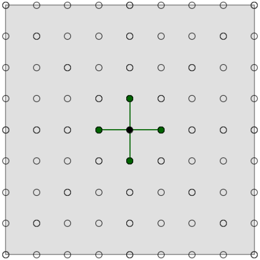
Chebyshev polynomials $\{T_k\}_{k\in\mathbb{N}}$

$$\cos(k\,\theta) = T_k\,(\cos(\theta)) \quad \text{where} \quad \cos(\theta) = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$$

$z = \cos(2\pi u) \in [-1, 1]$

$$\min_{u\in\mathbb{R}} f(u) \;=\; \min_{1-z^2\geq 0} \sum_{k\in\mathbb{N}} 2a_k\, T_k(z)$$

2D lattices & symmetry

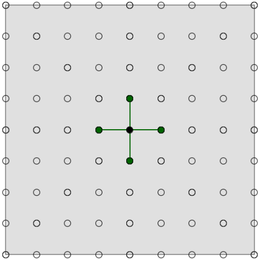


$\mathfrak{G} = \{+1, -1\}^2$

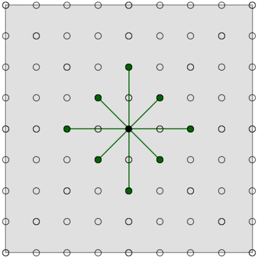
Invariance : $f(-u, v) = f(u, v) = f(u, -v)$

Trigonometric $\rightsquigarrow$ polynomial optimization

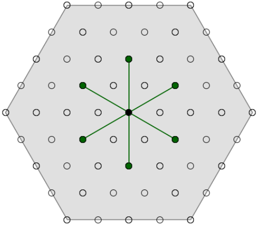
$$\min_{u,v\in\mathbb{R}} f(u, v) = \min_{z_1,z_2\in[-1,1]^2} a_{k,l}\, T_k(z_1)\, T_l(z_2)$$



