Peak minimization for compartmental models

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Outline

1. Motivation: The covid problem
2. Peak minimization on a SIR dynamic
3. General models of peak minimization
   - Planar dynamics with L^1 constraints
   - Reformulations
4. Conclusion

Outline

Context: Covid desease

High peaks overcrowd the healthy system.

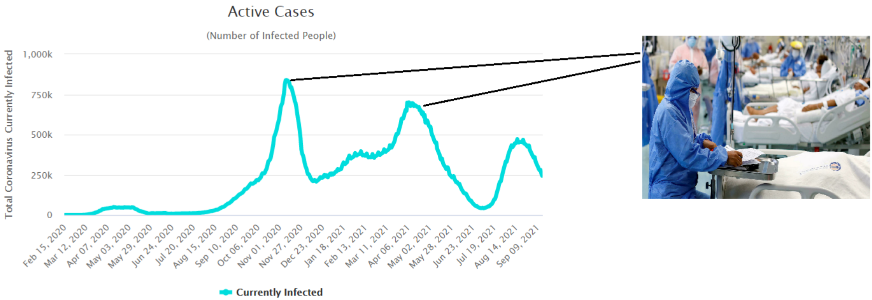

Figure: France's data from www.worldometers.info

Outline

SIR model

A classical SIR model corresponds to:

$$\begin{cases} \dot{S}(t) = -\beta S(t)I(t) \\ \dot{I}(t) = \beta S(t)I(t) - \gamma I(t) \\ \dot{R}(t) = \gamma I(t) \end{cases}$$

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where:

- $S(t)$: portion of susceptible individuals at time t.
- $I(t)$: portion of infected individuals at time t.
- $R(t)$: portion of recovered individuals at time t.
- β: transmission rate.
- γ: recovery rate.

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- β: transmission rate.
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And

$$S(t) + I(t) + R(t) = 1, \forall t \geq 0$$

Problem formulation

We consider the identical dynamic

$$\begin{aligned}\dot{S}(t) &= -(1-u(t))\beta S(t)I(t)\\ \dot{I}(t) &= (1-u(t))\beta S(t)I(t) - \gamma I(t)\end{aligned}$$

with the positive initial condition $(S(0), I(0)) = (S_0, I_0)$, and $S_0 + I_0 \leq 1$.

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$$\int_0^\infty u(t)dt \leq Q. \tag{1}$$

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We want:

$$\inf_{u(\cdot)\in\mathcal{U}} \max_{t\geq 0} I(t), \tag{2}$$

where $\mathcal{U}$ denotes the set of measurable functions $u(\cdot)$ that take values in $[0, 1]$ and satisfying (1).

Equivalently, one can consider the extended dynamics.

$$\begin{cases} \dot{S}(t) = -\beta S(t) I(1-u(t)), \\ \dot{I}(t) = \beta S(t) I(t)(1-u(t)) - \gamma I(t), \\ \dot{C}(t) = -u(t), \end{cases} \tag{3}$$

with the initial condition $(S(0), I(0), C(0)) = (S_0, I_0, Q)$ and the state constraint

$$C(t) \geq 0, \quad t \geq 0.$$

Assumptions

Assumption 1

The *basic reproduction number* $\mathcal{R}_0$ is larger than one.

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$$\mathcal{R}_0 := \frac{\beta}{\gamma} > 1.$$

Let us denote the *immunity threshold*

$$S_h := \mathcal{R}_0^{-1} = \frac{\gamma}{\beta} < 1.$$

Assumption 2

We consider the non trivial case:

$$S_0 > S_h.$$

The NSN (null-singular-null) strategy

The maximum of $I(\cdot)$ in the not controlled case is:

$$I_h := I_0 + S_0 - S_h - S_h \log\left(\frac{S_0}{S_h}\right). \tag{4}$$

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Definition 1

For $\bar{I} \in [I_0, I_h]$, consider the feedback control

$$\psi_{\bar{I}}(I, S) := \begin{cases} 1 - \frac{S_h}{S}, & \text{if } I = \bar{I} \text{ and } \ S > S_h, \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

We denote the L^1 norm associated to the NSN control

$$\mathcal{L}(\bar{I}) := \int_0^{+\infty} u^{\psi_{\bar{I}}}(t) dt, \quad \bar{I} \in [I_0, I_h],$$

where $u^{\psi_{\bar{I}}}(\cdot)$ is the control generated by the feedback (11).

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This control strategy consists in three phases:

1. No intervention until the prevalence I reaches $\bar{I}$ (null control).
2. Maintain the prevalence I equal to $\bar{I}$ by adjusting the interventions until S reaches S_h or the budget is entirely consumed (singular control).
3. No longer intervention when $S < S_h$ (null control).

