Keplerian orbits with a common focus and secular evolution in the R3BP with crossing singularities

Giovanni Federico Gronchi

Dipartimento di Matematica, Università di Pisa

e-mail: `gronchi@dm.unipi.it`

University of Nice Sophia-Antipolis

March 8, 2023

The Keplerian distance function

Let (E_j, v_j), $j = 1, 2$ be the orbital elements of two celestial bodies on Keplerian orbits with a common focus:

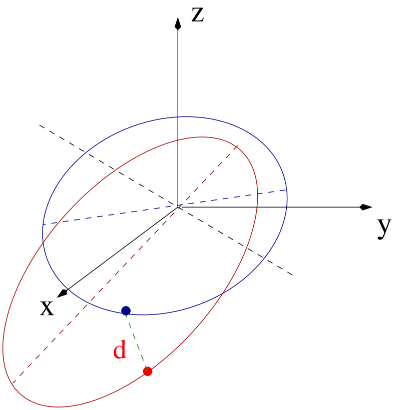
E_j represents the trajectory of a body,

v_j is a parameter along it.

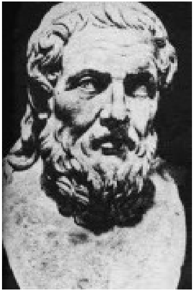
Set $V = (v_1, v_2)$. For a given two-orbit configuration $\mathcal{E} = (E_1, E_2)$, we introduce the Keplerian distance function

$$\mathbb{T}^2 \ni V \mapsto d(\mathcal{E}, V) = |\mathcal{X}_1 - \mathcal{X}_2|.$$

We are interested in the local minimum points of d and in particular in the absolute minimum d_{min}, called orbit distance, or MOID.



Is there still something that we do not know about distance of points on conic sections?



ἐθεώρουν σε σπεύδοντα μετασχεῖν τῶν πεπραγμένων ἡμῖν κωνικῶν (1)

(Apollonius of Perga, *Conics*, Book I)

(1) I observed you were quite eager to be kept informed of the work I was doing in conics.

Critical points of d^2

- The local minimum points of d can be found by computing all the critical points of d^2 (so that crossing points are also critical).

How many can they be?

- Apart from the case of two concentric coplanar circles, or two overlapping ellipses, d^2 has finitely many critical points...

... but they can be more than what we expect!

- There exist configurations with 12 critical points, and 4 local minima of d^2.
This is thought to be the maximum possible, but a proof is not known yet.

Computation of the local minima

There are several papers in the literature about the computation of the MOID, e.g. Sitarski (1968), Dybczyński et al. (1986) and more recently Hedo et al. (2018), Baluev and Mikryukov (2019).

The following papers introduced algebraic methods to compute all the critical points of d^2:

- Kholshevnikov and Vassiliev, CMDA (1999), with *Gröbner bases*;
- Gronchi, SJSC (2002), CMDA (2005), with *resultant theory*.

They are based on a polynomial formulation of the problem, which gives some advantages.

Algebraic formulation

The critical points equations is

$$\nabla_V d^2(\mathcal{E}, V) = 0. \tag{1}$$

By the coordinate change

$$s = \tan(v_1/2)\,; \qquad t = \tan(v_2/2)$$

we obtain from (1) a system of 2 polynomials in 2 unknowns

$$\begin{cases} p(s,t) &= f_4(t)\,s^4 + f_3(t)\,s^3 + f_2(t)\,s^2 + f_1(t)\,s + f_0(t) = 0 \\ q(s,t) &= g_2(t)\,s^2 + g_1(t)\,s + g_0(t) = 0 \end{cases}$$

each with total degree 6; precisely $p(s,t)$ has degree 4 in s and degree 2 in t, while $q(s,t)$ has degree 2 in s and degree 4 in t.

Computation of the solutions

From elimination theory we know that p and q have a common solution if and only if

$$\mathtt{Res}(p,q,s)(t) = \det \mathtt{S}(t) = 0\,;$$

where

$$\mathtt{S}(t) = \begin{pmatrix} f_4 & 0 & g_2 & 0 & 0 & 0 \\ f_3 & f_4 & g_1 & g_2 & 0 & 0 \\ 0 & f_3 & g_0 & g_1 & g_2 & 0 \\ f_1 & 0 & 0 & g_0 & g_1 & g_2 \\ f_0 & f_1 & 0 & 0 & g_0 & g_1 \\ 0 & f_0 & 0 & 0 & 0 & g_0 \end{pmatrix}.$$

$R(t) = \mathtt{Res}(p,q,s)(t)$ is a polynomial with degree 20; it has a factor $(1+t^2)^2$ giving 4 imaginary roots.

Scheme of the algorithm

We use an *interpolation method* to compute its coefficients:

- Evaluate the polynomial coefficients of the matrix $\mathtt{S}(t)$ at the 32–th roots of unit $\omega_k = e^{2\pi i \frac{k}{32}}, k = 0\ldots 31$ by a DFT
- Compute the determinant of the 32 Sylvester matrices and observe that

$$(\det \mathtt{S}(t))|_{t=\omega_k} = \det \mathtt{S}(\omega_k)\,, \quad k = 0\ldots 31$$

- Apply an IDFT to obtain the coefficients of $R(t)$ from its 32 evaluations
- Compute the real roots of $R(t)$
- Given $\bar{t} \in \mathbb{R}$: $R(\bar{t}) = 0$, search for $\bar{s} \in \mathbb{R}$ such that $(\bar{t}, \bar{s})$ is a solution.

  Hint! in some cases for each root $\bar{t}$ of $R(t)$ we can find more than one such $\bar{s}$.

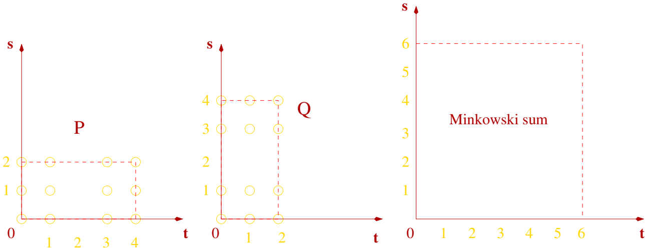
Maximal number of critical points

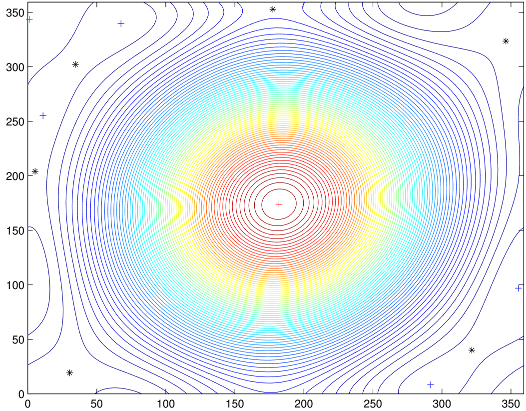
For the case of two bounded orbits we can prove the following:

If there are finitely many critical points of d^2, then they are at most 16 in the general case and at most 12 if one orbit is circular.

