Some optimisation problems for magnetic confinement in stellarator

Rémi Robin

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Joint work with Yannick Privat, Mario Sigalotti and Francesco Volpe
1. Introduction to stellarators

2. Magnetic forces on a surface

3. Coil Winding Surface optimization

4. Existence of surface optimizing some PDE shape functionals
Nuclear Fusion: principle

\[ ^2H + ^3H \rightarrow ^4He + 3.5 \text{ MeV} \]

\[ n + 14.1 \text{ MeV} \]

Figures from Wikipedia
Controlled nuclear fusion: motivations

Serious candidate for power plants.

**Avantages**

- abundant reagents\(^1\)
- No direct emission of greenhouse gases
- No highly radioactive wastes\(^1\)
- No risk of runaway reaction
- No military applications\(^2\)

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1. mostly true...
2. for magnetic technologies
Problem: Confine a 150 million Kelvin plasma.
Controlled nuclear fusion: magnetic confinement

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strategy: Plasma is made up of charged particles $\rightarrow$ react with external magnetic field.
 Controlled nuclear fusion : magnetic confinement

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strategy : Plasma is made up of charged particles $\Rightarrow$ react with external magnetic field.
Figure – Left : scheme of a Tokamak, right : simulation by Robin Roussel (LJLL).
Stellarator

Strategy: ensure confinement only with the external field.

\textbf{FIGURE} – Wendelstein 7-X, Max-Planck Institut für Plasmaphysik
**Figure** – Poincaré map, from *An introduction to symmetries in stellarators*, Imbert-Gérard et al.
Design of a stellarator

1. Find a good target magnetic field $B_T$ inside the plasma.
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2. Use a *Coil winding surface* to find a surface current distribution generating $B_T^3$

**Figure** – Coil winding surface and plasma surface of NCSX.

Design of a stellarator

1. Find a good target magnetic field $B_T$ inside the plasma.
2. Use a *Coil winding surface* to find a surface current distribution generating $B_T^3$.
3. (approach the current density by discrete coils)

**Figure** – Coil winding surface and plasma surface of NCSX.

An optimization problem:

$$\inf_{j \in L^2(\mathcal{X}(S))} \chi_B^2(j)$$

div $j = 0$

Cost function:

$$\chi_B^2(j) = \int_P |\text{BS}(j)(y) - B_T(y)|^2 dy$$

Biot–Savart law:

$$\forall y \notin S, \text{BS}(j)(y) = \int_S j(x) \times \frac{y - x}{|y - x|^3} dx$$
An inverse problem

$BS(\cdot)$ is continuous $L^2(\mathcal{X}(S)) \rightarrow C^k(P, \mathbb{R}^3)$. In particular,

$$L^2(\mathcal{X}(S)) \rightarrow L^2(P, \mathbb{R}^3)$$

$$j \mapsto BS(j)$$

is compact.
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is compact.

Solutions:

- Solve on a finite dimensional space
- Use a Tychonoff regularisation

$$\|j\|_{L^2}^2 = \int_S |j|^2 dS$$

Lemme

For $\lambda > 0$, the optimization problem

$$\inf_{j \in L^2(\mathcal{X}(S))} \chi_B^2(j) + \lambda \|j\|_{L^2}^2$$

admits a unique minimiser $j_S$ given by

$$j_S = (\lambda \text{Id} + BS^\dagger BS)^{-1} BS^\dagger B_T$$
click here
Magnetic forces: motivations

Building a stellarator is expensive. Compact stellarators require higher magnetic fields. Higher magnetic fields call for higher currents. Magnetic forces ($\mathbf{dF} = \mathbf{i} \mathbf{dl} \wedge \mathbf{B}$) increase quadratically. We have to optimize the magnetic forces. Problem: how to define the magnetic forces on a current-sheet?
Magnetic forces: motivations