The NSN (null-singular-null) strategy

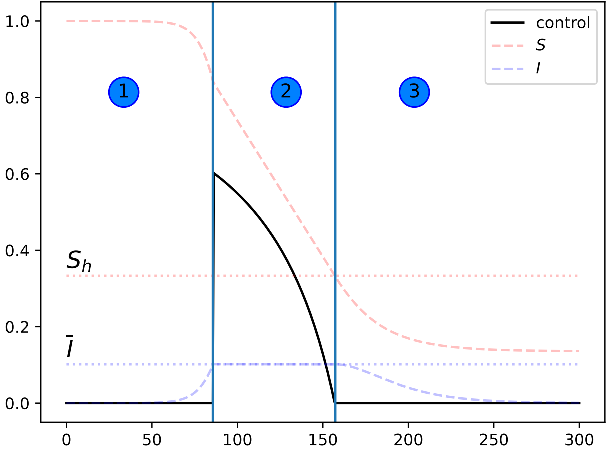

Figure: NSN strategy

The NSN (null-singular-null) strategy

Lemma 1

For any $\bar{I} \in [I_0, I_h]$*, the maximal value of the control* $u^{\psi_{\bar{I}}}(\cdot)$ *is given by*

$$u_{max}(\bar{I}) := 1 - \frac{S_h}{\bar{S}} < 1,$$

where $\bar{S}$ *is solution of*

$$\bar{S} - S_h \log \bar{S} = S_0 + I_0 - S_h \log S_0 - \bar{I}.$$

Moreover, any solution given by the NSN strategy verifies

$$\max_{t \geq 0} I(t) = \bar{I}.$$

Computing L^1 norm

Proposition 1

For $u^{\psi_{\bar{I}}}(\cdot)$ one has

$$\mathcal{L}(\bar{I}) = \frac{I_h - \bar{I}}{\beta S_h \bar{I}}, \quad \bar{I} \in [I_0, I_h]. \tag{6}$$

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Corollary

When $Q \leq \frac{I_h - I_0}{\beta S_h I_0}$, the smallest $\bar{I} \in [I_0, I_h]$ for which the solution with the NSN strategy is admissible, is given by the value

$$\bar{I}^\star(Q) := \frac{I_h}{Q\beta S_h + 1} \tag{7}$$

and one has

$$\mathcal{L}(\bar{I}^\star(Q)) = Q.$$

Main result

Proposition 2 (M-Rapaport)

Let Assumptions 1 and 2 be fulfilled. Then, the NSN feedback is optimal with

$$\bar{I} = \begin{cases} \bar{I}^\star(Q), & Q < \frac{I_h - I_0}{\beta S_h I_0}, \\ I_0, & Q \geq \frac{I_h - I_0}{\beta S_h I_0}, \end{cases}$$

where $\bar{I}^\star(Q)$ *is defined in* (7), *and* $\bar{I}$ *is the optimal value of problem* (2).

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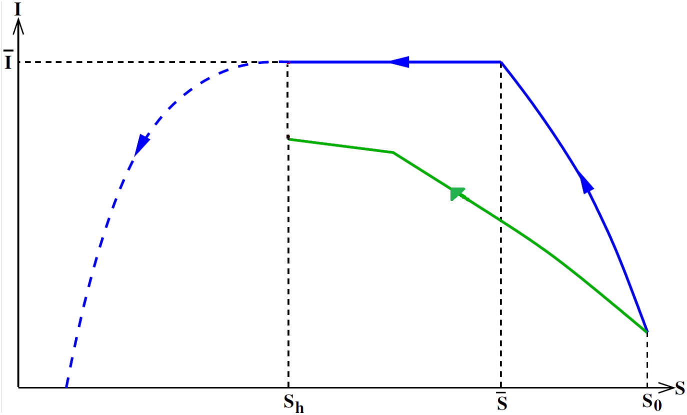
Sketch of proof: Non trivial case $Q < \frac{I_h - I_0}{\beta S_h I_0}$.
To remember:

$$\dot{C}(t) = -u(t).$$

and we pass to the (S, I) plane

Main result

NSN strategy: $(S^\star(\cdot), I^\star(\cdot), C^\star(\cdot))$ with $\bar{I} = \bar{I}^\star(Q)$, and control $u^\star(\cdot)$.
Any other solution: $(S(\cdot), I(\cdot), C(\cdot))$ with $\max_t I(t) < \bar{I}$.