The proof uses Bernstein's theorem, which says that an upper bound for the solutions in $\mathbb{C}^2$ is given by the mixed area of Newton's polygons of p e q:

$$\text{Mixed Area}(P, Q) = \text{Area}(P + Q) - \text{Area}(P) - \text{Area}(Q)$$





level curves of d^2, plane of the eccentric anomalies

$$\left\{ \begin{array}{l} + = max \\ + = min \\ * = saddle \end{array} \right.$$

By Morse theory

$$\#(max) + \#(min) = \#(saddles)$$

Q	e_1	q	e_2	i_M	$\omega_M^{(1)}$	$\omega_M^{(2)}$
0.585	0.415	0.462	0.615	80.0°	8.0°	176.0°

Conjecture

The following table gives a conjecture on the maximum number of critical points in case of bounded orbits:

$e_1 \neq 0$	$e_2 \neq 0$	12 points
$e_1 \neq 0$	$e_2 = 0$	10 points
$e_1 = 0$	$e_2 \neq 0$	10 points
$e_1 = 0$	$e_2 = 0$	8 points

This is still an open problem!

The local minimum distance maps

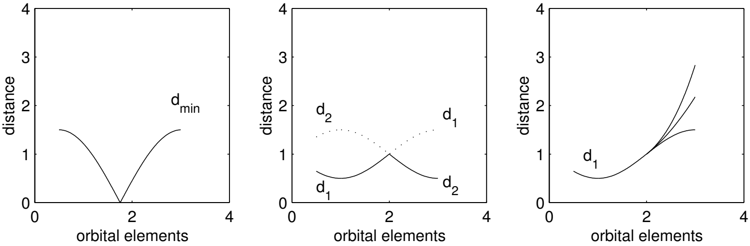
Gronchi and Tommei, DCDS-B (2007)

Let $V_h = V_h(\mathcal{E})$ be a local minimum point of $V \mapsto d^2(\mathcal{E}, V)$.
Consider the maps

$$\mathcal{E} \mapsto d_h(\mathcal{E}) = d(\mathcal{E}, V_h) \ ,$$
$$\mathcal{E} \mapsto d_{min}(\mathcal{E}) = \min_h d_h(\mathcal{E}) \ .$$

The map $\mathcal{E} \mapsto d_{min}(\mathcal{E})$ gives the MOID.

Singularities of d_h and d_{min}



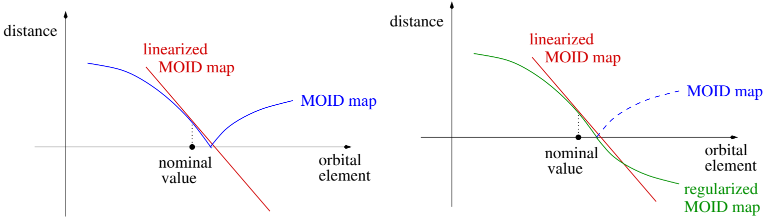
(i) d_h and d_{min} are not differentiable where they vanish;

(ii) two local minima can exchange their role as absolute minimum thus d_{min} loses its regularity without vanishing;

(iii) when a bifurcation occurs the definition of the maps d_h may become ambiguous after the bifurcation point.

Problems in computing the uncertainty of d_{min}

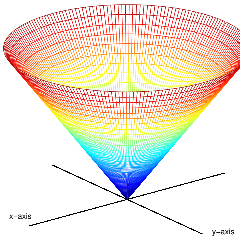
Given a nominal orbit configuration $\bar{\mathcal{E}}$, with its covariance matrix $\Gamma_{\bar{\mathcal{E}}}$, the covariance propagation of a function of $\mathcal{E}$, like d_{min}, is based on a linearization of the function near $\bar{\mathcal{E}}$.

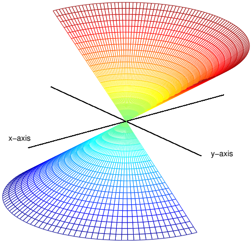


Remark: $d_{min}(\mathcal{E})$ is not smooth where it vanishes, thus usually the linearizzation of d_{min} in a neighborhood of the nominal orbit is not a good approximation (see fig. on the left)

Problem: can we give a sign to $d_{min}(\mathcal{E})$ so that its linearization becomes meaningful (see fig. on the right)?

Smoothing through change of sign



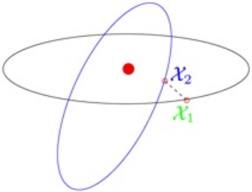


Toy problem:

$$f(x,y) = \sqrt{x^2+y^2} \qquad \tilde{f}(x,y) = \begin{cases} -f(x,y) & \text{for } x > 0 \\ f(x,y) & \text{for } x < 0 \end{cases}$$

Can we smooth the maps $d_h(\mathcal{E})$, $d_{min}(\mathcal{E})$ through a change of sign?

Local smoothing of d_h at a crossing singularity



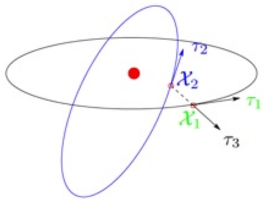
Smoothing d_h, the procedure for d_{min} is the same.

- Consider the points on the two orbits

$$\mathcal{X}_1^{(h)} = \mathcal{X}_1(E_1, v_1^{(h)})\,; \qquad \mathcal{X}_2^{(h)} = \mathcal{X}_2(E_2, v_2^{(h)})\;.$$

corresponding to the local minimum point $V_h = (v_1^{(h)}, v_2^{(h)})$ of d^2;

Local smoothing of d_h at a crossing singularity



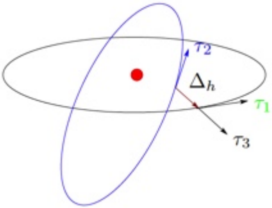
- introduce the tangent vectors to the trajectories E_1, E_2 at these points:

$$\tau_1 = \frac{\partial \mathcal{X}_1}{\partial v_1}(E_1, v_1^{(h)})\,, \qquad \tau_2 = \frac{\partial \mathcal{X}_2}{\partial v_2}(E_2, v_2^{(h)})\,,$$

and their cross product

$$\tau_3 = \tau_1 \times \tau_2;$$

Local smoothing of d_h at a crossing singularity



- define also

$$\Delta = \mathcal{X}_1 - \mathcal{X}_2, \qquad \Delta_h = \mathcal{X}_1^{(h)} - \mathcal{X}_2^{(h)}.$$

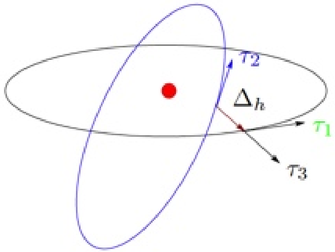
The vector Δ_h joins the points attaining a local minimum of d^2 and $|\Delta_h| = d_h$.

Note that $\Delta_h \times \tau_3 = 0$.

Smoothing the crossing singularity

smoothing rule:

$$\tilde{d}_h = \text{sign}(\tau_3 \cdot \Delta_h) d_h$$



$\mathcal{E} \mapsto \tilde{d}_h(\mathcal{E})$ is an analytic map in a neighborhood of most crossing configurations.