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- Higher magnetic fields call for higher currents
- Magnetic forces \( \vec{dF} = i \vec{dl} \wedge \vec{B} \) increase quadratically.

\[ \Rightarrow \quad \text{We have to optimize the magnetic forces.} \]

Problem: how to define the magnetic forces on a current-sheet?
Statement of the problem

Let $S$ be a surface and $j \in \mathcal{X}(S)$ a vector field on $S$. Biot–Savart

$$\forall y \not\in S, \quad \text{BS}(j)(y) = \int_S j(x) \times \frac{y - x}{|y - x|^3} dS(x)$$

⚠️ $\int_S \frac{1}{|x - y|^2} dx = \infty \quad \text{si } y \in S$
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$$\int_S \frac{1}{|x - y|^2} dx = \infty \quad \text{si } y \in S$$

There is a magnetic discontinuity on the surface given by

$$B_1^T - B_2^T = n_{12} \wedge j.$$
## Toward a definition

B does not blow-up near S.

### Average magnetic forces

We define

\[
L_\varepsilon(j)(y) = \frac{1}{2} (j \wedge \left[ B(j)(y + \varepsilon n(y)) + B(j)(y - \varepsilon n(y)) \right])
\]

\[
L(j) = \lim_{\varepsilon \to 0} L_\varepsilon(j)
\]
This definition raises several questions:

1. Under which assumptions on $j$ can we ensure that $L(j)$ is well defined?

2. Can we find an explicit expression of $L(j)$ (i.e. without a limit on $\varepsilon$)?

3. Which functional space does $L(j)$ belong to (for $j$ in a given functional space)?
A 3 scale problem

To compute $L$ from $L_\varepsilon$, we need 3 scales:

1. the discretisation-length of $S$: $h$,
2. the infinitesimal displacement $\varepsilon$,
3. the characteristic distance of variation of the magnetic field, $d_B$.

With:

- $h \ll \varepsilon$ as $\int_S |y + \varepsilon n(y) - x|^{-2} dS(x)$ blows up when $\varepsilon \to 0$.
- $\varepsilon \ll d_B$ to approximate $L$. 

Assume $j \in H^1$, then $L_\varepsilon(j)$ converge in $L^p(S, \mathbb{R}^3)$ for $1 \leq p < \infty$ as $\varepsilon \to 0$.

Besides, $L$ is a continuous (quadratic) $H^1 \rightarrow L^p(S, \mathbb{R}^3)$ given by

$$L(j)(y) = -\int_S \frac{1}{|y - x|} \left[ \text{div}_x(\pi_x j(y)) + \pi_x j(y) \cdot \nabla_x \right] j(x) \, dx$$

$$+ \int_S \langle j(y) \cdot n(x) \rangle \frac{\langle y - x \cdot n(x) \rangle}{|y - x|^3} j(x) \, dx$$

$$+ \int_S \frac{1}{|y - x|} \left[ \langle j(y) \cdot j(x) \rangle \text{div}_x(\pi_x) + \nabla_x \langle j(y) \cdot j(x) \rangle \right] \, dx$$

$$- \int_S \langle j(y) \cdot j(x) \rangle \frac{\langle y - x \cdot n(x) \rangle}{|y - x|^3} n(x) \, dx$$
Some ideas of the proof

- Use $A \wedge (B \wedge C) = (A \cdot C)B - (A \cdot B)C$
- Note that $\frac{y-x}{|y-x|^3} = -\nabla_x \frac{1}{|y-x|}$.
- Do an integration by part on the tangential component of the gradient.
- Use some estimates when $\varepsilon$ is small to eliminate the part responsible for the magnetic discontinuity.
- Tools: Hardy-Littlewood-Sobolev inequality and Sobolev embedding on compact manifold.
Optimization

We introduce the following costs:

- $\chi_B$ to ensure that we produce the magnetic field chosen:
  \[ \chi_B^2 = \int_{\partial P} \langle B(x) \cdot n(x) \rangle^2 \, dx \]