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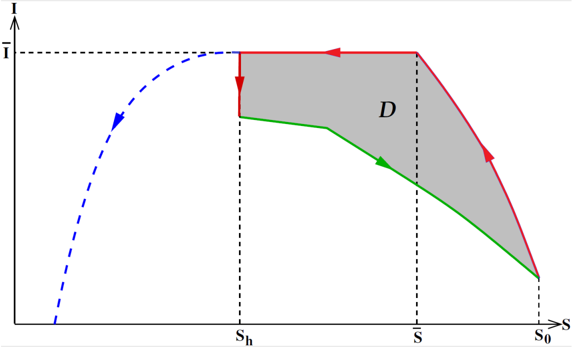

$$\Gamma := \{(\tilde{S}(\tau), \tilde{I}(\tau)),\ \tau \in [0, T]\} \cup \{(S(T + t_h - t), I(T + t_h - t)),\ \tau \in [T, T + t_h]\},$$

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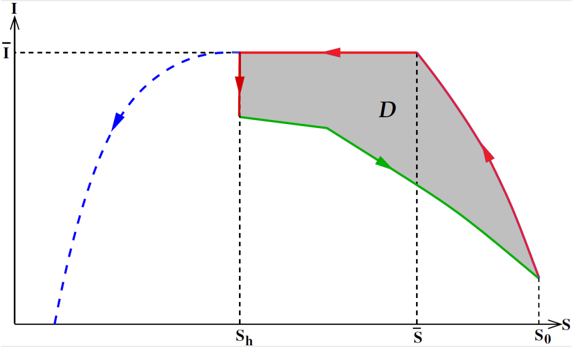

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Using Green theorem we proved:

$$\tilde{C}(T) - C(t_h) = \oint_\Gamma dC > 0$$

Outline

Formulation general problem

We consider the following dynamical system in a domain $\mathcal{D} \subset \mathbb{R}^{n+1}$.

$$\begin{cases} \dot{x} = f(x, y, u) \\ \dot{y} = g(x, y, u) \end{cases} \tag{8}$$

$\mathcal{U} := \{u(\cdot) : [0, T] \mapsto U, \text{mesurable}\}$ and $(x_0, y_0) \in \mathcal{D},\ T > 0$.

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The solutions set:

$$\mathcal{S} := \{(x(\cdot), y(\cdot)) \in \mathcal{AC}([0, T], \mathbb{R}^{n+1}), \quad \begin{array}{l} \text{sol. of (8) for } u(\cdot) \in \mathcal{U} \\ \text{with } (x(0), y(0)) = (x_0, y_0)\} \end{array}$$

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The optimal control problem:

$$\mathcal{P} : \quad \inf_{u(\cdot)\in\mathcal{U}} \left(\max_{t\in[0,T]} y(t) \right) = \inf_{(x(\cdot),y(\cdot))\in\mathcal{S}} \left(\max_{t\in[0,T]} y(t) \right)$$

State of art

- L^∞-criterion.

$$\inf_{u(\cdot)} \operatorname*{ess\,sup}_{t\in[t_0,T]} y(t)$$

where $y(t) = \eta(\xi(t))$ with $\xi(\cdot)$ solution of a controlled system $\dot{\xi} = \phi(\xi, u)$, $\xi(t_0) = \xi_0$.

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Outline

We consider a dynamics defined on an invariant domain $\mathcal{D}$ of $\mathbb{R}^2$

$$\left\{ \begin{array}{rcl} \dot{x} & = & f_1(x,y) + g_1(x,y)u \\ \dot{y} & = & f_2(x,y) + g_2(x,y)u \end{array} \right. \qquad u \geq 0 \tag{9}$$

with initial condition $(x_0, y_0) \in \mathcal{D}$, where f_1, f_2, g_1, g_2 are at least C^1. We consider the following optimal control problem:

$$\inf_{u(\cdot)} \sup_{t \geq 0} y(t), \tag{10}$$

subject to the constraint

$$\int_0^{+\infty} u(t)dt \leq K,$$

Let us define the sub-domains

$$\mathcal{D}_\pm := \{(x,y) \in \mathcal{D} \ ; \ \pm f_2(x,y) > 0\}, \ \mathcal{D}_0 := \{(x,y) \in \mathcal{D} \ ; \ f_2(x,y) = 0\}$$

and the function

$$\Delta(x,y) := f_2(x,y)g_1(x,y) - f_1(x,y)g_2(x,y).$$

Assumptions.