Uncertainty of the MOID

For a given orbit $\bar{\mathcal{E}}$, with its covariance matrix $\Gamma_{\bar{\mathcal{E}}}$, the covariance propagation formula

$$\Gamma_{\tilde{d}_{min}(\bar{\mathcal{E}})} = \left[\frac{\partial \tilde{d}_{min}}{\partial \mathcal{E}}(\bar{\mathcal{E}})\right] \Gamma_{\bar{\mathcal{E}}} \left[\frac{\partial \tilde{d}_{min}}{\partial \mathcal{E}}(\bar{\mathcal{E}})\right]^t$$

allows us to compute the covariance of the regularized MOID.

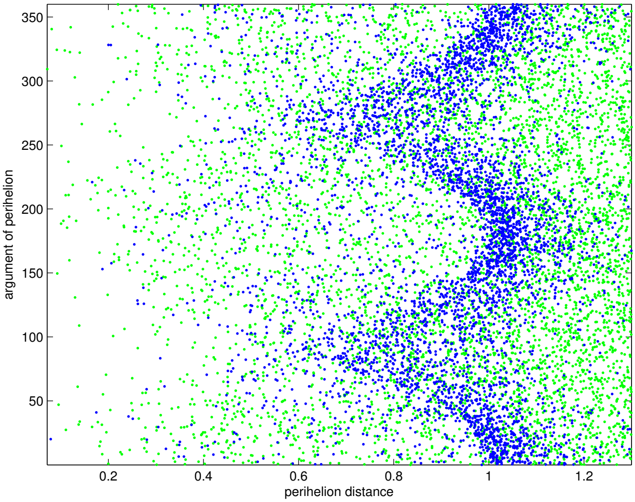
Using the orbit distance

to detect observational biases

in the discovery of NEAs

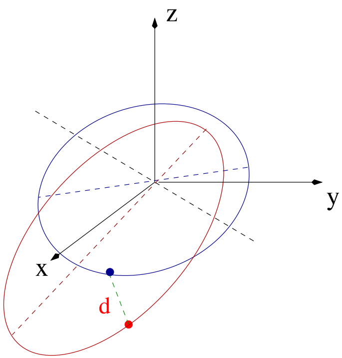
(q, ω) plot of all the known NEAs

Gronchi and Valsecchi, MNRAS (2014)



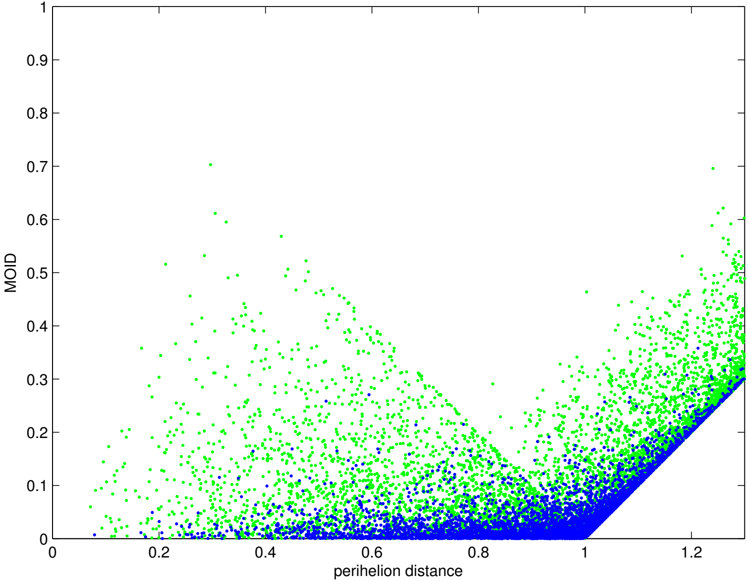
The blue dots are NEAs with $H > 22$.

Consider the orbits of the Earth and of a NEA. We denote by d_{min} the MOID between the trajectories of these two bodies.



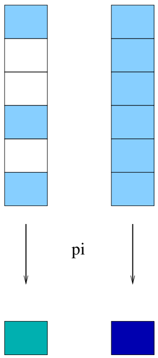
- Most NEAs with a small value of d_{min} are detected, sooner or later;
- small NEAs with a large value of d_{min} are likely to be unobserved.

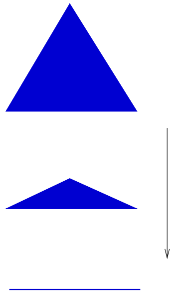
(q, d_{min}) plot of all the known NEAs



Projections

In all the previous plots we see projections on a plane of data from an N-dimensional space, with $N > 2$.





'Nothing was visible, nor could be visible, to us, except Straight Lines' (E. A. Abbot), *Flatland*.

The near-Earth asteroid class

We define the NEA class $\mathcal{N}$ as the set of cometary orbital elements $(q, e, I, \Omega, \omega)$ such that

$$q \in [0, q_{max}], \quad e \in [0, 1], \quad I \in [0, \pi], \quad \Omega \in [0, 2\pi], \quad \omega \in [0, 2\pi].$$

Here q is the perihelion distance and $q_{max} = 1.3$ au.

We use

$$q' = 1, \quad e' = 0, \quad I' = 0, \quad \Omega' = 0, \quad \omega' = 0$$

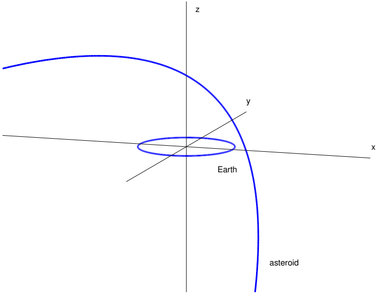
for the elements of the Earth.

Possible values of d_{min} as function of (q, ω)

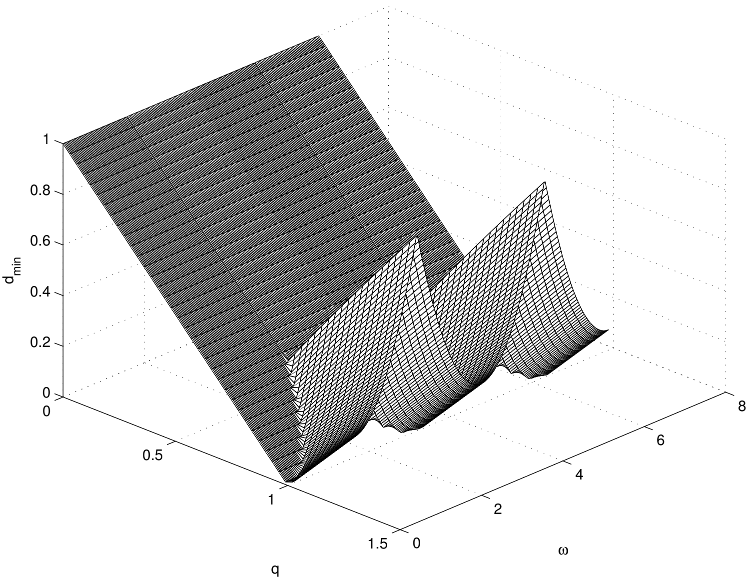
Let $\mathcal{D}_1 = \{(e, I) : 0 \leq e \leq 1, 0 \leq I \leq \pi\}$. For each choice of (q, ω), with $0 < q \leq q_{max}, 0 \leq \omega \leq 2\pi$, we have

