Some optimisation problems for magnetic confinement in stellarator

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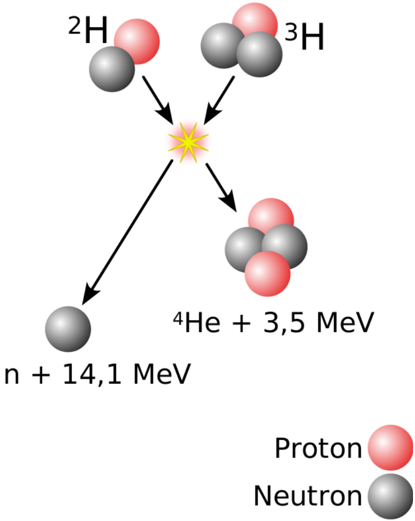
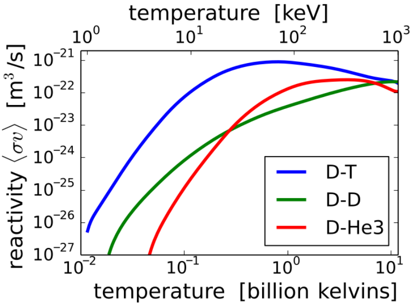
McTAO seminar, October 27^{th} 2022

Joint work with Yannick Privat, Mario Sigalotti and Francesco Volpe

Nuclear Fusion : principle





Figures from Wikipedia

Controlled nuclear fusion : motivations

Serious candidate for power plants.

Avantages

- abundant reagents [1]
- No direct emission of greenhouse gases
- No highly radioactive wastes [1]
- No risk of runaway reaction
- No military applications [2]

1. mostly true. . .
2. for magnetic technologies

Controlled nuclear fusion : magnetic confinement

Problem : Confine a 150 million Kelvin plasma.

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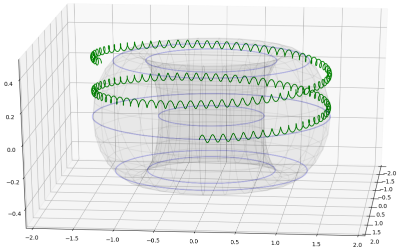
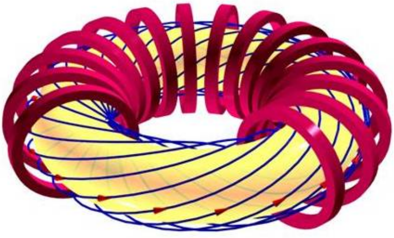




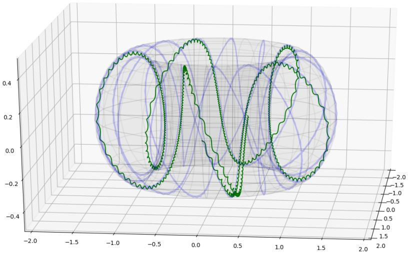


Figure – Left : scheme of a Tokamak, right : simulation by Robin Roussel (LJLL).

Stellarator

Strategy : ensure confinement only with the external field.

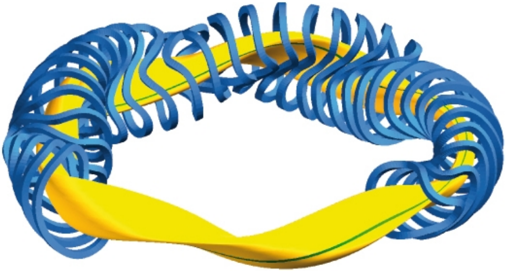
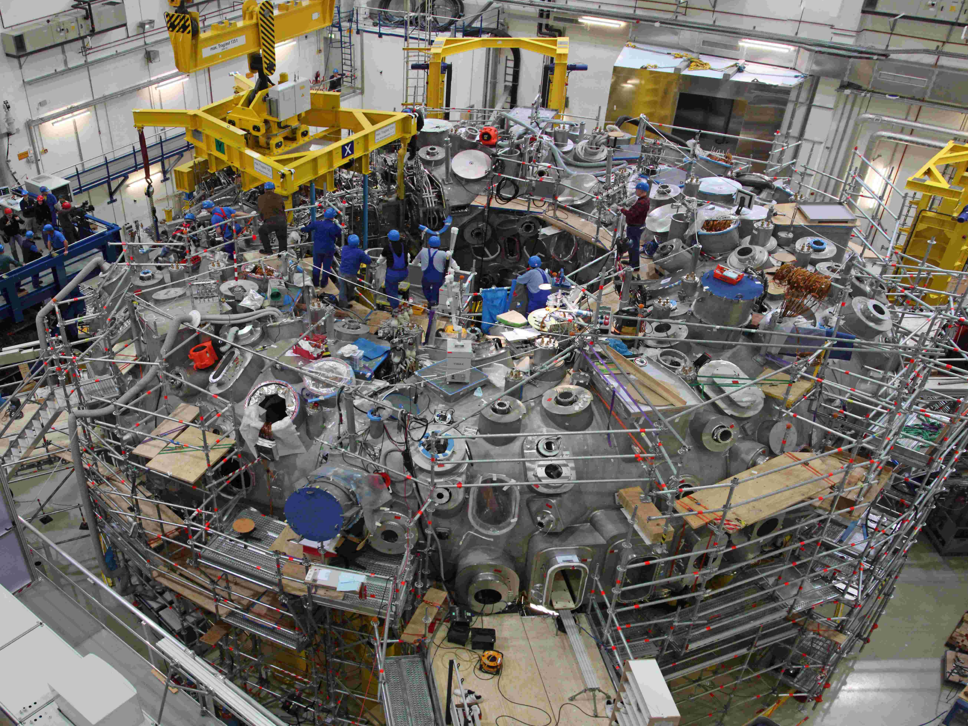