- A penalization term on $j$
  \[ \chi_j^2 = \int_S |j|^2 \, dx \]
  \[ \chi_{\nabla j}^2 = \int_S (|\nabla j_x|^2 + |\nabla j_y|^2 + |\nabla j_z|^2) \, dx \]

- A penalizing term on the Laplace forces, for example $L^p(S, \mathbb{R}^3)$
  \[ \chi_F^2 = |L(j)|_{L^p} = \left( \int_S |L(j)|_2^p \right)^{1/p} \, dx \]

Thus, we will minimize the new cost with relative weights $\lambda_1, \lambda_2, \gamma \geq 0$.
\[ \chi^2 = \chi_B^2 + \lambda_1 \chi_j^2 + \lambda_2 \chi_{\nabla j}^2 + \gamma \chi_F^2 \]
Theorem [R., Volpe]

Suppose $\lambda_1, \lambda_2, \gamma > 0$ and $p < \infty$ then

$$\inf_{j \in E} \chi_B^2 + \lambda_1 \chi_j^2 + \lambda_2 \nabla \chi_j^2 + \gamma |L(j)|_{L^p}$$

admits a minimizer.

We also introduce a cost to penalize only high values of the forces: $C_e = \int_S f_e(|L(j)|)$
\begin{align*}
\text{Case} & & \lambda_1 & & \lambda_2 & & \gamma & & \chi_F^2 \\
1 & & 1.5 \cdot 10^{-16} & & 0 & & 0 & & 0 \\
2 & & 0 & & 0 & & 10^{-17} & & \frac{|L(j)|_2^2}{C_e} \\
3 & & 0 & & 0 & & 10^{-16} & & \frac{C_e}{Ce} \\
4 & & 10^{-19} & & 10^{-19} & & 10^{-16} & & \frac{C_e}{Ce} \\
\end{align*}
CWS optimization

\[ \inf_{j \in L^2(\mathcal{X}(S))} \chi_B^2(j) + \lambda \|j\|_{L^2}^2 \]

\[ \text{div } j = 0 \]

**Figure** – Coil winding surface and plasma surface of NCSX.
CWS optimization

\[
\inf_{S \in \mathcal{O}_{\text{adm}}} \inf_{j \in L^2(\mathcal{X}(S))} \chi_B^2(j) + \lambda \|j\|_{L^2}^2
\]

**Figure** – Coil winding surface and plasma surface of NCSX.
Previous works

First approach by Paul et al. (2018)

- Finite dimensional approach (discretize then optimize)
- Regularity of the surface is ensured by non intrinsic cost (Fourier compression).
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Our contribution
- Existence of a minimizer of the shape optimisation problem,
- Computation of the shape gradient in the set of admissible shapes,
- Numerics based on our approach.
admissible shapes

Constraints on the set of admissible shapes $S \in \mathcal{O}_{\text{adm}}$:

1. $S$ is an orientable surface homotopic to the usual torus
2. $\text{dist}(S, P) \geq \delta$
3. $S$ is included inside a given compact set
4. $\mathcal{H}^2(S) \leq A_M$
click here
admissible shapes

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2. $\text{dist}(S, P) \geq \delta$
3. $S$ is included inside a given compact set
4. $\mathcal{H}^2(S) \leq A_M$
5. Lower bound on the reach of $S$
$V \subset \mathbb{R}^n$ closed, $\text{Sk}(V)$ the set of points in $\mathbb{R}^n$ whose orthogonal projection on $V$ is not unique.

$$U_h(V) = \{x \mid d(x, V) < h\}$$

$$\text{Reach}(V) = \sup\{h \mid U_h(V) \cap \text{Sk}(V) = \emptyset\}$$
Theorem [Privat, R., Sigalotti, JMPA, 2022]