$$\max_{(e,I)\in\mathcal{D}_1} d_{min} = \max\{q' - q, \delta(q, \omega)\}$$

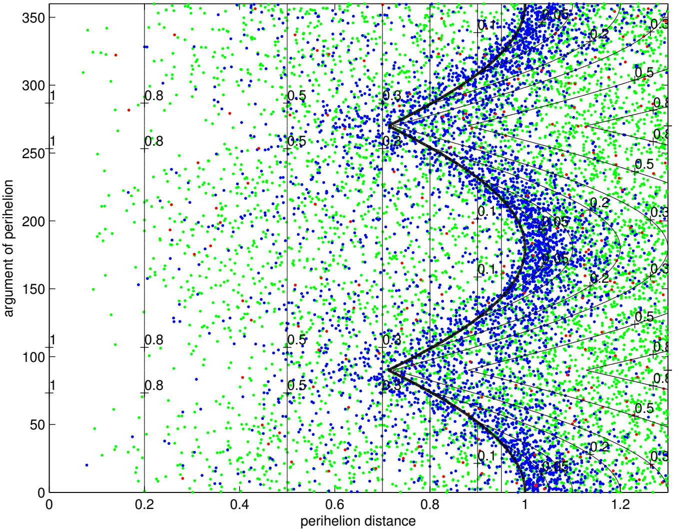
where $\delta(q, \omega)$ is the distance between the orbit of the Earth and a parabolic orbit $(e = 1)$ with $I = \pi/2$.



Maximal orbit distance as function of (q, ω)

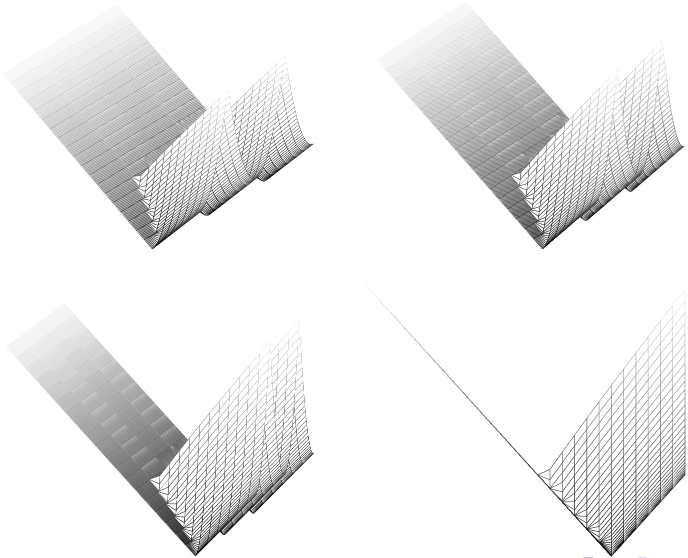


Distribution of NEAs in the plane (q, ω)

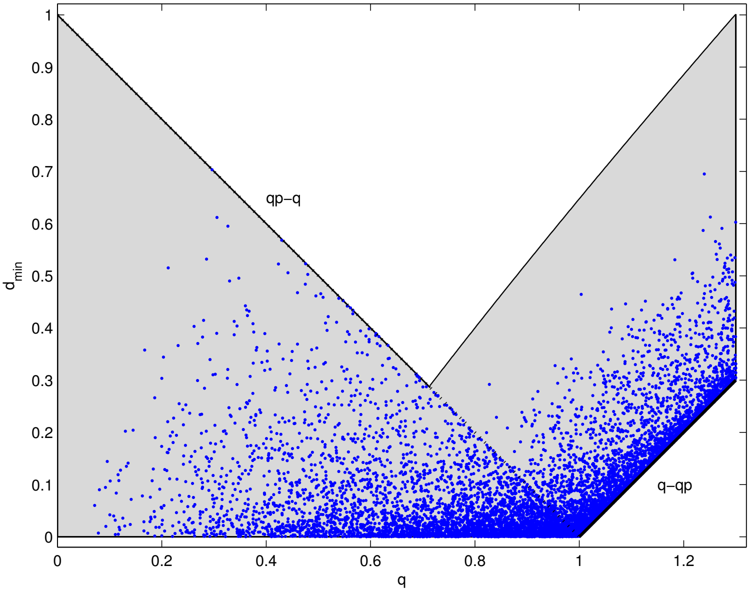


Blue dots are NEAs with $H > 22$, red dots with $H < 16$.

Distribution of NEAs in the plane (q, d_{min})



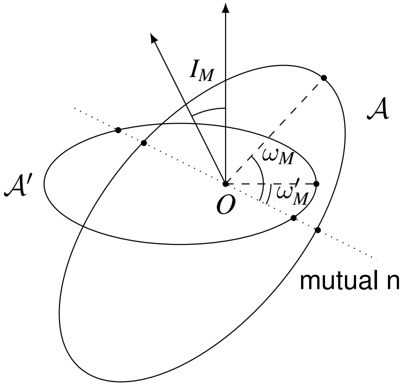
Distribution of NEAs in the plane (q, d_{min})



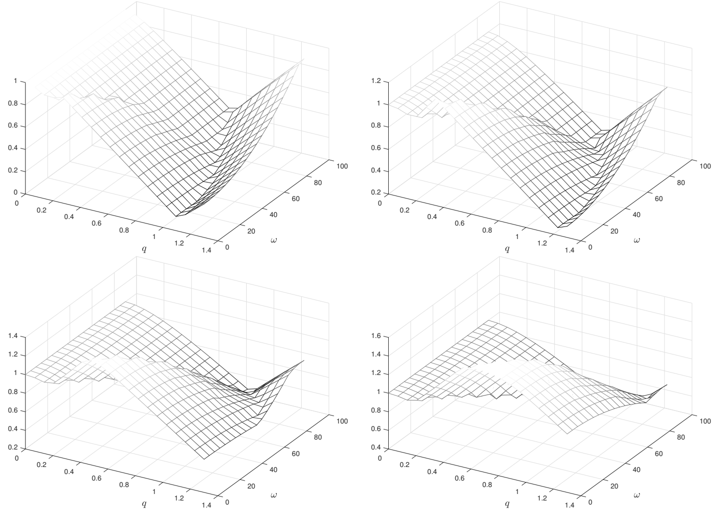
Problem: generalize this theory to the eccentric case $e' \in (0,1)$.

Gronchi and Niederman, CMDA (2020)

Mutual orbital elements: $\mathcal{E}_M = (q, e, q', e', I_M, \omega_M, \omega'_M)$



The eccentric case $e' \in (0,1)$



Graphic of $\max_{\widetilde{\mathcal{D}}_1} d_{\min}(q,\omega)$, with $\widetilde{\mathcal{D}}_1 = \{(e, I, \omega') : 0 \le e \le 1,\ 0 \le I \le \pi/2,\ 0 \le \omega' \le 2\pi\}$.
$e' = 0.1$ (top left), $e' = 0.2$ (top right), $e' = 0.3$ (bottom left), $e' = 0.4$ (bottom right). Here we set $q' = 1$.