FIGURE – Wendelstein 7-X, Max-Planck Institut für Plasmaphysik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Stellarator

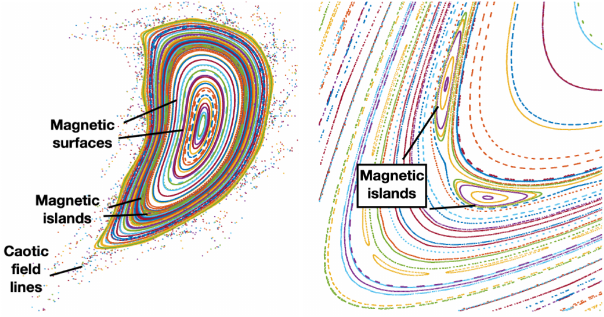

FIGURE – Poincaré map, from *An introduction to symmetries in stellarators*, Imbert-Gérard et al.

Design of a stellarator

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2. Use a *Coil winding surface* to find a surface current distribution generating B_T [3]

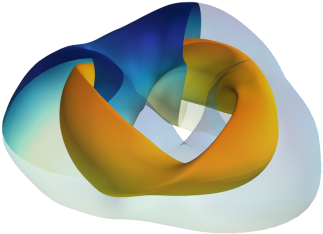

FIGURE – Coil winding surface and plasma surface of NCSX.

3. P. Merkel (1986)

Design of a stellarator

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2. Use a *Coil winding surface* to find a surface current distribution generating B_T [3]
3. (approach the current density by discrete coils)

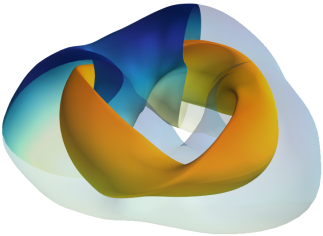

FIGURE – Coil winding surface and plasma surface of NCSX.

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Modelisation

An optimization problem :

$$\inf_{\substack{j \in L^2(\mathfrak{X}(S)) \\ \operatorname{div} j = 0}} \chi_B^2(j)$$

Cost function :

$$\chi_B^2(j) = \int_P |\mathrm{BS}(j)(y) - B_T(y)|^2 dy$$

Biot–Savart law :

$$\forall y \notin S, \mathrm{BS}(j)(y) = \int_S j(x) \times \frac{y - x}{|y - x|^3} dx$$

An inverse problem

$BS(\cdot)$ is continuous $L^2(\mathfrak{X}(S)) \to C^k(P, \mathbb{R}^3)$. In particular,

$$\begin{aligned} L^2(\mathfrak{X}(S)) &\to L^2(P, \mathbb{R}^3) \\ j &\mapsto BS(j) \end{aligned}$$

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Solutions :

- Solve on a finite dimensional space [4]
- Use a Tychonoff regularisation [5]

$$\|j\|_{L^2}^2 = \int_S |j|^2 dS$$

4. P. Merkel (1986)
5. M. Landreman (2017)

Lemme

For $\lambda > 0$, the optimization problem

$$\inf_{\substack{j \in L^2(\mathfrak{X}(S)) \\ \operatorname{div} j = 0}} \chi_B^2(j) + \lambda \|j\|_{L^2}^2$$

admits a unique minimiser j_S given by

$$j_S = (\lambda \operatorname{Id} + \mathrm{BS}^\dagger \, \mathrm{BS})^{-1} \mathrm{BS}^\dagger \, B_T$$

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Magnetic forces : motivations

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Magnetic forces : motivations

- Building a stellarator is expensive...
- compact stellarators require higher magnetic field
- Higher magnetic fields call for higher currents
- Magnetic forces ($d\vec{F} = i\vec{dl} \wedge \vec{B}$) increase quadratically.

$\implies$ We have to optimize the magnetic forces.

Problem : how to define the magnetic forces on a current-sheet ?