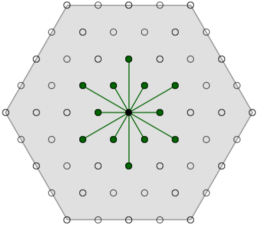
$A_1 \times A_1$



C_2

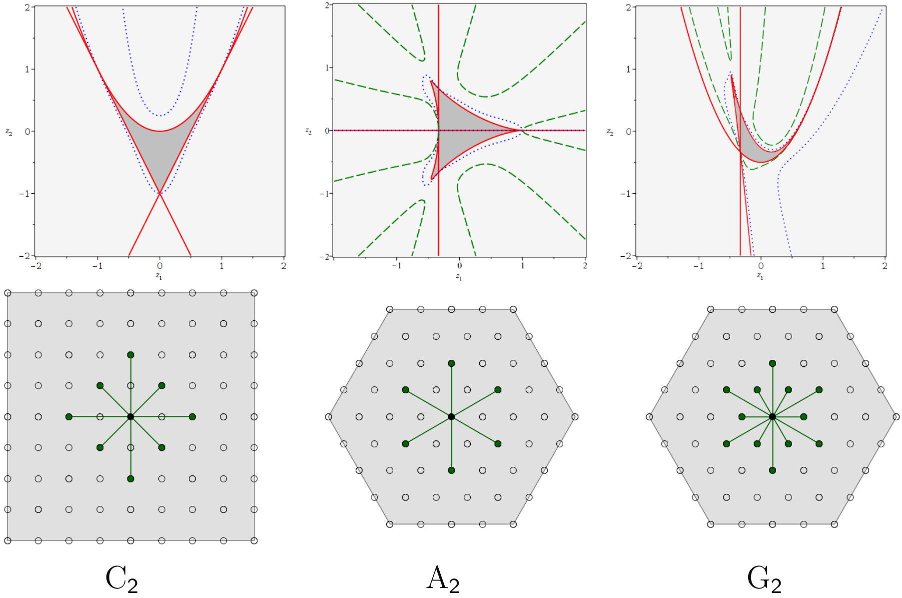


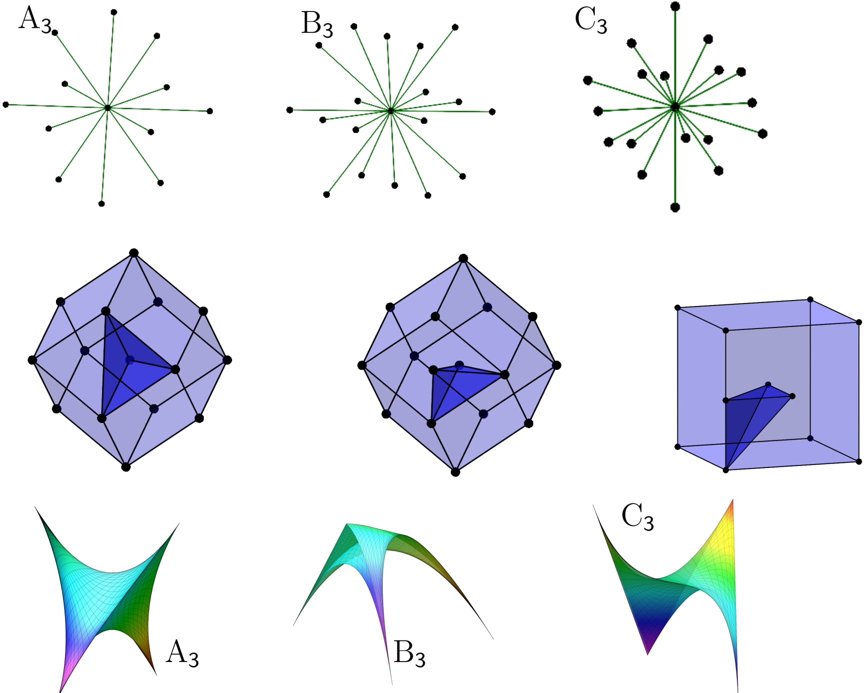
A_2



G_2

2D lattices & symmetry

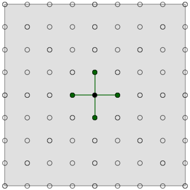



A_3
B_3
C_3
C_3
A_3
B_3

R root system

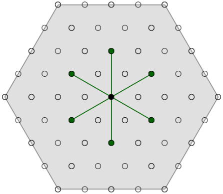
- R is finite and spans $\mathbb{R}^n$
- If $\rho, \tilde{\rho} \in \mathrm{R}$, then $\sigma_\rho(\tilde{\rho}) \in \mathrm{R}$, where $\sigma_\rho(u) := u - \langle u, \rho^\vee \rangle \rho$

♣ If $\rho, \tilde{\rho} \in \mathrm{R}$, then $\langle \tilde{\rho}, \rho^\vee \rangle \in \mathbb{Z}$, where $\rho^\vee := 2\,\frac{\rho}{\langle \rho\,, \rho \rangle}$



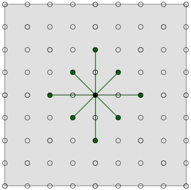
$\mathrm{A}_1 \times \mathrm{A}_1$

$\{+1, -1\}^2$



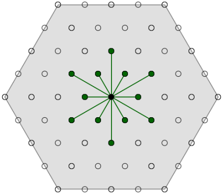
A_2

$\mathfrak{S}_3$



C_2

$\mathfrak{S}_2 \ltimes \{+1, -1\}^2$



G_2

$\mathfrak{S}_3 \ltimes \{+1, -1\}$

- The **Weyl group** $\mathfrak{G}$ is the group generated by the σ_ρ
- The **coroot lattice** Λ is the lattice generated by the $\rho^\vee$
- The **weight lattice** Ω is the dual lattice of Λ:

$$\omega \in \Omega \quad \Leftrightarrow \quad \langle \omega, \rho^\vee \rangle \in \mathbb{Z} \quad \forall \rho \in \mathrm{R}$$