1. With $u = 0$, the domain $\mathcal{D}_-$ is invariant and for any initial condition in $\mathcal{D}_+$, the solution enters the domain $\mathcal{D}_-$ in finite time.
2. For any $(x, y) \in \mathcal{D}_+$, one has $f_1(x, y) < 0$ and $f_2(x, y) + g_2(x, y) < 0$
3. For any (x, y) in $\mathcal{D}_+$, one has $\Delta(x, y) < 0$ and

$$\frac{\partial f_2(x, y)}{\partial x} > 0 \quad \text{and} \quad \frac{\partial}{\partial y}\left(\frac{f_2(x, y)}{\Delta(x, y)}\right) > 0$$

4. For any $(x, y) \in \mathcal{D}_0$, one has $g_2(x, y) < 0$ and

$$\text{sgn}(\nabla f_2(x, y).f(x, y)) + \text{sgn}(\nabla f_2(x, y).g(x, y)) = 0$$

(where the sgn function is defined as $\text{sgn}(0) = 0$ and $\text{sgn}(\xi) = \xi/|\xi|$ for $\xi \neq 0$).

Definition 2

For $\bar{y} \in [y_0, y_{max}]$, consider the feedback control

$$\psi_{\bar{y}}(x,y) := \begin{cases} k(x) := -\dfrac{f_2(x,\bar{y})}{g_2(x,\bar{y})}, & \text{if } y = \bar{y} \text{ and } (x,\bar{y}) \in \mathcal{D}_+, \\ 0, & \text{otherwise.} \end{cases} \tag{11}$$

Proposition 3

For any $\bar{y} \in [y_0, y_{max}]$, one has

$$\mathcal{L}(\bar{y}) := \int_0^{+\infty} u^{\psi_{\bar{y}}}(t)dt = \int_{x_h(\bar{y})}^{\bar{x}(\bar{y})} \frac{-f_2(x,\bar{y})}{\Delta(x,\bar{y})} dx \tag{12}$$

where $x_h(\bar{y}) := \max\{x \leq \bar{x}(\bar{y});\ f_2(x,\bar{y}) = 0\}$. Moreover, the map $\bar{y} \mapsto \mathcal{L}(\bar{y})$ is **decreasing**.

Proposition 4

Assume one has

$$\frac{\partial}{\partial y}\left(\frac{f_2(x,y)}{\Delta(x,y)}\right)+\frac{\partial}{\partial x}\left(\frac{f_1(x,y)}{\Delta(x,y)}\right)>0, \quad (x,y)\in\mathcal{D}_+,\ y\leq y_{max} \qquad (13)$$

If $\mathcal{L}(y_0)>K$*, then there exists* $y^*\in[y_0,y_{max}]$ *such that* $\mathcal{L}(y^*)=K$ *and the feedback* ψ_{y^*} *is optimal.*

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If $\mathcal{L}(y_0) > K$, then there exists $y^* \in [y_0, y_{max}]$ such that $\mathcal{L}(y^*) = K$ and the feedback ψ_{y^*} is optimal.

Examples 1

The SIR model presented.

Examples

Examples 2

The resource-consumer (or batch bioprocess) model where the control limits the contact between the resource and the consumer

$$\begin{cases} \dot{x} &= -\frac{xy}{1+x}(1-u) \\ \dot{y} &= \frac{xy}{1+x}(1-u) - \alpha y \end{cases} \qquad u \in [0,1]$$

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Examples 3

The same resource-consumer model as the previous example but with a ratio-dependent growth

$$\begin{cases} \dot{x} = -\frac{xy}{x+y}(1-u) \\ \dot{y} = \frac{xy}{x+y}(1-u) - \alpha y \end{cases} \qquad u \in [0,1]$$

Outline

Reformulation $\mathcal{P}_0$

The first basic reformulation is

$$\mathcal{P}_0 : \inf_{u(\cdot)\in\mathcal{U}} z(T)$$

for the extended dynamics in $\mathcal{D} \times \mathbb{R}$

$$\left\{ \begin{array}{l} \dot{x} = f(x, y, u) \\ \dot{y} = g(x, y, u) \\ \dot{z} = 0 \end{array} \right.$$

under the state constraint

$$\mathcal{C} : \quad z(t) - y(t) \geq 0, \; t \in [0, T]$$

where $(x(0), y(0)) = (x_0, y_0)$ and $z(0)$ is free .

Reformulation $\mathcal{P}_1$

The first basic reformulation is

$$\mathcal{P}_1 : \inf_{u(\cdot)\in\mathcal{U}} z(T)$$

for the extended dynamics in $\mathcal{D} \times \mathbb{R}$

$$\begin{cases} \dot{x} = f(x, y, u) \\ \dot{y} = g(x, y, u) \\ \dot{z} = \max(g(x, y, u), 0)(1 - v) \quad , v \in [0, 1] \end{cases}$$

under the state constraint

$$\mathcal{C} : \quad z(t) - y(t) \geq 0, \; t \in [0, T]$$

where $(x(0), y(0)) = (x_0, y_0)$ and $z(0) = y_0$.

