Optimal control of a bioeconomic model applied to the recovery of household waste

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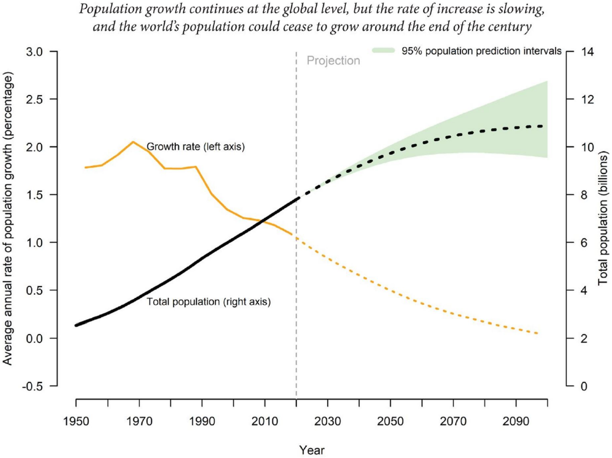
Content

Introduction



Selected Results of the 2019 UN World Population Projections
POPULATION AND DEVELOPMENT REVIEW 45(3): 687–694 (Sept. 2019)

Key findings on global waste generation

- The world generates **0.74 kilograms** of waste per capita per day
- Low-income countries: waste generation may **increase by more than three times by 2050**
- **Food/green waste make up more than 50% of total waste** in low- and middle-income countries
- **37% of waste** is disposed of in some sort of **landfill**
  Only **8%** of them include **gas collection systems**
- **Open dumping : 33%**
  Recycling & composting accounts for 19%
- **11% use modern incineration systems**

Data from : `https://landfillsolutions.eu/`

Landfillsolutions

Waste : global impact

- **Global waste generation is a problem** [https://landfillsolutions.eu/]

  1. Deterioration of the living environment
  2. Pollutes the air, soil, and water
  3. Blocks drainage system blockage and floods
  4. Enters the food chain and impacts health
  5. It chokes and causes death to animals

Undeniable facts: World produced **250 million tonnes of plastic waste** $\approx$ 12% of total MSW

- **Contribution to climate change:**

  Waste management causes **5%** of global greenhouse gas (**GHG**) emissions

  Growth of global waste generation may increase GHG emissions to 2.6 billion tonnes per year (**+62,5%**)

Waste-to-Energy

Definition 1.1 (Waste–to–energy process)

Energy recovery from waste is the conversion of non-recyclable waste materials into usable heat, electricity, or fuel through a variety of processes, including combustion, gasification, pyrolization, anaerobic digestion and landfill gas recovery EPA

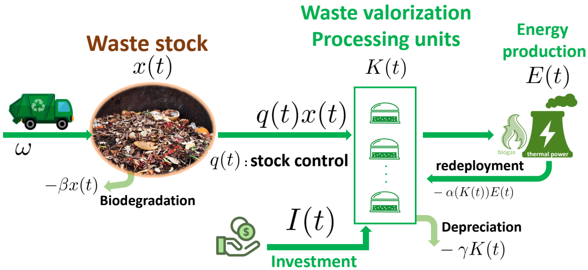
Waste management methods: landfills & incineration

Main issues:

1. **Mismanagement & logistical problems**
2. **High operating costs, profitability (need for optimization)**
3. **Significant levels of pollution, Health and environment risk...**

- https://www.conserve-energy-future.com/advantages-and-disadvantages-incineration.php

The mathematical model



Dynamic model:

$$\begin{cases} \dot{x}(t) = \omega - \big(\beta + K(t)q(t)\big)x(t) \\ \dot{K}(t) = I(t) - \gamma K(t) \\ \dot{E}(t) = \mu K(t)x(t)q(t) - \alpha E(t) \end{cases} \tag{1}$$