Statement of the problem

Let S be a surface and $j \in \mathfrak{X}(S)$ a vector field on S.
Biot–Savart

$$\forall y \notin S,\ \mathrm{BS}(j)(y) = \int_S j(x) \times \frac{y-x}{|y-x|^3} dS(x)$$

$$\int_S \frac{1}{|x-y|^2} dx = \infty \qquad \text{si } y \in S$$

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There is a magnetic discontinuity on the surface given by

$$B_T^1 - B_T^2 = n_{12} \wedge j.$$

Toward a definition

B does not blow-up near S.

Average magnetic forces

We define

$$L_\varepsilon(j)(y) = \frac{1}{2}\left(j \wedge \left[B(j)(y + \varepsilon n(y)) + B(j)(y - \varepsilon n(y))\right]\right)$$

$$L(j) = \lim_{\varepsilon \to 0} L_\varepsilon(j)$$

This definition raises several questions :

1. Under which assumptions on j can we ensure that $L(j)$ is well defined ?
2. Can we find an explicit expression of $L(j)$ (i.e. without a limit on ε) ?
3. Which functional space does $L(j)$ belong to (for j in a given functional space) ?

A 3 scale problem

To compute L from L_ε, we need 3 scales :

1. the discretisation-length of S : h,
2. the infinitesimal displacement ε,
3. the characteristic distance of variation of the magnetic field, d_B.

With :

- $h \ll \varepsilon$ as $\int_S |y + \varepsilon n(y) - x|^{-2} dS(x)$ blows up when $\varepsilon \to 0$.
- $\varepsilon \ll d_B$ to approximate L.

Theorem [R., Volpe, *Nuclear Fusion*, 2022]

Assume $j \in H^1$, then $L_\varepsilon(j)$ converge in $L^p(S, \mathbb{R}^3)$ for $1 \leq p < \infty$ as $\varepsilon \to 0$.

Besides, L is a continuous (quadratic) $H^1 \to L^p(S, \mathbb{R}^3)$ given by

$$
\begin{aligned}
L(j)(y) = & -\int_S \frac{1}{|y-x|}\left[\operatorname{div}_x(\pi_x j(y)) + \pi_x j(y) \cdot \nabla_x\right] j(x) dx \\
& + \int_S \langle j(y) \cdot n(x)\rangle \frac{\langle y - x \cdot n(x)\rangle}{|y-x|^3} j(x) dx \\
& + \int_S \frac{1}{|y-x|}\left[\langle j(y) \cdot j(x)\rangle \operatorname{div}_x(\pi_x) + \nabla_x \langle j(y) \cdot j(x)\rangle\right] dx \\
& - \int_S \langle j(y) \cdot j(x)\rangle \frac{\langle y - x \cdot n(x)\rangle}{|y-x|^3} n(x) dx
\end{aligned}
$$

Some ideas of the proof

- Use $A \wedge (B \wedge C) = (A \cdot C)B - (A \cdot B)C$
- Note that $\frac{y-x}{|y-x|^3} = -\nabla_x \frac{1}{|y-x|}$.
- Do an integration by part on the tangential component of the gradient.
- Use some estimates when ε is small to eliminate the part responsible for the magnetic discontinuity.
- Tools : Hardy-Littlewood-Sobolev inequality and Sobolev embeding on compact manifold.

Optimization

We introduce the following costs :

- χ_B to ensure that we produce the magnetic field chosen :

$$\chi_B^2 = \int_{\partial P} \langle B(x) \cdot n(x) \rangle^2 dx$$

- A penalization term on j

$$\chi_j^2 = \int_S |j|^2 dx$$

$$\chi_{\nabla j}^2 = \int_S (|\nabla j_x|^2 + |\nabla j_y|^2 + |\nabla j_z|^2) dx$$

- A penalizing term on the Laplace forces, for example $L^p(S, \mathbb{R}^3)$

$$\chi_F^2 = |L(j)|_{L^p} = \left(\int_S |L(j)|_2^p \right)^{1/p} dx$$

Thus, we will minimize the new cost with relative weights $\lambda_1, \lambda_2, \gamma \geq 0$.

$$\chi^2 = \chi_B^2 + \lambda_1 \chi_j^2 + \lambda_2 \chi_{\nabla j}^2 + \gamma \chi_F^2$$

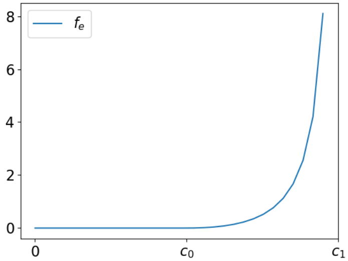
Theorem [R., Volpe]

Suppose $\lambda_1, \lambda_2, \gamma > 0$ and $p < \infty$ then

$$\inf_{j \in E} \chi_B^2 + \lambda_1 \chi_j^2 + \lambda_2 \chi_{\nabla j}^2 + \gamma |L(j)|_{L^p}$$

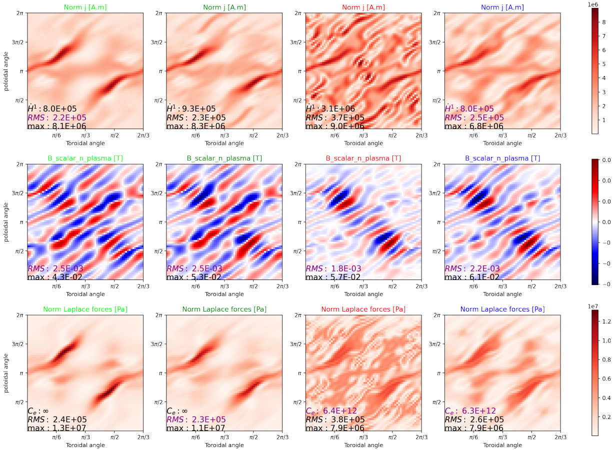
admits a minimizer.