Reformulation $\mathcal{P}_2$

The first basic reformulation is

$$\mathcal{P}_2 : \inf_{u(\cdot)\in\mathcal{U}} z(T)$$

for the extended dynamics in $\mathcal{D} \times \mathbb{R}$

$$\begin{cases} \dot{x} = f(x, y, u) \\ \dot{y} = g(x, y, u) \\ \dot{z} = \max(g(x, y, u), 0)(1 - v) \quad , v \in [0, 1] \end{cases}$$

under the state constraint

$$\mathcal{C}_m : \quad \max(y(t) - z(t), 0)(1 - v(t)) + z(t) - y(t) \geq 0, \quad \text{a.e. } t \in [0, T]$$

where $(x(0), y(0)) = (x_0, y_0)$ and $z(0) = y_0$.

Idea

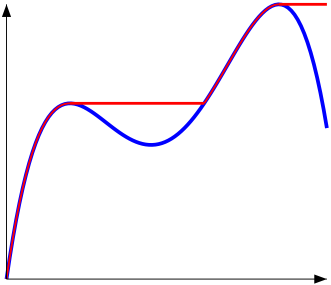

Figure: Illustration of the function z (red) corresponding to a function y (blue)

Reformulation $\mathcal{P}_3$

We posit $\Pi = (x, y, z) \in \mathcal{D} \times \mathbb{R}$ with dynamic:

$$\dot{\Pi} \in F(\Pi) := \bigcup_{(u,v)\in U\times[0,1]} \begin{bmatrix} f(x,y,u) \\ g(x,y,u) \\ h(x,y,z,u,v) \end{bmatrix} \tag{14}$$

and

$$h(x,y,z,u,v) = \max(g(x,y,u),0)(1 - v\mathbb{1}_{\mathbb{R}^+}(z-y)).$$

Let $\mathcal{S}_\ell := \{\Pi(\cdot) \in AC., \dot{\Pi} \in F(\Pi)$ and $\Pi(0) = (x_0, y_0, y_0)$

$$\mathcal{P}_3 : \quad \inf_{\Pi(\cdot)\in\mathcal{S}_\ell} z(T).$$

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Reformulation $\mathcal{P}_3^\theta$

A dynamic parameterized by $\theta > 0$

$$\left\{ \begin{array}{l} \dot{x} = f(x, y, u) \\ \dot{y} = g(x, y, u) \\ \dot{z} = h_\theta(x, y, z, u, v) \end{array} \right. \qquad (15)$$

with

$$h_{\theta}(x, y, z, u, v) = \max(g(x, y, u), 0)(1 - v\, e^{-\theta \max(y - z, 0)})$$

The family of Mayer problems

$$\mathcal{P}_3^\theta : \quad \inf_{\Pi(\cdot) \in \mathcal{S}_\theta} z(T)$$

where $\mathcal{S}_\theta$ denotes the set of absolutely continuous solutions $\Pi(\cdot) = (x(\cdot), y(\cdot), z(\cdot))$ of (15) for the initial condition $\Pi(0) = (x_0, y_0, y_0)$

Returning to SIR model

Remembering the dynamic

$$
\begin{aligned}
\dot{S}(t) &= -(1-u(t))\beta S(t)I(t) \\
\dot{I}(t) &= (1-u(t))\beta S(t)I(t) - \gamma I(t) \\
\dot{C}(t) &= -u(t),
\end{aligned}
$$

Returning to SIR model

Remembering the dynamic

$$\begin{aligned}\dot{S}(t) &= -(1-u(t))\beta S(t)I(t)\\ \dot{I}(t) &= (1-u(t))\beta S(t)I(t)-\gamma I(t)\\ \dot{C}(t) &= -u(t),\end{aligned}$$

with initial condition (S_0, I_0, Q) and $C(T) \geq 0$

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with initial condition (S_0, I_0, Q) and $C(T) \geq 0$ and we want

$$\min_{u} \max_{t\in[0,T]} I(t)$$

β	γ	T	Q	$S(0)$	$I(0)$	$\bar{I}$
0.21	0.07	300	28	$1-10^{-6}$	10^{-6}	0.1015

Numerical examples

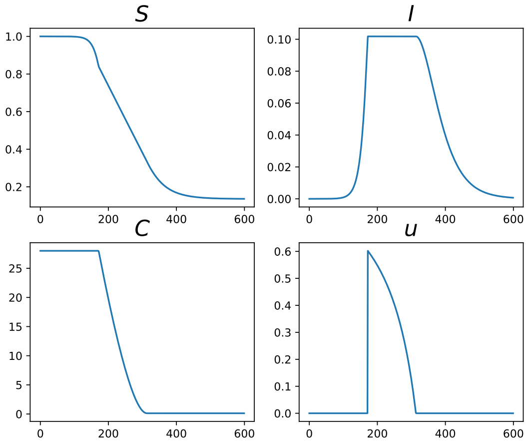

Figure: The optimal solution for the SIR problem using NSN strategy

Numerical solutions

To improve convergence we used the approximation:

$$\frac{\log\left(e^{\lambda\xi}+1\right)}{\lambda} \underset{\lambda\to+\infty}{\longrightarrow} \max(\xi,0), \quad \xi\in\mathbb{R}$$