The mathematical model

Dynamic model:

$$\begin{cases} \dot{x}(t) = \omega - \big(\beta + K(t)q(t)\big)x(t) \\ \dot{K}(t) = I(t) - \gamma K(t) \\ \dot{E}(t) = \mu K(t)x(t)q(t) - \alpha E(t) \end{cases}$$

- x : cumulative quantity of waste
- K : capital dedicated to the activity
- E : cumulative quantity of produced energy
- $\omega \geq 0$: constant waste streams entering the landfill
- $0 < \beta \leq 1$: coefficient of biodegradation
- $0 < \alpha \leq 1$: depreciation rate + loss of energy due to dessipation.
- $0 \leq q(t) \leq 1$: ratio of recovered waste (control)
- $I(t) \geq 0$: investment related to the activity (control)
- $0 < \gamma \leq 1$: capital depreciation rate.
  As depreciation is considered in the model, investments do not only include acquisition decisions but also maintenance efforts.
- μ : (constant) proportional conversion rate waste-to-energy

The mathematical model

Dynamic model:
$$\begin{cases} \dot{x}(t) = \omega - \big(\beta + K(t)q(t)\big)x(t) \\ \dot{K}(t) = I(t) - \gamma K(t) \\ \dot{E}(t) = \mu K(t)x(t)q(t) - \alpha E(t) \end{cases}$$

- This is an upgraded version of the model recently introduced and analyzed in **[1]**
- Its design is widely inspired from Fishery models as in **[2]**, particularly for the capital dynamics K, and the Cobb–Douglas production function in E

[1] Cherkaoui Dekkaki, O., El Khattabi, N., & Raissi, N. (**2022**). **Bioeconomic modeling of household waste recovery**. Mathematical Methods in the Applied Sciences, 45(1), 468-482..

[2] Clark, C. W., Clarke, F. H., Munro, G. R. (1979). **The Optimal Exploitation of Renewable Resource Stocks: Problems of Irreversible Investment**. Econometrica, 47(1), 25–47.

The optimal control problem (OCP)

[OCP] Find admissible controls (I, q) maximizing,

$$J := \int_0^T e^{-\delta t} \mathcal{J}(t) dt \tag{2}$$

$$\mathcal{J}(t) = pE(t) - cK(t)x(t)q(t) - I(t)(c_1 + c_2 I(t)) \tag{3}$$

over a finite fixed-time horizon $[0, T]$

[1] Kamien, M. I., Schwartz, N. L. (2012). **Dynamic optimization: the calculus of variations and optimal control in economics and management.** 2nd Edition, Advanced textbooks in economics, 31, Dover Publications, Inc.

[2] Moser, E., Grass, D., Tragler, G. (2016). **A non-autonomous optimal control model of renewable energy production under the aspect of fluctuating supply and learning by doing.** Or Spectrum, 38(3), 545-575.

[3] Reed, W. J. (1988). **Optimal harvesting of a fishery subject to random catastrophic collapse.** Mathematical Medicine and Biology: A Journal of the IMA, 5(3), 215-235.

OCP

- $T > 0$ be a finite time-horizon. In practice, T is rather large since it is more interesting to study long-term behavior in investment.
- T specifically designates the end of the limited-term agreement between the investor and the legal authority managing the incineration/landfill units
- Instantaneous yield energy $E(t)$ is supposed to be sold at a given constant unit price p
- Taking into account a given actualization rate δ, a constant unit cost of production c, and we distinguish between linear investment cost c_1 and quadratic adjustment cost c_2 that arise from installation efforts.