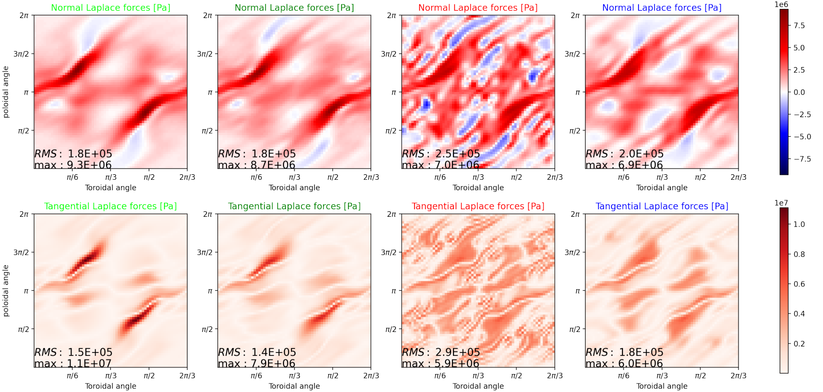
We also introduce a cost to penalize only high values of the forces : $C_e = \int_S f_e(|L(j)|)$



Case	λ_1 ($\mathrm{T^2\ m^2/A^2}$)	λ_2 ($\mathrm{T^2\ m^4/A^2}$)	γ ($\mathrm{T^2/Pa^2}$)	χ_F^2
1	$1.5 \cdot 10^{-16}$	0	0	0
2	0	0	10^{-17}	$\lvert L(j)\rvert^2_{L^2(S,\mathbb{R}^3)}$
3	0	0	10^{-16}	C_e
4	10^{-19}	10^{-19}	10^{-16}	C_e

(1)




Normal Laplace forces [Pa]
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Normal Laplace forces [Pa]
Normal Laplace forces [Pa]
Tangential Laplace forces [Pa]
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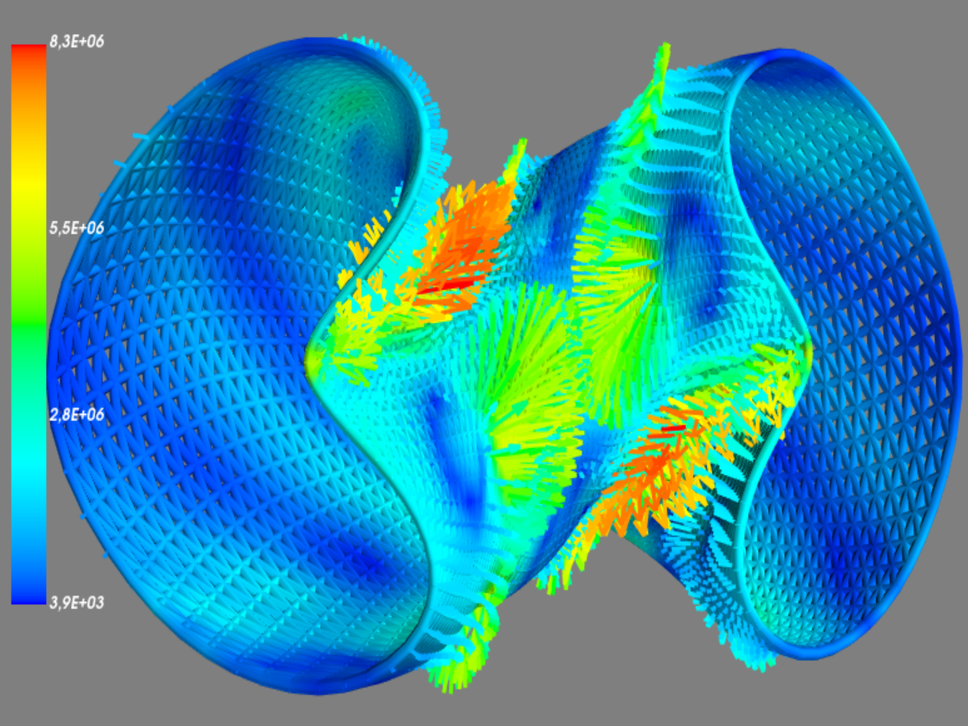
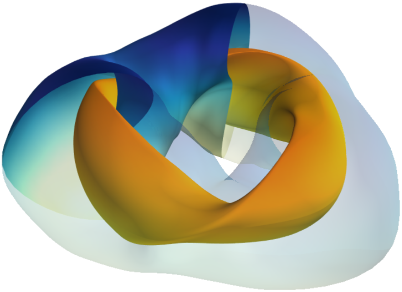
poloidal angle
Toroidal angle
RMS : 1.8E+05
max : 9.3E+06
RMS : 1.8E+05
max : 8.7E+06
RMS : 2.5E+05
max : 7.0E+06
RMS : 2.0E+05
max : 6.9E+06
RMS : 1.5E+05
max : 1.1E+07
RMS : 1.4E+05
max : 7.9E+06
RMS : 2.9E+05
max : 5.9E+06
RMS : 1.8E+05
max : 6.0E+06
1e6
1e7