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Using $\lambda = 100$ we obtain

problem	$\max_{t\in[0,T]} y(t)$	computation time
$\mathcal{P}_0$	0.1015	$10\,s$
$\mathcal{P}_1$	0.1015	$12\,s$
$\mathcal{P}_2$	0.1015	$13\,s$

Table: Comparison of performances for problems $\mathcal{P}_0$, $\mathcal{P}_1$, $\mathcal{P}_2$

Numerical solutions

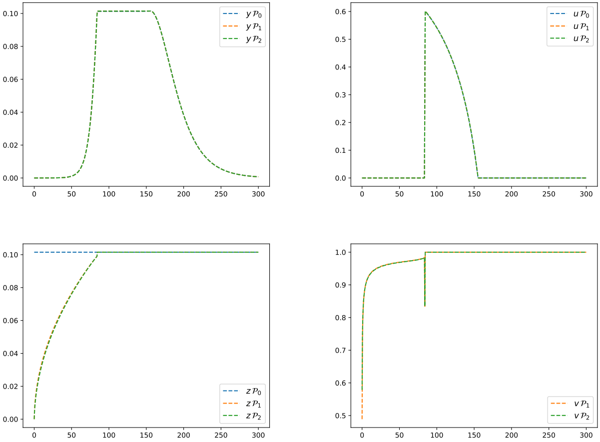

Figure: Comparisons of numerical results for the methods $\mathcal{P}_0$, $\mathcal{P}_1$, $\mathcal{P}_2$

Numerical solutions $\mathcal{P}_3^\theta$

The function h_θ is approximated by the expression

$$h_\theta(x,y,z,u,v) \simeq \frac{\log\left(e^{\lambda_1 g(x,y,u)}+1\right)}{\lambda_1}\left(1 - v e^{\frac{\theta}{\lambda_2}\log\left(e^{\lambda_2(y-z)}+1\right)}\right)$$

which depends on three parameters λ_1, λ_2 and θ.

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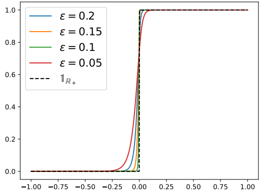
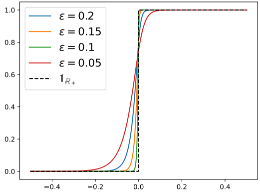
which depends on three parameters λ_1, λ_2 and θ. We can approximate indicator function depending of a parameter $\varepsilon = \varepsilon(\theta,\lambda_2)$





Numerical solutions

ε	θ	$z(T)$	$\max\limits_{t\in[0,T]} y(t)$	computation time
0.2	40.18	0.0684	0.1038	80 s
0.15	84.31	0.0823	0.1038	65 s
0.1	230.26	0.0954	0.1037	51 s
0.075	460.49	0.0993	0.1050	83 s
0.05	1198.29	0.1010	0.1036	97 s