Sets of admissible controls:

$$\mathcal{I} = \{I : [0, T] \to [0, I_{\max}] \ \mid \ I(\cdot) \in \mathcal{L}^{\infty}_{loc}([0, T])\},$$
$$\mathcal{Q} = \{q : [0, T] \to [0, 1] \ \mid \ q(\cdot) \in \mathcal{L}^{\infty}_{loc}([0, T])\},$$

$I_{\max} > 0$: maximum possible amount of instantaneous investment
$\mathcal{I}, \mathcal{Q}$: subsets of $\mathcal{L}^{\infty}_{loc}(\mathbb{R}^+)$, the space of locally integrable functions on every compact set on $\mathbb{R}^+$

The PMP application

H: the *current-value Hamiltonian*:

$$H : \mathbb{R}^3 \times \mathbb{R}^3 \times U \to \mathbb{R}$$
$$(\xi, \lambda, u) \mapsto H(\xi, \lambda, u) = \mathcal{L}(\xi, u) + \langle \lambda, f(\xi, u) \rangle$$

$\lambda = (\lambda_1, \lambda_2, \lambda_3)$: pseudo-covector. Using the system's dynamics and the criterion:

$$H = pE - cKxq - I(c_1 + c_2 I) + \lambda_1 (\omega - (\beta + qK)x) + \lambda_2 (I - \gamma K) + \lambda_3 (\mu x K q - \alpha E)$$

Consequently:

$$H = \quad h(X, \Lambda) + \tilde{h} K x q + h^{\dagger}(I) \tag{4}$$

- $h(X, \Lambda) = pE + \lambda_1(w - \beta x) - \lambda_2 \gamma K - \lambda_3 \alpha E$
- $\tilde{h} = -c - \lambda_1 + \mu \lambda_3$
- $h^{\dagger}(I) = -I(c_1 + c_2 I) + \lambda_2 I$

Pseudo-costates and transversality conditions

An absolutely continuous pseudo-covector $\lambda = (\lambda_1, \lambda_2, \lambda_3)$:

$$\begin{cases} \dot{\lambda}_1 = \delta\lambda_1 - \frac{\partial H}{\partial x} \\ \dot{\lambda}_2 = \delta\lambda_2 - \frac{\partial H}{\partial K} \\ \dot{\lambda}_3 = \delta\lambda_3 - \frac{\partial H}{\partial E} \end{cases}$$

Using (4), we end up with:

$$\begin{cases} \dot{\lambda}_1 = (\delta+\beta)\lambda_1 - \tilde{h}Kq \\ \dot{\lambda}_2 = (\delta+\gamma)\lambda_2 - \tilde{h}xq \\ \dot{\lambda}_3 = (\delta+\alpha)\lambda_3 - p \;\Rightarrow\; \lambda_3(t) = \frac{p}{\delta+\alpha}\left[1 - e^{-(\alpha+\gamma)(T-t)}\right] \end{cases}$$

Transversality conditions:

$$\lambda_i(T) = 0, \quad \forall i = 1,2,3 \tag{5}$$

PMP maximization condition

PMP aims to determine the admissible controls s.t., for almost all $t \in [0, T]$:

$$(I(t), q(t)) \in_{I \in \mathcal{I},\ I(t) \in [0, I_{\max}];\ q \in \mathcal{Q},\ q(t) \in [0,1]} H$$

Proposition: For almost all $t \in [0, T]$, T fixed final-time:

i) The optimal control $I^*(t)$ satisfies,

$$I^*(t) = \begin{cases} 0 \ \text{ if } \lambda_2(t) \le c_1 \\ \min\left\{\frac{\lambda_2(t) - c_1}{2c_2}, I_{\max}\right\} \ \text{ if } \lambda_2(t) > c_1 \end{cases} \tag{6}$$

ii) The optimal control $q^*(t)$ satisfies

$$q^*(t) = \begin{cases} 0, \ \text{ if } \tilde{h} = -c - \lambda_1 + \mu\lambda_3 < 0 \\ 1 \ \text{ if } \tilde{h} > 0, \\ q_s(t) \ \text{ if } \tilde{h} \equiv 0 \text{ over } [t_1, t_2],\ t_1 < t_2, \end{cases} \tag{7}$$

Single arcs in the optimal ratio of recovered waste

$I = [t_1, t_2]$, s.t. $t_1 < t_2$, the singular arc $q_s(t)$ occurs, *i.e.*,

$$\tilde{h}(t) = 0 \iff -c - \lambda_1(t) + \mu\lambda_3(t) = 0, \ \forall t \in [t_1, t_2]. \tag{8}$$