8,3E+06
5,5E+06
2,8E+06
3,9E+03

CWS optimization

$$\inf_{\substack{j \in L^2(\mathfrak{X}(S)) \\ \operatorname{div} j = 0}} \chi_B^2(j) + \lambda \|j\|_{L^2}^2$$



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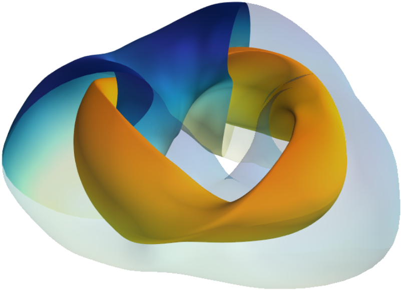

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Previous works

First approach by Paul et al.(2018)

- Finite dimensional approach (discretize then optimize)
- Regularity of the surface is ensured by non intrinsic cost (Fourier compression).

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Our contribution

- Existence of a minimizer of the shape optimisation problem,
- Computation of the shape gradient in the set of admissible shapes,
- Numerics based on our approach.

admissible shapes

Constraints on the set of admissible shapes $S \in \mathcal{O}_{\text{adm}}$:

1. S is an orientable surface homotopic to the usual torus
2. $dist(S, P) \geq \delta$
3. S is included inside a given compact set
4. $\mathcal{H}^2(S) \leq A_M$

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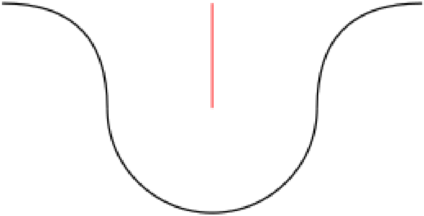
5. Lower bound on the reach of S

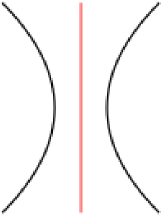
Reach

$V \subset \mathbb{R}^n$ closed, $\mathrm{Sk}(V)$ the set of points in $\mathbb{R}^n$ whose orthogonal projection on V is not unique.

$$U_h(V) = \{x \mid d(x, V) < h\}$$

$$\mathrm{Reach}(V) = \sup\{h \mid U_h(V) \cap \mathrm{Sk}(V) = \emptyset\}$$





Reach

Theorem [Privat, R. , Sigalotti, *JMPA*, 2022]

The shape optimization problem

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Shape gradient

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- Nevertheless the range of $\mathscr{F}_S^0$ by φ^ε does not coincide with $\mathscr{F}^0_{S^\varepsilon}$.

$$\Phi^\varepsilon : \mathscr{F}_S \longrightarrow \mathscr{F}_{S^\varepsilon}$$

$$X \longmapsto \frac{1}{[J(\mu_S, \mu_S^\varepsilon)\varphi^\varepsilon] \circ \varphi^{-\varepsilon}}(\mathrm{Id} + \varepsilon D\theta)X \circ \varphi^{-\varepsilon}$$

Shape gradient

For every $\theta \in W^{2,\infty}(\mathbb{R}^3, \mathbb{R}^3)$, we get

$$\boxed{\langle dC(S), \theta \rangle = \int_S \theta \cdot (X_1 - \mathrm{div}_S (X_2)_{i:})\, d\mu_S}$$

where

$$X_1 = -2\widehat{Z}_P(\mathrm{BS}_S\, j_S - B_T, j_S)$$
$$X_2 = -2Z_P(\mathrm{BS}_S\, j_S - B_T) j_S^T + 2\lambda j_S j_S^T - \lambda |j_S|^2 (I_3 - \nu\nu^T),$$

where $i \in \{1, 2, 3\}$, $(X_2)_{i:}$ is the i-line of X_2 and ν is the unit normal outward vector on $S = \partial V$.
And

$$Z_P(k) = \int_P K(\cdot, y) \times k(y)\, d\mu_P(y)$$
$$\widehat{Z}_P(k, j)(x) = \int_P D_x \left(\frac{x - y}{|x - y|^3} \right)^T (k(y) \times j(x))\, d\mu_P(y), \qquad \forall x \in S.$$

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Perspectives

- Optimisation on specific set of surfaces and optimization of Stellacode [6]
- Magnetic forces and shape optimization together
- Optimization of the plasma

6. https://rrobin.pages.math.cnrs.fr/stellacode/

A shape functional

For Ω regular enough,

$$F(\Omega) = \int_{\partial\Omega} j(x, \nu_{\partial\Omega}(x), B_{\partial\Omega}(x))\, d\mu_{\partial\Omega}(x),$$

- $\nu_{\partial\Omega}$ is the normal outward vector,
- $B_{\partial\Omega}(x)$ is either a geometric quantity (mean curvature, Gauss curvature ...) or the solution of a PDE defined on Ω or $\partial\Omega$.