Table: Comparison of performances for problem $\mathcal{P}_3^\theta$

Numerical solutions

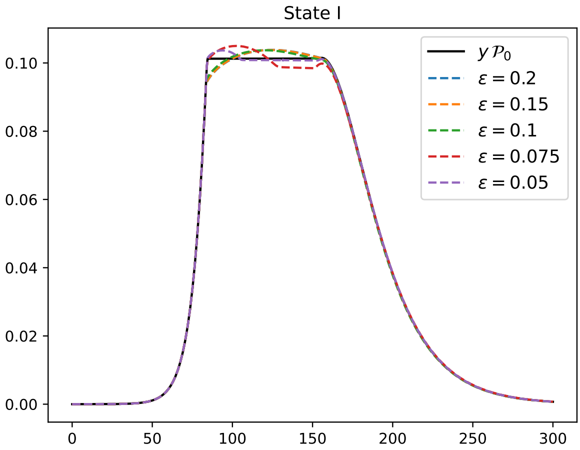


Figure: Comparison of the numerical results for problem $\mathcal{P}_3^\theta$

Numerical solutions

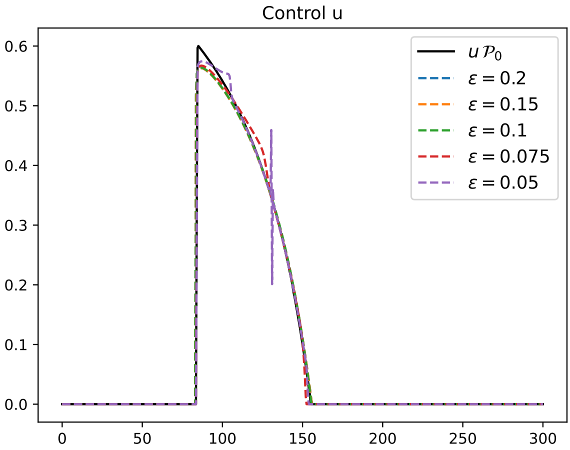


Figure: Comparison of the numerical results for problem $\mathcal{P}_3^{\theta}$

Numerical solutions

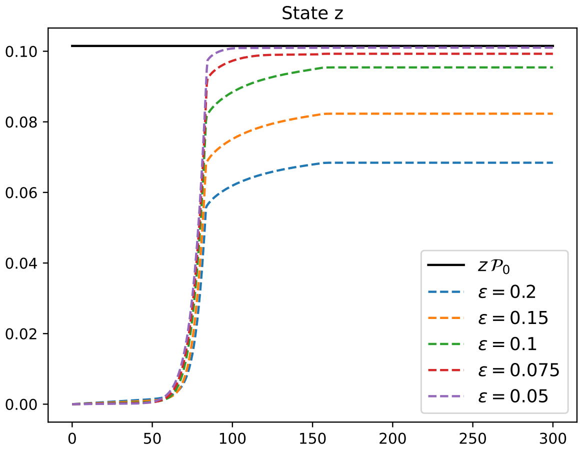


Figure: Comparison of the numerical results for problem $\mathcal{P}_3^\theta$

A simple SIR-vector model (inspired from Wei, Li, Matcheva 2007)

The host population follow a SIR dynamic and we call $V(t)$ to the portion infectious vector (ex: mosquitoes) at time t.

$$\begin{aligned}
\dot{S}(t) &= -\beta S(t)V(t) \\
\dot{I}(t) &= \beta S(t)V(t) - \gamma I(t) \\
\dot{V}(t) &= \alpha I(t)(1 - V(t)) - \mu V(t) - u(t)V(t)
\end{aligned}$$

$$\min_u \max_t I(t), \quad \int_0^T u \leq Q$$

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β	γ	α	μ	T	Q	$S(0)$	$I(0)$	$V(0)$	$\bar{I}$
0.21	0.07	0.12	0.02	300	28	0.999	0.001	0.005	0.06

Solution

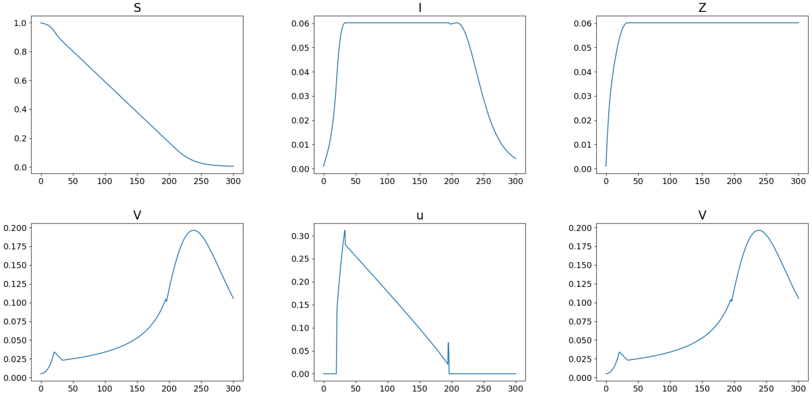
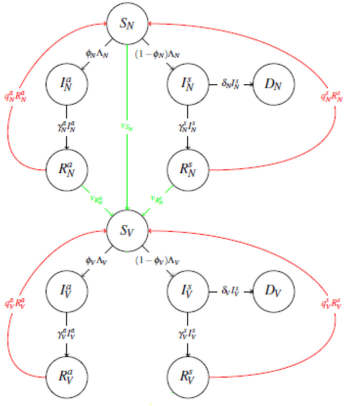