The nodal distance

Let

$$r_+ = \frac{q(1+e)}{1+e\cos\omega}, \qquad r_- = \frac{q(1+e)}{1-e\cos\omega},$$
$$r'_+ = \frac{q'(1+e')}{1+e'\cos\omega'}, \qquad r'_- = \frac{q'(1+e')}{1-e'\cos\omega'}$$

and introduce the ascending and descending nodal distances:

$$d^+_{\rm nod} = r'_+ - r_+, \qquad d^-_{\rm nod} = r'_- - r_-.$$

The (minimal) nodal distance $\delta_{\rm nod}$ is the minimum between the absolute values of the ascending and descending nodal distances:

$$\delta_{\rm nod} = \min\{|d^+_{\rm nod}|, |d^-_{\rm nod}|\}. \tag{2}$$

Note that $\delta_{\rm nod}$ does not depend on the mutual inclination I.

Optimal bounds for $\delta_{\rm nod}$ when $e' \in (0,1)$

Let

$$\mathcal{D}_1 = \{(e,\omega') : 0 \le e \le 1, 0 \le \omega' \le \pi\},$$
$$\mathcal{D}_2 = \{(q,\omega) : 0 < q \le q_{\max}, 0 \le \omega \le \pi/2\}.$$

For each choice of $(q,\omega) \in \mathcal{D}_2$ we have

$$\max_{(e,\omega')\in\mathcal{D}_1} \delta_{\rm nod} = \max\{u^{\omega}_{\rm int}, u^{\omega}_{\rm ext}, u^{\omega}_{\rm link}\},$$

Optimal bounds for δ_{nod} when $e' \in (0, 1)$

where[1]

$$u^{\omega}_{\text{int}}(q,\omega) = p' - q,$$

$$u^{\omega}_{\text{ext}}(q,\omega) = \min\Big\{\frac{2q}{1-\cos\omega} - \frac{p'}{1-\hat{\xi}'_*},\ \frac{2q}{1+\cos\omega} - q'\Big\},$$

with

$$\hat{\xi}'_* = \min\{\xi'_*, e'\}, \qquad \xi'_*(q,\omega) = \frac{4q\cos\omega}{p'\sin^2\omega + \sqrt{p'^2\sin^4\omega + 16q^2\cos^2\omega}},$$

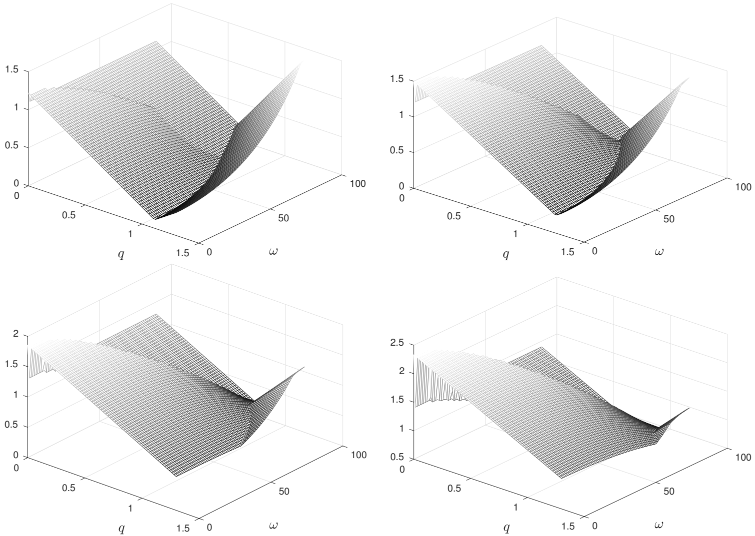
and

$$u^{\omega}_{\text{link}}(q,\omega) = \min\left\{Q' - \frac{q(1+\hat{e}_*)}{1+\hat{e}_*\cos\omega}, \frac{2q}{1-\cos\omega} - q'\right\}, \qquad (3)$$

with

$$\hat{e}_* = \max\{0, \min\{e_*, 1\}\}, \qquad e_*(q,\omega) = \frac{2(p' - q(1-e'^2))}{q(1-e'^2) + \sqrt{q^2(1-e'^2)^2 + 4p'\cos^2\omega(p' - q(1-e'^2))}}.$$

[1] we admit infinite values for the considered functions



Graphics of $(q,\omega) \mapsto \max_{(e,\omega')\in\mathcal{D}_1} \delta_{\rm nod}(q,\omega)$ for $e'=0.1$ (top left), $e'=0.2$ (top right), $e'=0.3$ (bottom left), $e'=0.4$ (bottom right). Here we set $q'=1$.

Secular evolution of crossing orbits

The restricted three-body problem

Three-body problem: Sun, Earth, asteroid

Restricted problem: the asteroid does not influence the motion of the two larger bodies

Equations of motion of the asteroid:

$$\ddot{\boldsymbol{y}} = -G\left[m_\odot \frac{(\boldsymbol{y} - \boldsymbol{y}_\odot(t))}{|\boldsymbol{y} - \boldsymbol{y}_\odot(t)|^3} + m_\oplus \frac{(\boldsymbol{y} - \boldsymbol{y}_\oplus(t))}{|\boldsymbol{y} - \boldsymbol{y}_\oplus(t)|^3}\right]$$

- $\boldsymbol{y}$ is the unknown position of the asteroid;
- $\boldsymbol{y}_\odot(t), \boldsymbol{y}_\oplus(t)$ are known functions of time, solutions of the two-body problem Sun-Earth.

The restricted three–body problem

In heliocentric coordinates

$$\ddot{\boldsymbol{x}} = -k^2\left[\frac{\boldsymbol{x}}{|\boldsymbol{x}|^3} + \mu\left(\frac{(\boldsymbol{x}-\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|^3} - \frac{\boldsymbol{x}'}{|\boldsymbol{x}'|^3}\right)\right]$$

- $\boldsymbol{x} = \boldsymbol{y} - \boldsymbol{y}_\odot$, $\boldsymbol{x}' = \boldsymbol{y}_\oplus - \boldsymbol{y}_\odot$;
- $k^2 = Gm_\odot$, $\mu = \frac{m_\oplus}{m_\odot}$ is a small parameter;
- $-k^2\mu\frac{(\boldsymbol{x}-\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|^3}$ is the direct perturbation of the planet on the asteroid;
- $k^2\mu\frac{\boldsymbol{x}'}{|\boldsymbol{x}'|^3}$ is the indirect perturbation, due to the interaction Sun-planet.

Hint! We can model the dynamics of an asteroid in the solar system by summing up the contribution of each planet to the perturbation.

Canonical formulation of the problem

Use Delaunay's variables $\mathcal{Y} = (L, G, Z, \ell, g, z)$ for the motion of the asteroid:

$$\left\{ \begin{array}{l} L = k\sqrt{a} \\ G = L\sqrt{1-e^2} \\ Z = G\cos I \end{array} \right. \qquad \left\{ \begin{array}{l} \ell = n(t-t_0) \\ g = \omega \\ z = \Omega \end{array} \right.$$

These are canonical variables, representing the osculating orbit, solution of the 2-body problem Sun-asteroid.

Denote by $\mathcal{Y}' = (L', G', Z', \ell', g', z')$ Delaunay's variables for the planet.