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Existence of minimizers

Can we find $\Omega^* \in \mathcal{O}_{\text{adm}}$ such that

$$F(\Omega^*) = \inf_{\Omega \in \mathcal{O}_{\text{adm}}} F(\Omega)?$$

Uniform ball property

$\Omega \in \mathcal{O}_{\text{adm}}$ if an only if $\Omega \subset D$ compact, $\forall x \in \partial\Omega$, $\exists d_x \in \mathbb{R}^n$

$$\|d_x\|_{\mathbb{R}^d} = 1,\ \ B_{r_0}(x - r_0 d_x) \subset \Omega \text{ and } B_{r_0}(x + r_0 d_x) \subset \mathbb{R}^n \backslash \Omega.$$

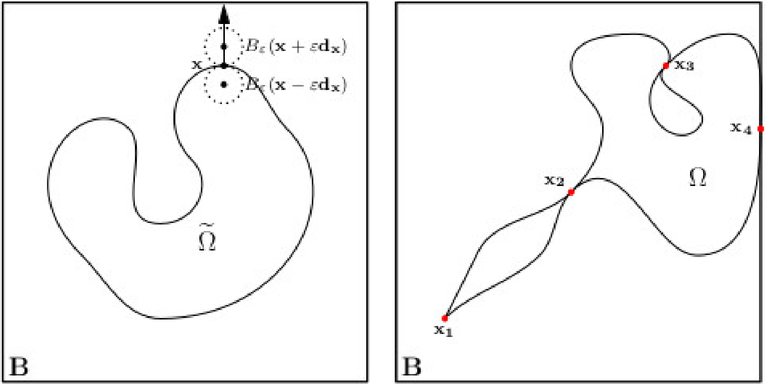

Figure taken from Dalphin.

Existing results

Theorem (Guo-Yang, 2013)

Let j be a continuous function from $\mathbb{R}^d \times \mathcal{S}^{d-1}$ to $\mathbb{R}$, then the following optimization problem

$$\inf_{\Omega \in \mathcal{O}_{adm}} \int_{\partial\Omega} j(x, \nu(x)) d\mu_{\partial\Omega}(x)$$

admits a minimiser.

Theorem (Dalphin, 2018)

Let j be a continuous function from $\mathbb{R}^d \times \mathcal{S}^{d-1} \times \mathbb{R}$ and convex with respect to the last variable, then the following optimization problem

$$\inf_{\Omega \in \mathcal{O}_{adm}} \int_{\partial\Omega} j(x, \nu(x), H_{\partial\Omega}(x)) d\mu_{\partial\Omega}(x)$$

admits a minimiser.

Let $h \in L^2(D)$, $g \in H^2(D)$, and define u_Ω as the solution of

$$\begin{cases} \Delta u_\Omega = h & \text{in } \Omega, \\ u_\Omega = g & \text{in } \partial\Omega. \end{cases}$$

Theorem (Dalphin, 2020)

Let j be a continuous function from $\mathbb{R}^d \times \mathcal{S}^{d-1} \times \mathbb{R} \times \mathbb{R}^d$, then the following optimization problem

$$\inf_{\Omega \in \mathcal{O}_{adm}} \int_{\partial\Omega} j(x, \nu(x), u_\Omega(x), \nabla u_\Omega(x))\, d\mu_{\partial\Omega}(x)$$

admits a minimiser.

The direct method of calculus of variations

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1. Define a (sequential) topology on $\mathcal{O}_{\text{adm}}$.
2. Take a minimizing sequence and use a compactness result
3. Prove the lower-semicontinuity of the functional

Signed distance

Distances functions

$$d_\Omega(x) = \inf_{y\in\Omega} \|x - y\|$$

$$b_\Omega(x) = d_\Omega(x) - d_{\mathbb{R}^d\setminus\Omega}(x)$$

Some properties

- For $x \in \partial\Omega$, $\nabla b_\Omega(x)$ is the unit outward normal vector,
- For $x \in \partial\Omega$, $\mathrm{Tr}(\nabla^2 b_\Omega(x))$ is the mean curvature,
- etc.