If $\tilde{h} \equiv 0$ over I, then $\dot{\tilde{h}}(t) = 0$, *i.e.*, $-\dot{\lambda}_1(t) + \mu\dot{\lambda}_3(t) = 0$:

$$-(\delta + \beta)\lambda_1 + \mu(\delta + \alpha)\lambda_3 - \mu p = 0 \tag{9}$$

Next, the second derivative of $\tilde{h}$ fulfills the equality:

$$\ddot{\tilde{h}} = -(\delta + \beta)^2\lambda_1 + \mu(\delta + \alpha)^2\lambda_3 - \mu(\delta + \alpha)p \tag{10}$$

which does not **explicitly involve** q

Through a process of successive derivation:

$$\tilde{h}^{(n)} = -(\delta + \beta)^n\lambda_1 + \mu(\delta + \alpha)^n\lambda_3 - \mu(\delta + \alpha)^{n-1}p \tag{11}$$

which **do not involve** q for all $n \geq 1$. Consequently, λ_1 and λ_3 must satisfy over the singular arc of q the $(n + 1)$-equations

- if $\alpha \neq \beta$, then the existence of a singular arc is ruled out. Indeed, when, $(\delta+\alpha)^n \neq (\delta+\beta)^n$, for all $n \geq 1$, leading to an overdetermined and inconsistent systems of $(n+1)$-linear equations:

$$\begin{cases} 0 = -\lambda_1 + \mu\lambda_3 - c \\ 0 = -(\delta+\beta)\lambda_1 + \mu(\delta+\alpha)\lambda_3 - \mu p \\ \cdots\cdots\cdots \\ 0 = -(\delta+\beta)^n\lambda_1 + \mu(\delta+\alpha)^n\lambda_3 - \mu(\delta+\alpha)^{n-1}p \end{cases}$$

- if $\alpha = \beta$ then the previous system reduces to,

$$\begin{cases} 0 = -c - \lambda_1 + \mu\lambda_3, \\ 0 = -\lambda_1 + \mu\lambda_3 - \frac{\mu}{\delta+\alpha}p, \end{cases}$$

which is an exact determined system of linear equation, leading to,

$$-c + \frac{\mu}{\delta+\alpha}p = 0. \tag{12}$$

If (12) or $\alpha = \beta$ do not hold, then the optimal q^* cannot have a singular phase

Characterization of the final-time

At the final-time:

- Since $\lambda_2(T) = 0 \leq c_1$, then there exists an $\varepsilon > 0$ s.t. the optimal control I^* activated over $[T - \varepsilon, T]$ is a *bang-0*, *i.e.*,

$$\exists \varepsilon > 0, \quad \text{s.t.,} \quad I^*(t) = 0, \quad \text{for} \quad t \in [T - \varepsilon, T]$$

- Since $\lambda_1(T) = 0$ and $\lambda_3(T) = 0$, if follows that $\tilde{h}(T) = -c < 0$. Consequently there exists an $\varepsilon > 0$ s.t. the optimal control q activated over $[T - \varepsilon, T]$ is a *bang-0*, *i.e.*,

$$\exists \varepsilon > 0, \quad \text{s.t.,} \quad q^*(t) = 0, \quad \text{for} \quad t \in [T - \varepsilon, T]$$

The results derived from the PMP are illustrated using a direct optimization method

Direct optimization

Numerical direct methods that we use are implemented in `Bocop` that transforms the OCP into a nonlinear programming problem (NLP) in finite-dimension, through the discretization step of the controls and the state variables

Table 1: Discretization scheme and `Bocop` settings

Discretization method	`Lobatto IIIC` (implicit, 4-stage, order 6)
Time steps	4000
NLP tolerance	10^{-30}