Canonical formulation of the problem

Hamilton's equations are

$$\dot{\mathcal{Y}} = \mathbb{J}\,\nabla_{\mathcal{Y}}\mathcal{H}\,,$$

where

$$\mathcal{H} = \mathcal{H}_0 + \epsilon\mathcal{H}_1, \qquad \epsilon = \mu k^2, \qquad \mathbb{J} = \begin{bmatrix} \mathcal{O}_3 & -\mathcal{I}_3 \\ \mathcal{I}_3 & \mathcal{O}_3 \end{bmatrix}.$$

$$\mathcal{H}_0 = -\frac{k^4}{2L^2} \qquad \text{(unperturbed part)},$$

$$\mathcal{H}_1 = -\left(\frac{1}{|\mathcal{X} - \mathcal{X}'|} - \frac{\mathcal{X}\cdot\mathcal{X}'}{|\mathcal{X}'|^3}\right) \qquad \text{(perturbing function)}.$$

Here $\mathcal{X}, \mathcal{X}'$ denote $\boldsymbol{x}, \boldsymbol{x}'$ as functions of $\mathcal{Y}$, $\mathcal{Y}'$.

The averaging method

The averaging principle is used to study the qualitative behavior of solutions of ODEs in perturbation theory, see Arnold, Kozlov, Neishtadt (1997).

unperturbed $\begin{cases} \dot{\phi} = \omega(I) \\ \dot{I} = 0 \end{cases}$ $\qquad \phi \in \mathbb{T}^n, I \in \mathbb{R}^m$

perturbed $\begin{cases} \dot{\phi} = \omega(I) + \epsilon f(\phi, I, \epsilon) \\ \dot{I} = \epsilon g(\phi, I, \epsilon) \end{cases}$

averaged $\dot{J} = \epsilon G(J)\,, \quad G(J) = \dfrac{1}{(2\pi)^n} \displaystyle\int_{\mathbb{T}^n} g(\phi, J, 0)\, d\phi_1 \dots d\phi_n$

Averaged equations

Gronchi and Milani, CMDA (1998)

Averaged Hamilton's equations:

$$\dot{\overline{Y}} = \epsilon\,\mathbb{J}\,\overline{\nabla_Y \mathcal{H}_1}\,, \tag{4}$$

with $Y = (G, Z, g, z)$.
If no orbit crossing occurs, (4) are equal to

$$\dot{\overline{Y}} = \epsilon\,\mathbb{J}\,\nabla_Y \overline{\mathcal{H}_1} \tag{5}$$

with

$$\overline{\mathcal{H}}_1 = \frac{1}{(2\pi)^2}\int_{\mathbb{T}^2} \mathcal{H}_1\, d\ell\, d\ell' = -\frac{1}{(2\pi)^2}\int_{\mathbb{T}^2} \frac{1}{|\mathcal{X} - \mathcal{X}'|}\, d\ell\, d\ell'$$

The average of the indirect term of $\mathcal{H}_1$ is zero.

Crossing singularities

If there is an orbit crossing, then averaging on the fast angles ℓ, ℓ' produces a singularity in the averaged equations:

we take into account every possible position on the orbits, thus also the collision configurations:

$$\overline{\mathcal{H}}_1 = -\frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} \frac{1}{|\mathcal{X} - \mathcal{X}'|} \, d\ell \, d\ell'$$

and

$$\left|\mathcal{X}(E_1, v_1^{(h)}) - \mathcal{X}'(E_2, v_2^{(h)})\right| = 0 \; .$$

Near-Earth asteroids and crossing orbits

(433) Eros: the first near-Earth asteroid (NEA, with $q = a(1-e) \leq 1.3$ AU), discovered in 1898; it can cross the trajectory of Mars.

from NEAR mission (NASA)

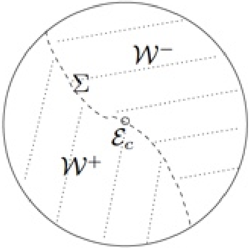
Today (March 8, 2023) we know about 31500 NEAs: several of them cross the orbit of the Earth during their evolution.

Derivative jumps

Let $\mathcal{E}_c$ be a non–degenerate crossing configuration for d_h, with only 1 crossing point.
Given a neighborhood $\mathcal{W}$ of $\mathcal{E}_c$, we set

$$\mathcal{W}^+ = \mathcal{W} \cap \{\tilde{d}_h > 0\}\,,$$

$$\mathcal{W}^- = \mathcal{W} \cap \{\tilde{d}_h < 0\}\,.$$

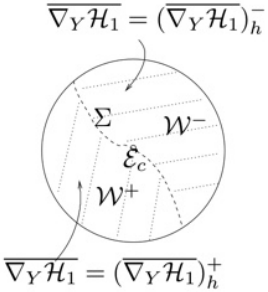


The averaged vector field $\overline{\nabla_Y \mathcal{H}_1}$ is not defined on $\Sigma = \{d_H = 0\}$.

Derivative jumps

Gronchi and Tardioli, DCDS-B (2013)

The averaged vector field $\overline{\nabla_Y \mathcal{H}_1}$ can be naturally extended to two Lipschitz–continuous vector fields $(\overline{\nabla_Y \mathcal{H}_1})_h^{\pm}$ on a neighborhood $\mathcal{W}$ of $\mathcal{E}_c$. The components of the extended fields, restricted to $\mathcal{W}^+$, $\mathcal{W}^-$ respectively, correspond to $\frac{\partial \overline{\mathcal{H}_1}}{\partial y_k}$.



Derivative jumps

Moreover the following relations hold:

$$\begin{aligned}\mathrm{Diff}_h\left(\frac{\partial\overline{\mathcal{H}_1}}{\partial y_k}\right) &\overset{def}{=} \left(\frac{\partial\overline{\mathcal{H}_1}}{\partial y_k}\right)_h^- - \left(\frac{\partial\overline{\mathcal{H}_1}}{\partial y_k}\right)_h^+ = \\ &= -\frac{1}{\pi}\left[\frac{\partial}{\partial y_k}\left(\frac{1}{\sqrt{\det(\mathcal{A}_h)}}\right)\tilde{d}_h + \frac{1}{\sqrt{\det(\mathcal{A}_h)}}\frac{\partial\tilde{d}_h}{\partial y_k}\right],\end{aligned}$$

where y_k is a component of Delaunay's elements Y.

Extraction of the singularity

$$d^2(\mathcal{E}, V) = d_h^2(\mathcal{E}) + (V - V_h) \cdot \mathcal{A}_h(\mathcal{E})(V - V_h) + \mathcal{R}_3^{(h)}(\mathcal{E}, V)\,,$$

where

$$2\mathcal{A}_h(\mathcal{E}) = \frac{\partial^2 d^2}{\partial V^2}(\mathcal{E}, V_h(\mathcal{E}))$$

is the Hessian matrix of d^2 in V_h and $\mathcal{R}_3^{(h)}$ is Taylor's remainder in the integral form.