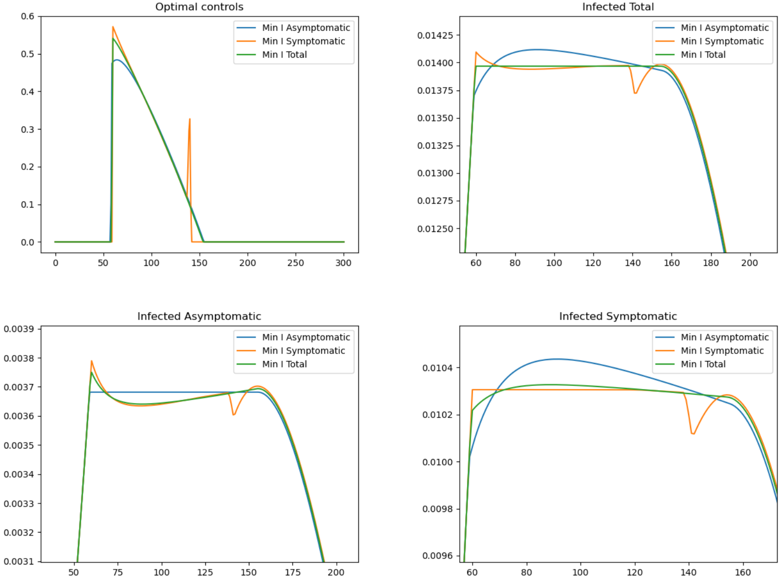
Figure: Solutions using reformulations

A model including vaccines

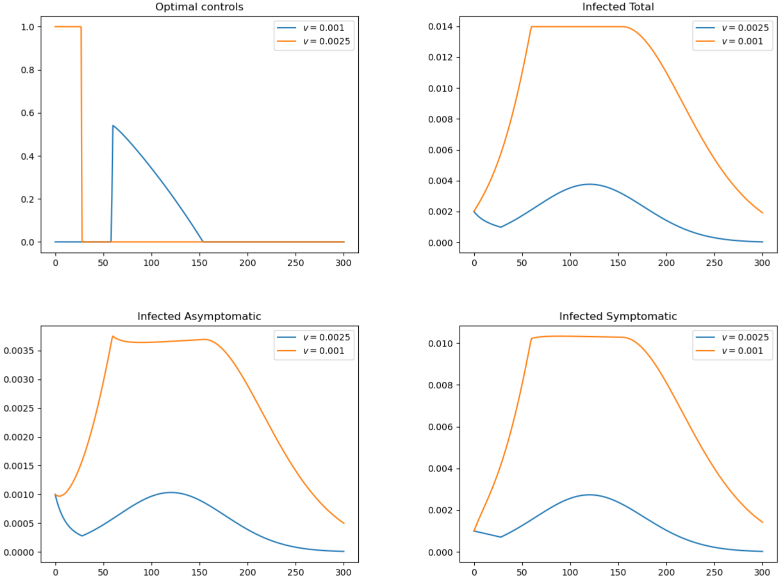


$$\Lambda_i(t) = (1 - f_i(t)) \left(\sum_{j \in \{N, V, V_r\}} ((1 - u(t))\beta^a_{i,j} I^a_j(t) + (1 - \mu)\beta^s_{i,j} I^s_j(t)) \right) S_i(t)$$

Solution


Optimal controls
Min I Asymptomatic
Min I Symptomatic
Min I Total
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0
50
100
150
200
250
300
Infected Total
Min I Asymptomatic
Min I Symptomatic
Min I Total
0.01425
0.01400
0.01375
0.01350
0.01325
0.01300
0.01275
0.01250
60
80
100
120
140
160
180
200
Infected Asymptomatic
Min I Asymptomatic
Min I Symptomatic
Min I Total
0.0039
0.0038
0.0037
0.0036
0.0035
0.0034
0.0033
0.0032
0.0031
50
75
100
125
150
175
200
Infected Symptomatic
Min I Asymptomatic
Min I Symptomatic
Min I Total
0.0104
0.0102
0.0100
0.0098
0.0096
60
80
100
120
140
160

Solution increasing vaccination speed



Summary

Formulation	$\mathcal{P}_0$	$\mathcal{P}_1$ or $\mathcal{P}_2$	$\mathcal{P}_3$	$\mathcal{P}_3^\theta$
suitable to direct methods	yes	yes	no	yes
suitable to HJB methods	no	yes	yes	yes
suitable to shooting methods	no	no/yes	no	yes
provides approximations from below	no	no	no	yes

Table: Comparison of the different formulations

Outline

Conclusion and ongoing work

- We have proved that the NSN strategy minimize the peak of infected over a SIR model with a L^1 constraint on the control.

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- We are interested on the study of generalize the NSN strategy on more general planar dynamics. Preliminary results were exhibited.
- We have proposed several reformulations which can be use for general cases of peaks minimization.
- The study of necessary optimality conditions using this reformulations will be the matter of a future work.

Thanks

Gracias

Merci