The state and the control variables (1) are discretized in `Bocop` using a `Lobatto` method, which is based on `Runge-Kutta` schemes (of the type `Lobatto-IIIC`, order 6 implementing an implicit trapezoidal rule)

Example 1

Table 2: Model parameters and criterion settings in Example 1.

w	50
β	0.25
α	0.1
γ	0.2
δ	0.1
μ	0.8
p	1
c	2
c_1	2
c_2	3
T (final-time)	50

Initial condition : $(x^0, K^0, E^0) = (1, 1, 0)$

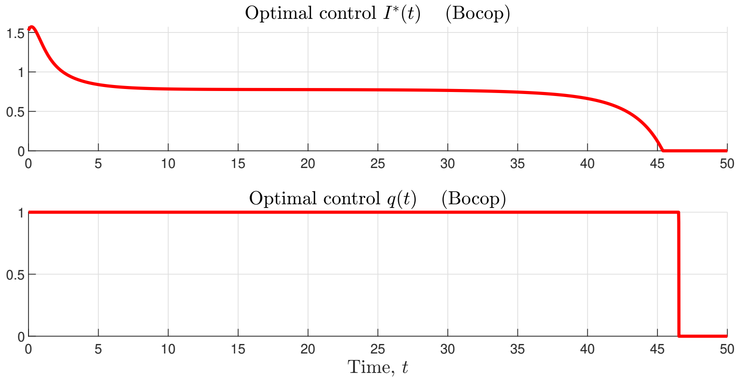


Figure 1: The optimal controls $I^*(t)$ and $q^*(t)$, obtained using Bocop in Example 1 (Tab. 1-2), when the final-time is $T = 50$.

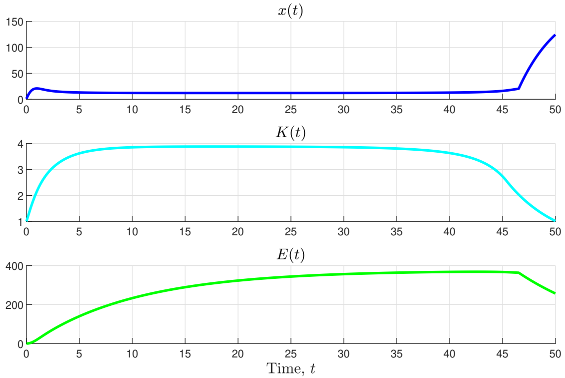


Figure 2: The optimal trajectories $x(t)$, $K(t)$ and $E(t)$, associated with the optimal controls in Fig. 1.

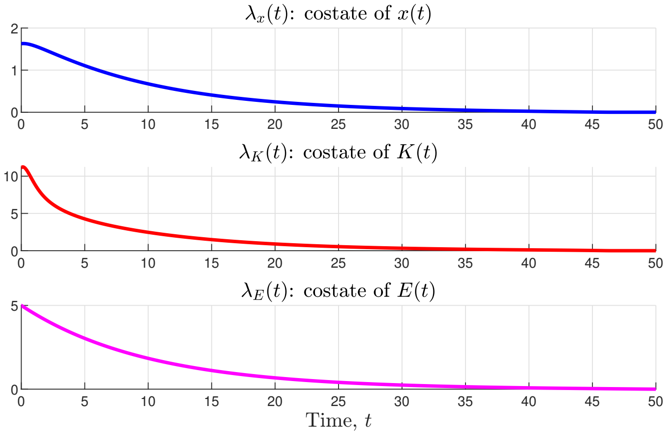


Figure 3: The co-state trajectories derived from Bocop. Using these co-states, it is possible to recover the pseudo-costates