Multivariate Chebysev polynomials

$\Omega = \mathbb{Z}\,\omega_1 \oplus \ldots \oplus \mathbb{Z}\,\omega_n$ the weight lattice of a root system R $\quad \mathfrak{G}$-invariant

For $\omega \in \Omega$

$$\begin{array}{rccc} \mathfrak{e}_\omega : & \mathbb{R}^n & \to & \mathbb{C} \\ & u & \mapsto & e^{2\pi \mathrm{i}\langle \omega, u\rangle} \end{array}$$

form an orthogonal basis of Λ-periodic functions

Generalized cosines

$$\mathfrak{c}_\omega = \frac{1}{|\mathfrak{G}|}\sum_{g\in\mathfrak{G}} \mathfrak{e}_{g\omega} \qquad \text{for } \omega \in \mathbb{N}\,\omega_1 \oplus \ldots \oplus \mathbb{N}\,\omega_n$$

form a linear basis of the $\mathfrak{G}$-invariant functions

$\mathfrak{c}_{\omega_1}, \ldots, \mathfrak{c}_{\omega_n}$ are [Bourbaki]

- algebraically independent
- generate the ring of invariant trigonometric polynomials

Generalized Chebyshev polynomial T_α, $\alpha \in \mathbb{N}^n$

is the unique element of $\mathbb{Q}[z_1, \ldots, z_n]$ satisfying

$$\mathfrak{c}_{\alpha_1\omega_1+\ldots+\alpha_n\omega_n} = T_\alpha\left(\mathfrak{c}_{\omega_1}, \ldots, \mathfrak{c}_{\omega_n}\right)$$

Optimization of trigonometric polynomials with symmetry

$\Omega = \mathbb{Z}\,\omega_1 \oplus \ldots \oplus \mathbb{Z}\,\omega_n$ is $\mathfrak{G}$-invariant

$$f(u) = \sum_{\omega \in \Omega^+} a_\omega \, \mathfrak{c}_\omega(u)$$

$$\min_{u \in \mathbb{R}^n} f(u) \;=\; \min_{z \in \mathcal{T}} \sum_{\alpha \in \mathbb{N}^n} \tilde{a}_\alpha \, T_\alpha(z)$$

$$\mathcal{T} = \mathfrak{c}\,(\mathbb{R}^n) \quad \text{where} \quad \mathfrak{c} = (\mathfrak{c}_{\omega_1}, \ldots, \mathfrak{c}_{\omega_n}) : \mathbb{R}^n \to \mathbb{R}^n$$

For root systems $\mathcal{A}_{n-1}$, $\mathcal{B}_n$, $\mathcal{C}_n$, $\mathcal{D}_n$, $\mathcal{G}_2$

$(z_1, \ldots, z_n) \in \mathcal{T} \quad \Leftrightarrow \qquad e_1 = (1, 0, \ldots, 0)$

$$\begin{pmatrix} T_0 - T_{2e_1} & T_{e_1} - T_{3e_1} & T_0 - T_{4e_1} & \iddots \\ T_{e_1} - T_{3e_1} & T_0 - T_{4e_1} & 2T_{e_1} - T_{3e_1} - T_{5e_1} & \iddots \\ T_0 - T_{4e_1} & 2T_{e_1} - T_{3e_1} - T_{5e_1} & 2T_0 + T_{2e_1} - T_{6e_1} - 2T_{4e_1} + T_{2e_1} & \iddots \\ \iddots & \iddots & \iddots & \iddots \end{pmatrix} \succeq 0$$

i.e.,

$$\left(-T_{(i+j)e_1} + \sum_{k=1}^{\lceil (i+j)/2 \rceil - 1} \left(4\binom{i+j-2}{k-1} - \binom{i+j}{k}\right) T_{(i+j-2k)\,e_1} + \begin{cases} 2\binom{i+j-2}{(i+j)/2-1} - \frac{1}{2}\binom{i+j}{(i+j)/2}, & i+j \text{ even} \\ 0, & i+j \text{ odd} \end{cases} \right)_{1 \le i,j \le n} \succeq 0$$