The shape optimization problem

$$\inf_{S \in \mathcal{O}_{adm}} \inf_{j \in L^2(\mathcal{X}(S))} \chi_B^2 + \lambda \|j\|_{L^2}^2$$

admits a minimizer.
Shape gradient

- Let $\theta \in W^{2,\infty}(\mathbb{R}^3, \mathbb{R}^3)$ be a perturbation.
Shape gradient

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- $\varphi^\varepsilon = \text{Id} + \varepsilon \theta$ induces a diffeomorphism from $S$ to $S^\varepsilon$
Shape gradient

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- We want to study $\lim_{\varepsilon \to 0} \frac{C(S^\varepsilon) - C(S)}{\varepsilon}$.
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- We want to study $\lim_{\varepsilon \to 0} \frac{C(S^\varepsilon) - C(S)}{\varepsilon}$.

$$\frac{\partial \tilde{C}(S, j_S)}{\partial S} = \frac{\partial \tilde{C}}{\partial S}(S, j_S) + \frac{\partial \tilde{C}}{\partial j} \frac{\partial j_S}{\partial S}(S, j_S).$$
Shape gradient

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- \( \varphi^\varepsilon = \text{Id} + \varepsilon \theta \) induces a diffeomorphism from \( S \) to \( S^\varepsilon \).
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- 
  \[
  \frac{\partial \tilde{\mathcal{C}}(S,j_s)}{\partial S} = \frac{\partial \tilde{\mathcal{C}}}{\partial S}(S,j_s) + \frac{\partial \tilde{\mathcal{C}}}{\partial j_s} \frac{\partial j_s}{\partial S}(S,j_s).
  \]
- The differential of \( \varphi^\varepsilon = \text{Id} + \varepsilon \theta \) provides a diffeomorphism from \( \mathcal{X}(S) \) to \( \mathcal{X}(S^\varepsilon) \).
Shape gradient

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- The differential of $\varphi^\varepsilon = \text{Id} + \varepsilon \theta$ provides a diffeomorphism from $X(S)$ to $X(S^\varepsilon)$.
- Nevertheless the range of $\mathcal{H}_S^0$ by $\varphi^\varepsilon$ does not coincide with $\mathcal{H}_{S^\varepsilon}^0$. 

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\begin{align*}
\frac{\partial \tilde{C}(S, j_S)}{\partial S} &= \frac{\partial \tilde{C}}{\partial S}(S, j_S) + \frac{\partial \tilde{C}}{\partial j} \frac{\partial j_S}{\partial S}(S, j_S).
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- The differential of $\varphi^\varepsilon = \text{Id} + \varepsilon \theta$ provides a diffeomorphism from $\mathcal{X}(S)$ to $\mathcal{X}(S^\varepsilon)$.
- Nevertheless the range of $\mathcal{F}^0_S$ by $\varphi^\varepsilon$ does not coincide with $\mathcal{F}^0_{S^\varepsilon}$.

\[
\Phi^\varepsilon : \mathcal{F}_S \longrightarrow \mathcal{F}_{S^\varepsilon}
\]

\[
X \longmapsto \frac{1}{[J(\mu_S, \mu^\varepsilon_S)\varphi^\varepsilon] \circ \varphi^{-\varepsilon}} (\text{Id} + \varepsilon D\theta)X \circ \varphi^{-\varepsilon}
\]
Shape gradient

For every $\theta \in W^{2,\infty}(\mathbb{R}^3, \mathbb{R}^3)$, we get

$$\left\langle dC(S), \theta \right\rangle = \int_{S} \theta \cdot (X_1 - \text{div}_S(X_2)_i:) \, d\mu_S$$

where

$$X_1 = -2\hat{Z}_P(BS_j - B_T, j_S)$$
$$X_2 = -2Z_P(BS_j - B_T)j_S^T + 2\lambda jSj_S^T - \lambda|j_S|^2(I_3 - \nu\nu^T),$$

where $i \in \{1, 2, 3\}$, $(X_2)_i:$ is the $i$-line of $X_2$ and $\nu$ is the unit normal outward vector on $S = \partial V$.