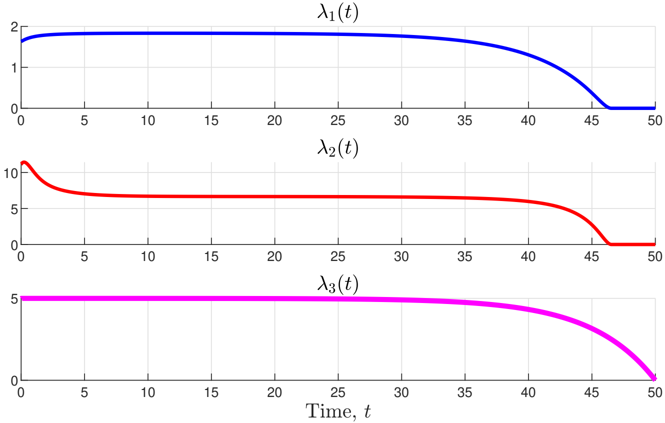


Figure 4: Optimal pseudo-costate trajectories $(\lambda_1, \lambda_2, \lambda_3)$ derived from the *current-value Hamiltonian*. These trajectories are reconstituted using the optimal co-states $(\lambda_x, \lambda_K, \lambda_E)$ in Fig. 3.

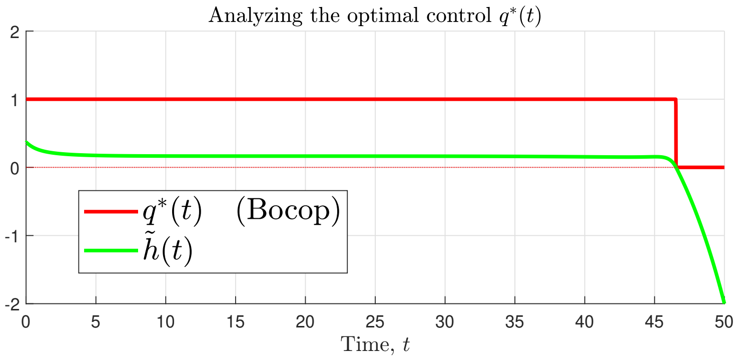


Figure 5: Optimal $q^*(t)$ satisfies the necessary optimality conditions derived from the PMP: $q^*(t)$ is *bang-1* when $\tilde{h}(t)$ is positive, while $q^*(t)$ is *bang-0* when $\tilde{h}(t)$ is negative. The last control phase is a *bang-0*

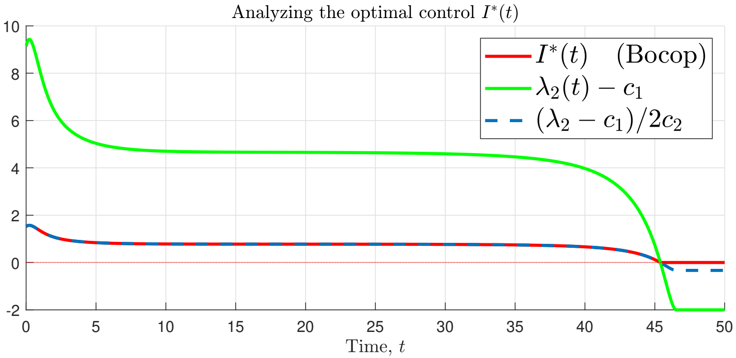


Figure 6: Optimal $I^*(t)$ satisfies the necessary optimality conditions derived from the PMP: $I^*(t)$ (in red) maximizes $h^\dagger(I)$, it coincides with $(\lambda_2 - c_1)/2c_2$ (in blue) when it is positive. $I^*(t)$ follows the description in Proposition 1. The last phase is a *bang-0*

Strategic redeployment plan for the produced energy

Upgraded model:

$$\begin{cases} \dot{x}(t) = \omega - \big(\beta + K(t)q(t)\big)x(t) \\ \dot{K}(t) = I(t) - \gamma K(t) \\ \dot{E}(t) = \mu K(t)x(t)q(t) - \alpha(K(t))E(t) \end{cases} \tag{13}$$

We focus on the typical case:

$$\alpha(K) = a + nK \quad \text{where,} \quad a > 0, \quad n \geq 0 \tag{14}$$

The objective is the same as in the previous **OCP** :