Introduce the approximated distance

$$\delta_h = \sqrt{d_h^2 + (V - V_h) \cdot \mathcal{A}_h(V - V_h)}\,.$$

Extraction of the singularity

Consider the following decomposition:

$$\mathcal{W}\setminus\Sigma \ni \mathcal{E} \mapsto \int_{\mathbb{T}^2} \frac{\partial}{\partial y_k}\frac{1}{d}\,d\ell d\ell'$$
$$= \int_{\mathbb{T}^2} \frac{\partial}{\partial y_k}\Big(\frac{1}{d}-\frac{1}{\delta_h}\Big)\,d\ell d\ell' + \int_{\mathbb{T}^2} \frac{\partial}{\partial y_k}\frac{1}{\delta_h}\,d\ell d\ell'$$

We can prove that:

i) the two maps $\mathcal{W}^{\pm} \ni \mathcal{E} \mapsto \int_{\mathbb{T}^2} \frac{\partial}{\partial y_k}\frac{1}{\delta_h}\,d\ell d\ell'$ admits two different analytic extensions to $\mathcal{W}$;

ii) the map $\mathcal{W}\setminus\Sigma \ni \mathcal{E} \mapsto \int_{\mathbb{T}^2} \frac{\partial}{\partial y_k}\Big(\frac{1}{d}-\frac{1}{\delta_h}\Big)\,d\ell d\ell'$ admits a Lipschitz–continuous extension to $\mathcal{W}$.

Idea of the proof of i)

$$\mathcal{W} \setminus \Sigma \ni \mathcal{E} \mapsto \int_{\mathbb{T}^2} \frac{\partial}{\partial y_k} \frac{1}{\delta_h} \, d\ell \, d\ell' = \frac{\partial}{\partial y_k} \int_{\mathbb{T}^2} \frac{1}{\delta_h} \, d\ell \, d\ell'$$

Moreover, set

$$\mathcal{D} = \{V \in \mathbb{T}^2 : (V - V_h) \cdot \mathcal{A}_h (V - V_h) \leq r\}.$$

We have

$$\int_{\mathcal{D}} \frac{1}{\delta_h} \, d\ell \, d\ell' = \frac{2\pi}{\sqrt{\det \mathcal{A}_h}} \left(\sqrt{d_h^2 + r} - d_h\right)$$

Idea of the proof of i)

Moreover we have

$$\int_{\mathbb{T}^2} \frac{\partial}{\partial y_k}\frac{1}{\delta_h}\,d\ell\,d\ell' = \frac{\partial}{\partial y_k}\Big(\frac{2\pi}{\sqrt{\det \mathcal{A}_h}}\Big)\big(\sqrt{d_h^2+r^2}-d_h\big) + $$
$$+\frac{2\pi}{\sqrt{\det \mathcal{A}_h}}\frac{d_h}{\sqrt{d_h^2+r^2}}\frac{\partial d_h}{\partial y_k} - \frac{2\pi}{\sqrt{\det \mathcal{A}_h}}\frac{\partial d_h}{\partial y_k} + \frac{\partial}{\partial y_k}\int_{\mathbb{T}^2\setminus\mathcal{D}}\frac{1}{\delta_h}\,d\ell\,d\ell'$$

so that we take

$$\Big(\int_{\mathbb{T}^2}\frac{\partial}{\partial y_k}\frac{1}{\delta_h}\,d\ell\,d\ell'\Big)_h^{\pm} = \frac{\partial}{\partial y_k}\Big(\frac{2\pi}{\sqrt{\det \mathcal{A}_h}}\Big)\big(\sqrt{d_h^2+r^2}\mp\tilde{d}_h\big) + $$
$$+\frac{2\pi}{\sqrt{\det \mathcal{A}_h}}\frac{\tilde{d}_h}{\sqrt{d_h^2+r^2}}\frac{\partial \tilde{d}_h}{\partial y_k} \mp \frac{2\pi}{\sqrt{\det \mathcal{A}_h}}\frac{\partial \tilde{d}_h}{\partial y_k} + \frac{\partial}{\partial y_k}\int_{\mathbb{T}^2\setminus\mathcal{D}}\frac{1}{\delta_h}\,d\ell\,d\ell'$$

Generalized solutions

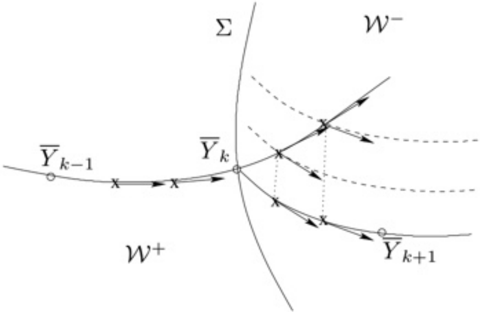
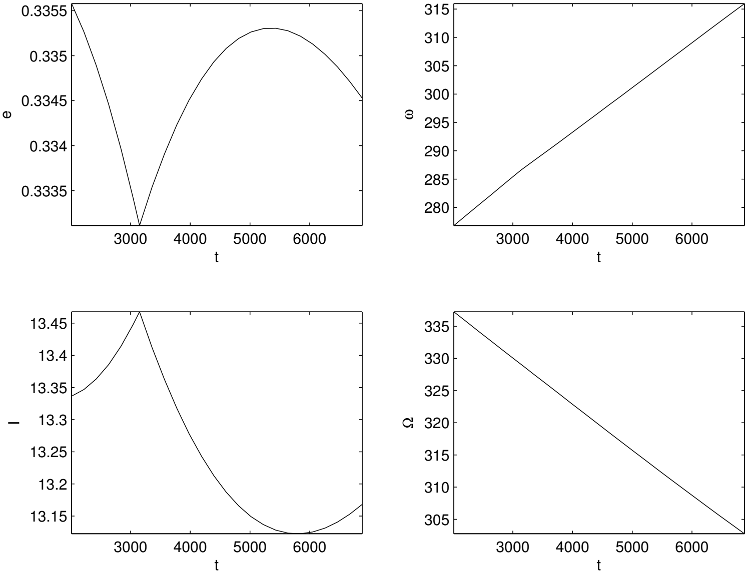


Figure: Runge-Kutta-Gauss method and continuation of the solutions of equations (4) beyond the singularity.

The averaged solutions are piecewise–smooth

Averaged evolution of (1620) Geographos



Proper elements for NEAs: (1620) Geographos

NEODyS-2

Near Earth Objects - Dynamic Site

Sponsored by esa, Università di Pisa, SpaceDyS

Go to NEA ▶

Home | Objects | Observatories | Search | Risk page | NEA elements | Related sites | Info & Credits

(1620) Geographos ▷ PROPER ELEMENTS

[Help]

- Summary
- Ephemerides
- Obs prediction
- Orbital info
- MOID
- Proper elements
- Observational info
- Close approaches
- Physical info
- NEOCC Orbit animation
- Additional notes

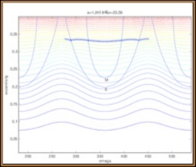
Proper elements and proper frequencies

Prop. ele.

e_{min}	0.3287
e_{max}	0.3359
i_{min}	13.301
i_{max}	13.935
ω_{min}	0
ω_{max}	0
D	0

Prop. freq.

g	4.403
s	-21.972
lf	0



(Click on the image to enlarge)

Encounter conditions

	Venus	Earth	Mars	Jupiter	Saturn
Crossings	0	4	4	0	0

u	θ	φ	λ	u_y
0.3888	86.6198	309.67	208.597	0.0229
0.3888	86.6198	50.3296	71.299	0.0229
0.3888	86.6198	129.67	41.2507	0.0229
0.3888	86.6198	230.33	278.582	0.0229

Note: "crossings" is the number of crossings per revolution or libration period of omega.