And

$$Z_P(k) = \int_P K(\cdot, y) \times k(y) \, d\mu_P(y)$$
$$\hat{Z}_P(k, j)(x) = \int_P D_x \left( \frac{x - y}{|x - y|^3} \right)^T (k(y) \times j(x)) \, d\mu_P(y), \quad \forall x \in S.$$
Click here
Perspectives

- Optimisation on specific set of surfaces and optimization of Stellacode\(^6\)
- Magnetic forces and shape optimization together
- Optimization of the plasma

A shape functional

For $\Omega$ regular enough,

$$F(\Omega) = \int_{\partial \Omega} j(x, \nu_{\partial \Omega}(x), B_{\partial \Omega}(x)) \, d\mu_{\partial \Omega}(x),$$

- $\nu_{\partial \Omega}$ is the normal outward vector,
- $B_{\partial \Omega}(x)$ is either a geometric quantity (mean curvature, Gauss curvature . . .) or the solution of a PDE defined on $\Omega$ or $\partial \Omega$. 
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Existence of minimizers

Can we find $\Omega^* \in \mathcal{O}_{adm}$ such that

$$F(\Omega^*) = \inf_{\Omega \in \mathcal{O}_{adm}} F(\Omega)?$$
Uniform ball property

\[ \Omega \in \mathcal{O}_{\text{adm}} \text{ if and only if } \Omega \subset D \text{ compact, } \forall x \in \partial \Omega, \exists d_x \in \mathbb{R}^n \]

\[ \|d_x\|_{\mathbb{R}^d} = 1, \ B_r(x - r_0 d_x) \subset \Omega \text{ and } B_r(x + r_0 d_x) \subset \mathbb{R}^n \setminus \Omega. \]

Figure taken from Dalphin.
Existing results

**Theorem (Guo-Yang, 2013)**

Let $j$ be a continuous function from $\mathbb{R}^d \times S^{d-1}$ to $\mathbb{R}$, then the following optimization problem

$$\inf_{\Omega \in \mathcal{O}_{adm}} \int_{\partial \Omega} j(x, \nu(x)) d\mu_{\partial \Omega}(x)$$

admits a minimiser.

**Theorem (Dalphin, 2018)**

Let $j$ be a continuous function from $\mathbb{R}^d \times S^{d-1} \times \mathbb{R}$ and convex with respect to the last variable, then the following optimization problem

$$\inf_{\Omega \in \mathcal{O}_{adm}} \int_{\partial \Omega} j(x, \nu(x), H_{\partial \Omega}(x)) d\mu_{\partial \Omega}(x)$$

admits a minimiser.
Let $h \in L^2(D)$, $g \in H^2(D)$, and define $u_\Omega$ as the solution of

$$\begin{cases}
\Delta u_\Omega = h & \text{in } \Omega, \\
u = g & \text{in } \partial \Omega.
\end{cases}$$

**Theorem (Dalphin, 2020)**

Let $j$ be a continuous function from $\mathbb{R}^d \times \mathbb{S}^{d-1} \times \mathbb{R} \times \mathbb{R}^d$, then the following optimization problem

$$\inf_{\Omega \in \mathcal{O}_{adm}} \int_{\partial \Omega} j(x, \nu(x), u_\Omega(x), \nabla u_\Omega(x)) \, d\mu_{\partial \Omega}(x)$$

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The direct method of calculus of variations

1. Define a (sequential) topology on $\mathcal{O}_{\text{adm}}$. 
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2. Take a minimizing sequence and use a compactness result.
The direct method of calculus of variations

1. Define a (sequential) topology on $\mathcal{O}_{adm}$.
2. Take a minimizing sequence and use a compactness result.
3. Prove the lower-semicontinuity of the functional.
Signed distance