Uniform reach property

Definition

$$\mathrm{Reach}(\Omega) = \sup\{h > 0 \mid d_\Omega \text{ is differentiable in } U_h(\Omega) \setminus \Omega\}.$$

Assume $\mathrm{Reach}(\partial\Omega) = r_0 > 0$, we have

- if $\mathcal{H}^d(\partial\Omega) = 0$, then $\partial\Omega$ is a $\mathscr{C}^{1,1}$ hypersurface of $\mathbb{R}^d$ and satisfies the uniform ball property.
- For $h < r_0$, ∇b_Ω is $\frac{2}{r_0 - h}$-Lipschitz continuous on the tubular neighborhood $U_h(\partial\Omega)$.
- The restriction of ∇b_Ω to $\partial\Omega$ is $\frac{1}{r_0}$-Lipschitz continuous.

A new framework

R-convergence in $\mathcal{O}_{\text{adm}}$

Given $(\Omega_n)_{n\in\mathbb{N}} \in \mathcal{O}_{\text{adm}}^{\mathbb{N}}$, we say that $(\Omega_n)_{n\in\mathbb{N}}$ R-converges to $\Omega_\infty \in \mathcal{O}_{\text{adm}}$ and we write $\Omega_n \xrightarrow{R} \Omega_\infty$ if

$$b_{\Omega_n} \to b_{\Omega_\infty} \quad \begin{cases} \text{in } \mathscr{C}(\overline{D}), \\ \text{in } \mathscr{C}^{1,\alpha}(U_r(\partial\Omega_\infty)),\ \forall r < r_0,\ \forall \alpha \in [0,1), \\ \text{weakly-star in } W^{2,\infty}(U_r(\partial\Omega_\infty)),\ \forall r < r_0. \end{cases}$$

Theorem

$\mathcal{O}_{adm}$ is sequentially compact for the R-convergence.

Tubular neighboorhood

For $0 < h < r_0$, consider

$$T_{\partial\Omega} : \begin{array}{rcl} (-h,h)\times\partial\Omega & \to & U_h(\partial\Omega) \\ (t,x) & \mapsto & x + t\nabla b_\Omega(x). \end{array}$$

Since $T_{\partial\Omega}$ is Lipschitz continuous, it is differentiable at almost every (t_0, x_0), with

$$d_{(t_0,x_0)}T_{\partial\Omega}(s,y) = y + s\nabla b_\Omega(x_0) + t_0 d_{x_0}\nabla b_\Omega(y), \qquad \forall (s,y) \in \mathbb{R}\times T_{x_0}\partial\Omega.$$

Lemma

For every $\varepsilon > 0$, there exists $h > 0$ such that for all $\Omega \in \mathcal{O}_{adm}$,

$$1-\varepsilon \le \det(d_{(t_0,x_0)}T_{\partial\Omega}) \le 1+\varepsilon, \quad \text{for a.e. } (t_0,x_0) \in (-h,h)\times\partial\Omega.$$

Lemma

If $\Omega_n \xrightarrow{R} \Omega_\infty$ *then*

1. $\mathcal{H}^{d-1}(\partial\Omega_n)$ *converges toward* $\mathcal{H}^{d-1}(\partial\Omega_\infty)$ *as* $n \to +\infty$.
2. $\mathcal{H}^{d}(\Omega_n)$ *converges toward* $\mathcal{H}^{d}(\Omega_\infty)$ *as* $n \to +\infty$.
3. *If all the* $\partial\Omega_n$ *belong to the same homotopic class, then* $\partial\Omega_\infty$ *also belongs such a class.*

Corollary

$$\{\Omega \in \mathcal{O}_{\text{adm}} \mid a \leq \mathcal{H}^{d-1}(\partial\Omega) \leq b,\ \partial\Omega \text{ is homotopic to } \partial\Omega_0\}$$

is sequentially compact

Let j be a continuous function from $\mathbb{R}^d \times \mathcal{S}^{d-1} \times \mathbb{R}$ and convex with respect to the last variable.

$$F(\Omega) = \int_{\partial\Omega} j(x, \nu(x), H_{\partial\Omega}(x)) d\mu_{\partial\Omega}(x)$$

Theorem

F is a lower-semicontinuous shape functional for the R-convergence, i.e., for every sequence $(\Omega_n)_{n\in\mathbb{N}} \in \mathcal{O}_{\text{adm}}^{\mathbb{N}}$ that R-converges toward Ω_∞, one has

$$\liminf_{n\to+\infty} F(\Omega_n) \geq F(\Omega_\infty).$$

As a consequence, the shape optimization problem

$$\inf_{\Omega\in\mathcal{O}_{\text{adm}}} F(\Omega)$$

has a solution.