$\mathcal{T}$ is a semi-algebraic set defined by the inequalities $\chi_1(z), \ldots, \chi_n(z) \geq 0$

Comments

- We obtain an explicit and unified formula

R	$\mathcal{A}_{n-1}$	$\mathcal{B}_n$	$\mathcal{C}_n$	$\mathcal{D}_n$	$\mathcal{G}_2$
$\mathfrak{G}$	$\mathfrak{S}_n$	$\mathfrak{S}_n \ltimes \{\pm 1\}^n$	$\mathfrak{S}_n \ltimes \{\pm 1\}^n$	$\mathfrak{S}_n \ltimes \{\pm 1\}^n_+$	$\mathfrak{S}_3 \ltimes \{\pm 1\}$

  We use the interconnection between $\mathfrak{S}_n$ and the roots of polynomials. [Procesi Schwarz 85] would lead to different matrices.

- Missing $\mathcal{F}_4$, $\mathcal{E}_6$, $\mathcal{E}_7$, $\mathcal{E}_8$

It makes sense to work out a Lasserre-type hierarchy

- based on the Hol-Scherer potivistellensatz for matrices
- in the basis $\{T_\alpha\}_{\alpha\in\mathbb{N}^n}$

$$T_\alpha\, T_\beta \;=\; \sum_{A\in\mathfrak{G}} T_{\alpha+A\beta} \;=\; T_{\alpha+\beta} \;+ \sum_{\langle\gamma,\rho_0\rangle<\langle\alpha+\beta,\rho_0\rangle} c_\gamma\, T_\gamma$$

- with a weighted degree given by ρ_0, the highest root

Explicit semi-algebraic description of the orbit space of Weyl group actions

1. Trigonometric and Chebysev polynomials
2. $\mathcal{T}$ as the orbit space of a multiplicative action
3. Case $\mathcal{C}_n$, $\mathcal{B}_n$, $\mathcal{D}_n$,
4. Case $\mathcal{A}_{n-1}$
5. Spectral bounds on the chromatic number of some infinite graphs

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Spectral bound on the chromatic number of infinite graphs

Consider the infinite graph (V, E) where $V = \mathbb{R}^n$ or $\Omega = \mathbb{Z}\,\omega_1 \oplus \ldots \oplus \mathbb{Z}\,\omega_n$
$S \subset V$ centrally symmetric and $(v_1, v_2) \in E$ if $v_2 - v_1 \in S$

[Bachoc, DeCorte, de Oliveira Filho, Vallentin 14]

The chromatic number χ of the graph is bounded by

$$2^n \geq \chi \geq 1 - \frac{\sup_{u\in\mathbb{R}^n} \hat{\nu}(u)}{\inf_{u\in\mathbb{R}^n} \hat{\nu}(u)}$$

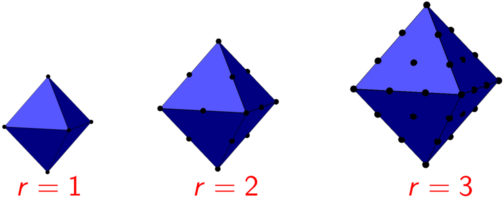
where ν is a measure on S and $\hat{\nu}(u) = \int e^{-2\pi i\langle x,u\rangle} d\nu(x)$

Known results :

- $n = 2$ for S the Euclidean sphere and $V = \mathbb{R}^n$ [Hardwinger, Nelson 50]
- S Voronoi cell in lattice Ω [Dutour Sikiric, Madore, Moustrou, Pecher]
- S polytope, $V = \mathbb{R}^n$ or Ω : partial results.