Maximizing the criterion (2)-(3) under similar considerations

To illustrate the effect of the function α we compare the following situations:

- **Example 2–A.** $n = 0$
- **Example 2–B.** $n > 0$

Table 3: Model parameters and criterion settings in **Ex. 2–A** and **2–B**

w	10
β	0.5
a	0.1
n (Example 2–A)	0
n (Example 2–B)	0.015
γ	0.2
δ	0.2
μ	0.9
p	1
c	1
c_1	0.25
c_2	0.25
T (final time)	40

Numerical results in Example 2–A. ($n = 0$)

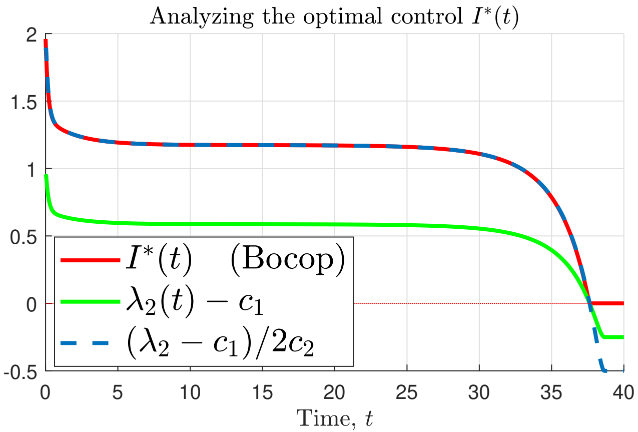


Figure 7: The optimal control $I^*(t)$ in **Example 2-A** satisfies the necessary optimality condition derived from the PMP.

Numerical results in Example 2–A. ($n = 0$)

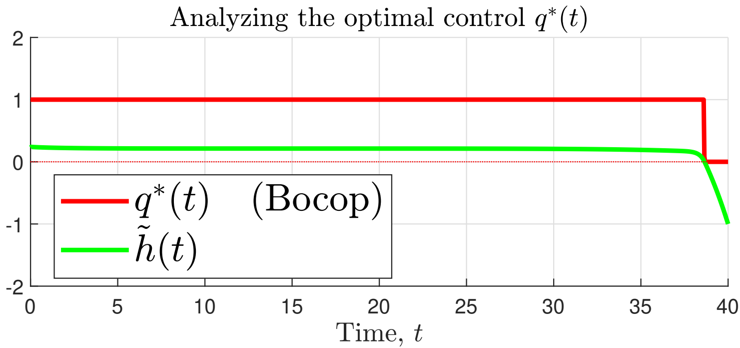


Figure 8: The optimal control $q^*(t)$ in **Example 2-A** satisfies the necessary optimality conditions derived from the PMP.

Numerical results in Example 2–A. ($n = 0$)

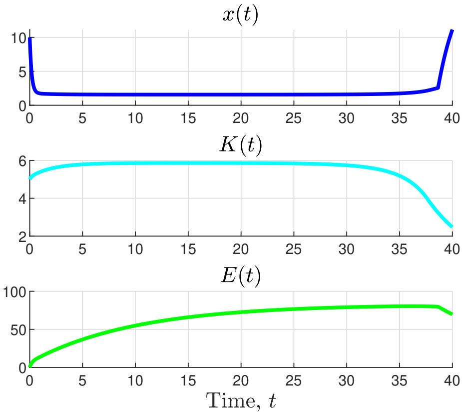


Figure 9: Optimal trajectories $x(t)$, $K(t)$ and $E(t)$, associated with the optimal controls in Fig. 7, and Fig. 8 in **Example 2-A**.