NEODyS ∴ Objects ∴ Geographos ∴ Proper elements

Contact »

NEODyS-2

Near Earth Objects - Dynamic Site

Sponsored by esa · Università di Pisa · SpaceDyS

Go to NEA ▶

Home | Objects | Observatories | Search | Risk page | NEA elements | Related sites | Info & Credits

(2102) Tantalus ▷ PROPER ELEMENTS

[Help]

- Summary
- Ephemerides
- Obs prediction
- Orbital info
- MOID
- Proper elements
- Observational info
- Close approaches
- Physical info
- NEOCC Orbit animation

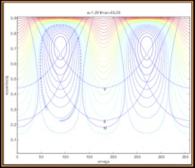
Proper elements and proper frequencies

Prop. ele.

e_{min}	0.2235
e_{max}	0.8553
i_{min}	36.179
i_{max}	64.59
ω_{min}	47.732
ω_{max}	132.277
D	84.545

Prop. freq.

g	0
s	-15.95
lf	-27.614



(Click on the image to enlarge)

Encounter conditions

	Venus	Earth	Mars	Jupiter	Saturn
Crossings	4	4	0	0	0

u	θ	φ	λ	u_y
1.129	117.708	124.004	235.645	-0.525
1.129	117.708	345.305	91.1116	-0.525
1.129	117.708	165.305	86.6798	-0.525
1.129	117.708	304.004	302.147	-0.525

Note: "crossings" is the number of crossings per revolution or libration period of omega.

NEODyS ∴ Objects ∴ Tantalus ∴ Proper elements

Contact »

Secular evolution of the orbit distance

Define the secular evolution of the minimal distances

$$\overline{d}_h(t) = \tilde{d}_h(\bar{\mathcal{E}}(t))\,, \qquad \overline{d}_{min}(t) = \tilde{d}_{min}(\bar{\mathcal{E}}(t))$$

in an open interval containing a crossing time t_c.

Proposition: Assume t_c is a crossing time and $\mathcal{E}_c = \overline{\mathcal{E}}(t_c)$ is a non-degenerate crossing configuration with only one crossing point, i.e. $d_h(\mathcal{E}_c) = 0$. Then there exists an interval (t_a, t_b), $t_a < t_c < t_b$ such that $\overline{d}_h \in C^1((t_a, t_b); \mathbb{R})$.

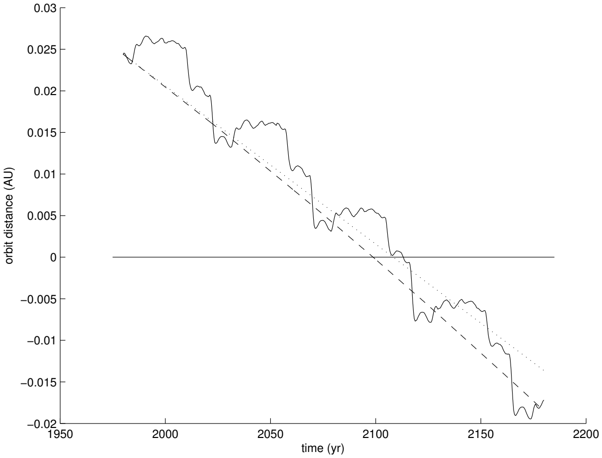
Secular evolution of the orbit distance

idea of the proof:

$$\begin{aligned}\lim_{t\to t_c^-} \dot{\overline{d}}_h(t) - \lim_{t\to t_c^+} \dot{\overline{d}}_h(t) &= \mathrm{Diff}_h\big(\overline{\nabla_Y \mathcal{H}_1}\big)\cdot \epsilon\, \mathbb{J}_2 \nabla_Y \tilde{d}_h\Big|_{\mathcal{E}=\mathcal{E}_c} \\ &= -\frac{\epsilon}{\pi\sqrt{\det \mathcal{A}_h}}\,\{\tilde{d}_h, \tilde{d}_h\}_Y\Big|_{\mathcal{E}=\mathcal{E}_c} = 0\,,\end{aligned}$$

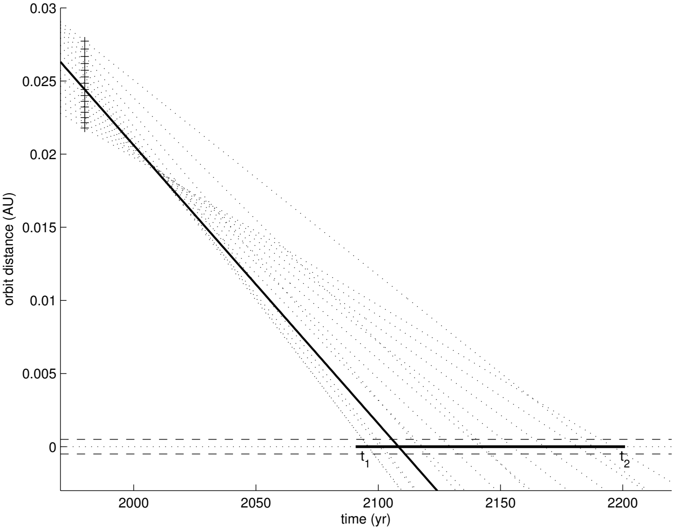
The secular evolution of $\tilde{d}_{min}$ is more regular than that of the orbital elements in a neighborhood of a planet crossing time.

Evolution of the orbit distance for 1979 XB



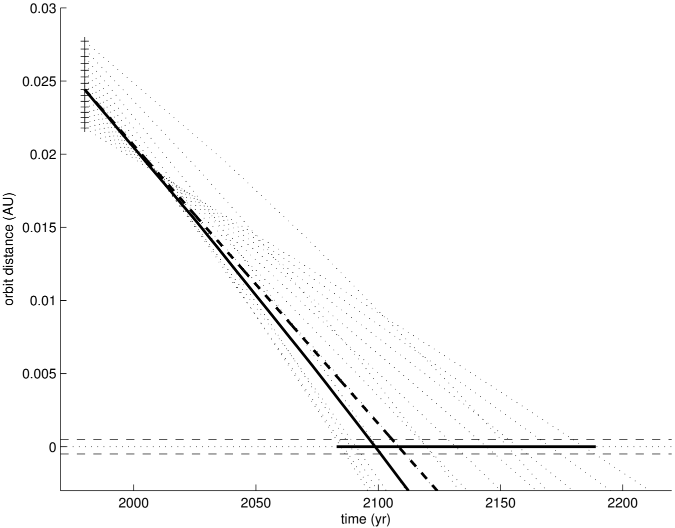
Transition through a planet crossing for 1979 XB

linearized secular evolution



Transition through a planet crossing for 1979 XB

nonlinear secular evolution



Thank you for your attention!