Distances functions

\[ d_\Omega(x) = \inf_{y \in \Omega} \| x - y \| \]

\[ b_\Omega(x) = d_\Omega(x) - d_{\mathbb{R}^d \setminus \Omega}(x) \]

Some properties

- For \( x \in \partial \Omega \), \( \nabla b_\Omega(x) \) is the unit outward normal vector,
- For \( x \in \partial \Omega \), \( \text{Tr}(\nabla^2 b_\Omega(x)) \) is the mean curvature,
- etc.
Uniform reach property

Definition

\[ \text{Reach}(\Omega) = \sup\{ h > 0 \mid d_\Omega \text{ is differentiable in } U_h(\Omega) \setminus \Omega \}. \]

Assume \( \text{Reach}(\partial \Omega) = r_0 > 0 \), we have

- if \( \mathcal{H}^d(\partial \Omega) = 0 \), then \( \partial \Omega \) is a \( C^{1,1} \) hypersurface of \( \mathbb{R}^d \) and satisfies the uniform ball property.

- For \( h < r_0 \), \( \nabla b_\Omega \) is \( \frac{2}{r_0 - h} \)-Lipschitz continuous on the tubular neighborhood \( U_h(\partial \Omega) \).

- The restriction of \( \nabla b_\Omega \) to \( \partial \Omega \) is \( \frac{1}{r_0} \)-Lipschitz continuous.
A new framework

**R-convergence in $\mathcal{O}_{adm}$**

Given $(\Omega_n)_{n \in \mathbb{N}} \in \mathcal{O}_{adm}^\mathbb{N}$, we say that $(\Omega_n)_{n \in \mathbb{N}}$ $R$-converges to $\Omega_\infty \in \mathcal{O}_{adm}$ and we write $\Omega_n \xrightarrow{R} \Omega_\infty$ if

\[
\begin{align*}
    b_{\Omega_n} &\to b_{\Omega_\infty} \\
\end{align*}
\]

\[
\begin{cases}
    \text{in } C(\overline{D}), \\
    \text{in } C^{1,\alpha}(U_r(\partial \Omega_\infty)), \quad \forall r < r_0, \quad \forall \alpha \in [0, 1), \\
    \text{weakly-star in } W^{2,\infty}(U_r(\partial \Omega_\infty)), \quad \forall r < r_0.
\end{cases}
\]

**Theorem**

$\mathcal{O}_{adm}$ is sequentially compact for the $R$-convergence.
For $0 < h < r_0$, consider

$$T_{\partial \Omega} : (-h, h) \times \partial \Omega \rightarrow U_h(\partial \Omega)$$

$$(t, x) \mapsto x + t \nabla b_\Omega(x).$$

Since $T_{\partial \Omega}$ is Lipschitz continuous, it is differentiable at almost every $(t_0, x_0)$, with

$$d_{(t_0, x_0)} T_{\partial \Omega}(s, y) = y + s \nabla b_\Omega(x_0) + t_0 d_{x_0} \nabla b_\Omega(y), \quad \forall (s, y) \in \mathbb{R} \times T_{x_0} \partial \Omega.$$

**Lemma**

*For every $\varepsilon > 0$, there exists $h > 0$ such that for all $\Omega \in \mathcal{O}_{adm}$,*

$$1 - \varepsilon \leq \det(d_{(t_0, x_0)} T_{\partial \Omega}) \leq 1 + \varepsilon, \quad \text{for a.e. } (t_0, x_0) \in (-h, h) \times \partial \Omega.$$
Lemma

If $\Omega_n \xrightarrow{R} \Omega_\infty$ then

1. $\mathcal{H}^{d-1}(\partial\Omega_n)$ converges toward $\mathcal{H}^{d-1}(\partial\Omega_\infty)$ as $n \to +\infty$.

2. $\mathcal{H}^{d}(\Omega_n)$ converges toward $\mathcal{H}^{d}(\Omega_\infty)$ as $n \to +\infty$.