$V = \mathbb{Z}^n$ and $S = \mathbb{S}^1_r$ the cross-polytopes

$\mathbb{S}^1_r := \{u \in \mathbb{Z}^n \,|\, |u_1| + \ldots + |u_n| = r\}$ [Füredi, Kang'04]



Symmetry: $\mathcal{B}_n$ or $\mathcal{C}_n$

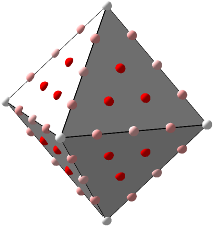
$$2^n \geq \chi_m(\mathbb{Z}^n, \mathbb{S}^1_r) \geq 1 - \frac{1}{F(r)}$$

$$F(r) := \max \left\{ \min_{z \in \mathcal{T}} \sum_{\alpha \in S_r^+} f_\alpha \, T_\alpha(z) \;\middle|\; \sum_{\alpha \in S_r^+} f_\alpha = 1, f_\alpha \geq 0 \right\}$$

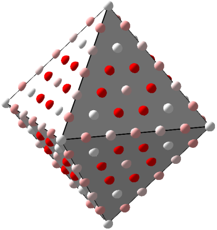
Analytical bounds (with Chebyshev polynomials) [HMMR 23]

$\chi_m(\mathbb{Z}^2, \mathbb{B}^1_{2\,r}) = 4$ $\chi_m(\mathbb{Z}^n, \mathbb{B}^1_{2\,r+1}) = 2$ $\chi_m(\mathbb{Z}^n, \mathbb{B}^1_2) = 2n$

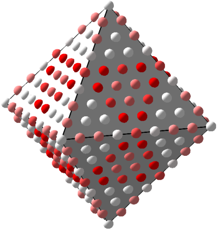
Numerical bounds for $n = 3$ and $r = 4, 6, \ldots$ [HMMR]



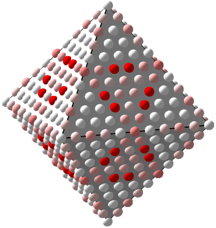
$1 - \frac{1}{F(4)} \geq 6.28148$



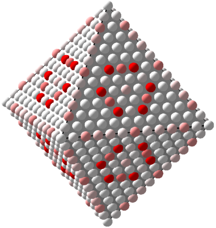
$1 - \frac{1}{F(6)} \geq 6.30269$



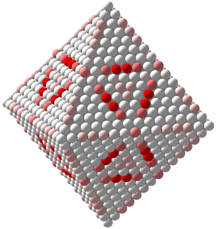
$1 - \frac{1}{F(8)} \geq 6.30229$



$1 - \frac{1}{F(10)} \geq 6.30502$

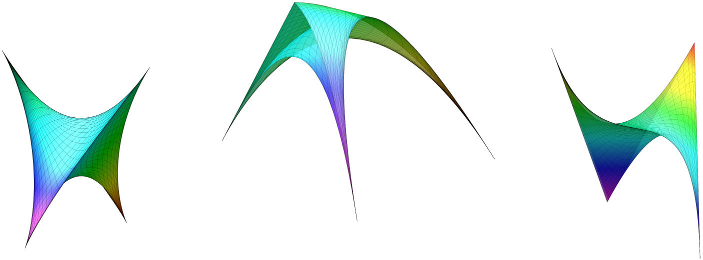


$1 - \frac{1}{F(12)} \geq 6.30229$



$1 - \frac{1}{F(14)} \geq 6.30156$

Thanks



E. Hubert, T. Metzlaff, C. Riener; Orbit spaces of Weyl groups acting on compact tori: a unified and explicit polynomial description. https://hal.science/hal-03590007

E. Hubert, T. Metzlaff, P. Moustrou, C. Riener; Optimization of trigonometric polynomials with symmetry and spectral bounds for set avoiding graphs. https://hal.science/hal-03768067