Numerical results in Example 2–B. ($n > 0$)

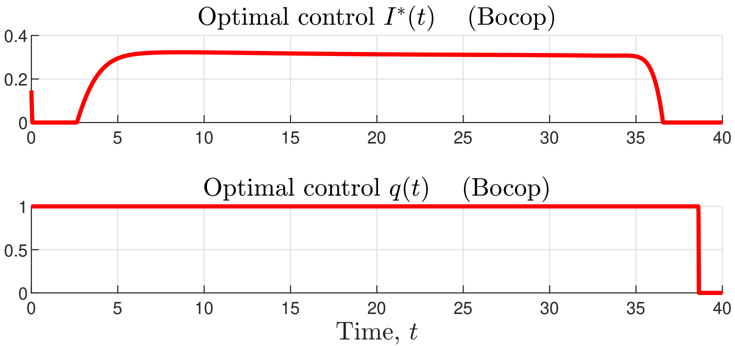


Figure 10: Optimal controls $I^*(t)$ and $q^*(t)$, given by Bocop in Example 2-B

Numerical results in Example 2–B. ($n > 0$)

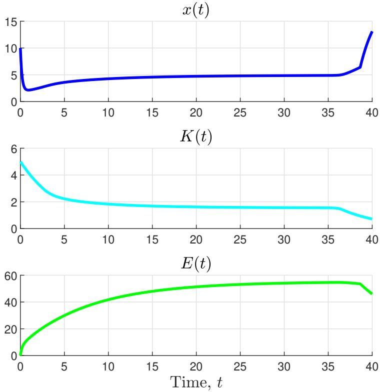


Figure 11: The optimal trajectories $x(t)$, $K(t)$ and $E(t)$ in **Example 2–B.** associated with the optimal controls in Fig. 10.

Numerical results in Example 2–B. ($n > 0$)

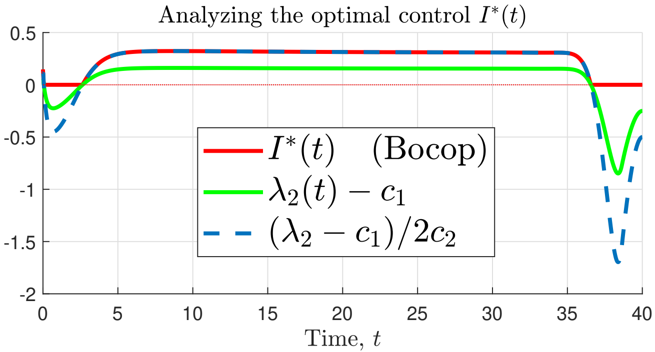


Figure 12: The optimal control $I^*(t)$ in **Example 2–B.** has a more complicated structure, but it satisfies similar necessary optimality conditions as those derived in **Example 2–A.**

Numerical results in Example 2–B. ($n > 0$)

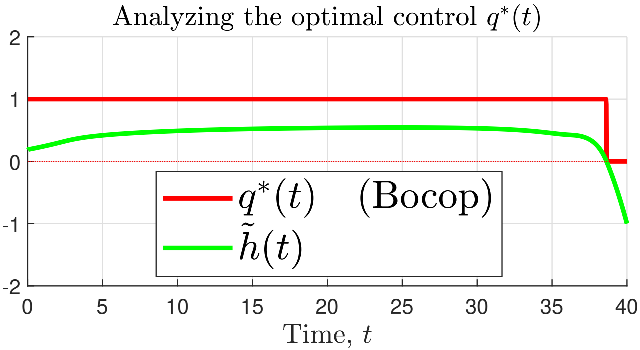


Figure 13: Optimal $q^*(t)$ in **Example 2–B.** also satisfies similar necessary optimality conditions derived from the PMP

Concluding discussion and open problems

To investigate the behaviors observed in Example 2–B., we need to adapt the PMP-analysis performed for **Model** (1) :

- We prove that the **PMP maximization conditions** are similar to those given in **Proposition 1** based on (1).
- The co-states have more complicated dynamics when $n > 0$, resulting in richer behaviors that reflect the control structure given in Fig. 12.
- It also appears that the analysis of the singular control q_s is more complicated, requiring the use of **second order optimality conditions**[1]which deserve a separated study

[1] Legendre clebsch conditions for systems with non-affine controls?