3. If all the $\partial\Omega_n$ belong to the same homotopic class, then $\partial\Omega_\infty$ also belongs such a class.

Corollary

\[ \{\Omega \in \mathcal{O}_{\text{adm}} \mid a \leq \mathcal{H}^{d-1}(\partial\Omega) \leq b, \partial\Omega \text{ is homotopic to } \partial\Omega_0\} \]

is sequentially compact
Let \( j \) be a continuous function from \( \mathbb{R}^d \times S^{d-1} \times \mathbb{R} \) and convex with respect to the last variable.

\[
F(\Omega) = \int_{\partial \Omega} j(x, \nu(x), H_{\partial \Omega}(x)) \, d\mu_{\partial \Omega}(x)
\]

\textbf{Theorem}

\( F \) is a lower-semicontinuous shape functional for the \( R \)-convergence, i.e., for every sequence \( (\Omega_n)_{n \in \mathbb{N}} \in \mathcal{O}_{\text{adm}}^{\mathbb{N}} \) that \( R \)-converges toward \( \Omega_\infty \), one has

\[
\liminf_{n \to +\infty} F(\Omega_n) \geq F(\Omega_\infty).
\]

As a consequence, the shape optimization problem

\[
\inf_{\Omega \in \mathcal{O}_{\text{adm}}} F(\Omega)
\]

has a solution.
\[ F(\Omega_n) = \int_{\partial \Omega_n} j(x, \nabla b_{\Omega_n}(x), H_{\partial \Omega_n}(p_n(y))) d\mu_{\partial \Omega_n}(x) \]
\[ = \frac{1}{2h} \int_{U_h(\partial \Omega_n)} j(p_n(y), \nabla b_{\Omega_n}(p_n(y)), H_{\partial \Omega_n}(p_n(y))) \det(dT_n^{-1}(y) T_n) dy. \]

\[ F(\Omega_n) = \frac{1}{2h} \int_{U_{h-t}(\partial \Omega_\infty)} j(p_n(y), \nabla b_{\Omega_n}(p_n(y)), H_{\partial \Omega_n}(p_n(y))) \det(dT_n) dy \]
\[ + \frac{1}{2h} \int_{U_h(\partial \Omega_n) \setminus U_{h-t}(\partial \Omega_\infty)} j(p_n(y), \nabla b_{\Omega_n}(p_n(y)), H_{\partial \Omega_n}(p_n(y))) \det(dT_n) dy. \]
Definition

Let $f \in \mathcal{C}^0(D)$. We consider $v_{\partial \Omega}$ the solution of the equation

$$\Delta_{\partial \Omega} v_{\partial \Omega}(x) = f(x) \quad \text{in} \quad \partial \Omega,$$
Let \( f \in C^0(D) \). We consider \( v_{\partial \Omega} \) the solution of the equation

\[
\Delta_{\partial \Omega} v_{\partial \Omega}(x) = f(x) \quad \text{in } \partial \Omega,
\]

\( v_{\partial \Omega} \) is the unique minimiser of

\[
\mathcal{E}_{\partial \Omega} : H^1_*(\partial \Omega) \ni u \mapsto \frac{1}{2} \int_{\partial \Omega} |\nabla_{\partial \Omega} u(x)|^2 d\mu_{\partial \Omega} - \int_{\partial \Omega} f(x)u(x)d\mu_{\partial \Omega}
\]

(2)

**Lemma [Privat, R., Sigalotti, 2022]**

For any \( \Omega \in \mathcal{O}_{\text{adm}} \), Eq. (2) admits one and only one minimiser.
\[
F(\Omega) = \int_{\partial \Omega} j(x, \nu(x), \nu_{\partial \Omega}(x), \nabla_{\partial \Omega} \nu_{\partial \Omega}(x)) \, d\mu_{\partial \Omega}(x),
\]
where \( j : \mathbb{R}^d \times S^{d-1} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \) is assumed to be continuous.

**Theorem [Privat, R., Sigalotti, 2022]**

The shape functional \( F \) is lower-semicontinuous for the \( R \)-convergence, i.e., for every sequence \( (\Omega_n)_{n \in \mathbb{N}} \in \mathcal{O}_{adm}^\mathbb{N} \) that \( R \)-converges toward \( \Omega_\infty \), one has

\[
\liminf_{n \to +\infty} F(\Omega_n) \geq F(\Omega_\infty). \tag{3}
\]

As a consequence, the shape optimization problem

\[
\inf_{\Omega \in \mathcal{O}_{adm}} F(\Omega)
\]

has a solution.
1. Transport $v_{\partial \Omega_n}$ to $\partial \Omega_\infty$ thanks to the orthogonal projector on $\partial \Omega_n$.

2. The sequence obtained is bounded $H^1(\partial \Omega_\infty)$, extract and called $v^* \in H^1_*(\partial \Omega_\infty)$ the limit.

3. Check that $v^* = v_{\partial \Omega_\infty}$.

4. Passing to the limit is similar to the previous case.
In a nutshell

Hypersurfaces with a uniform Reach condition enjoy nice properties:

- Sequential compactness for the $R$-convergence.
- Many functionals involving geometric or PDE related cost are lower-semicontinuous for the $R$-convergence.
- Proofs are (relatively) straightforward.
A few references

- **P. Merkel.** “Solution of stellarator boundary value problems with external currents”. In: *Nuclear Fusion* 27.5 (1987), p. 867-871

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- **Y. Privat, R. Robin et M. Sigalotti.** “Optimal shape of stellarators for magnetic confinement fusion”. In: *Journal de Mathématiques Pures et Appliquées* 163 (2022), p. 231-264


software: Stellacode
https://rrobin.pages.math.cnrs.fr/stellacode/
Hodge decomposition

On a closed Riemannian manifold $M$

$$L^2_p(M) = B_p \oplus B^*_p \oplus \mathcal{H}_p,$$

where

- $B_p$ is the $L^2$-closure of $\{d\alpha \mid \alpha \in \Omega^{p-1}(M)\}$
- $B^*_p$ is the $L^2$-closure of $\{d^*\beta \mid \beta \in \Omega^{p+1}(M)\}$
- $\mathcal{H}_p$ is the set $\{\omega \in \Omega^p(M) \mid \Delta_H \omega = 0\}$ of harmonic $p$-forms with $\Delta_H$ the Hodge Laplacian
In vacuo Maxwell equations on a toroidal 3D domain

Let $P$ be a toroidal domain. Let $\Gamma$ be a toroidal loop inside $P$ and denote by $I_p$ the electric current-flux across any surface enclosed by $\Gamma$ (also equal to the circulation of $B$ along $\Gamma$).

Lemma

Let $B \in C^\infty(P, \mathbb{R}^3)$ such that $\text{div} \, B = 0$ and $\text{curl} \, B = 0$ in $P$. Let $g$ be the normal magnetic field on $\partial P$. Then $g$ and $I_p$ determine completely the magnetic field $B$ in $P$. Besides, there exists a constant $C > 0$ such that for every other magnetic field $\tilde{B}$ with the same total poloidal currents, $|B - \tilde{B}|_{H^{1/2}(P, \mathbb{R}^3)} \leq C |g - \tilde{g}|_{L^2(\partial P)}$ where $\tilde{g}$ is the normal component of $\tilde{B}|_{\partial P}$.

Idea: consider the cochain complex

$$C^\infty(P) \xrightarrow{\text{grad}} C^\infty(P, \mathbb{R}^3) \xrightarrow{\text{curl}} C^\infty(P, \mathbb{R}^3) \xrightarrow{\text{div}} C^\infty(P).$$