$$F(\Omega_n) = \int_{\partial\Omega_n} j(x, \nabla b_{\Omega_n}(x), H_{\partial\Omega_n}(p_n(y))) d\mu_{\partial\Omega_n}(x)$$
$$= \frac{1}{2h} \int_{U_h(\partial\Omega_n)} j(p_n(y), \nabla b_{\Omega_n}(p_n(y)), H_{\partial\Omega_n}(p_n(y))) \, \det(d_{T_n^{-1}(y)} T_n) \, dy.$$

$$F(\Omega_n) = \frac{1}{2h} \int_{U_{h-t}(\partial\Omega_\infty)} j(p_n(y), \nabla b_{\Omega_n}(p_n(y)), H_{\partial\Omega_n}(p_n(y))) \, \det(dT_n) \, dy$$
$$+ \frac{1}{2h} \int_{U_h(\partial\Omega_n) \setminus U_{h-t}(\partial\Omega_\infty)} j(p_n(y), \nabla b_{\Omega_n}(p_n(y)), H_{\partial\Omega_n}(p_n(y))) \, \det(dT_n) \, dy.$$

Definition

Let $f \in \mathscr{C}^0(D)$. We consider $v_{\partial\Omega}$ the solution of the equation

$$\Delta_{\partial\Omega} v_{\partial\Omega}(x) = f(x) \quad \text{in } \partial\Omega,$$

Definition

Let $f \in \mathscr{C}^0(D)$. We consider $v_{\partial\Omega}$ the solution of the equation

$$\Delta_{\partial\Omega} v_{\partial\Omega}(x) = f(x) \quad \text{in } \partial\Omega,$$

$v_{\partial\Omega}$ is the unique minimiser of

$$\mathscr{E}_{\partial\Omega} : H^1_*(\partial\Omega) \ni u \mapsto \frac{1}{2}\int_{\partial\Omega} |\nabla_{\partial\Omega} u(x)|^2 d\mu_{\partial\Omega} - \int_{\partial\Omega} f(x)u(x) d\mu_{\partial\Omega} \tag{2}$$

Lemma [Privat, R., Sigalotti, 2022]

For any $\Omega \in \mathcal{O}_{\text{adm}}$, Eq. (2) admits one and only one minimiser.

$$F(\Omega) = \int_{\partial\Omega} j(x, \nu(x), v_{\partial\Omega}(x), \nabla_{\partial\Omega} v_{\partial\Omega}(x))\, d\mu_{\partial\Omega}(x),$$

where $j : \mathbb{R}^d \times \mathcal{S}^{d-1} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is assumed to be continuous.

Theorem [Privat, R., Sigalotti, 2022]

The shape functional F is lower-semicontinuous for the R-convergence, i.e., for every sequence $(\Omega_n)_{n\in\mathbb{N}} \in \mathcal{O}_{\text{adm}}^{\mathbb{N}}$ that R-converges toward Ω_∞, one has

$$\liminf_{n\to+\infty} F(\Omega_n) \geq F(\Omega_\infty). \tag{3}$$

As a consequence, the shape optimization problem

$$\inf_{\Omega\in\mathcal{O}_{\text{adm}}} F(\Omega)$$

has a solution.

1. Transport $v_{\partial\Omega_n}$ to $\partial\Omega_\infty$ thanks to the orthogonal projector on $\partial\Omega_n$
2. The sequence obtained is bounded $H^1_*(\partial\Omega_\infty)$, extract and called $v^* \in H^1_*(\partial\Omega_\infty)$ the limit.
3. Check that $v^* = v_{\partial\Omega_\infty}$.
4. Passing to the limit is similar to the previous case.

In a nutshell

Hypersurfaces with a uniform Reach condition enjoy nice properties :

- Sequential compactness for the R-convergence.
- Many functionals involving geometric or PDE related cost are lower-semicontinuous for the R-convergence.
- Proofs are (relatively) straightforward.

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software : Stellacode
`https://rrobin.pages.math.cnrs.fr/stellacode/`

Cohomology and divergence free vector fields on the torus

Hodge decomposition

On a closed Riemannian manifold M

$$L_p^2(M) = B_p \oplus B_p^* \oplus \mathscr{H}_p,$$

where

- B_p is the L^2-closure of $\{d\alpha \mid \alpha \in \Omega^{p-1}(M)\}$
- B_p^* is the L^2-closure of $\{d^*\beta \mid \beta \in \Omega^{p+1}(M)\}$
- $\mathscr{H}_p$ is the set $\{\omega \in \Omega^p(M) \mid \Delta_H\omega = 0\}$ of harmonic p-forms with Δ_H the Hodge Laplacian

In vacuo Maxwell equations on a toroidal 3D domain

Let P a be toroidal domain. Let Γ be a toroidal loop inside P and denote by I_p the electric current-flux across any surface enclosed by Γ (also equal to the circulation of B along Γ).

Lemma

Let $B \in \mathcal{C}^\infty(P, \mathbb{R}^3)$ such that $\operatorname{div} B = 0$ and $\operatorname{curl} B = 0$ in P. Let g be the normal magnetic field on ∂P. Then g and I_p determine completely the magnetic field B in P. Besides, there exists a constant $C > 0$ such that for every other magnetic field $\tilde{B}$ with the same total poloidal currents, $|B - \tilde{B}|_{H^{1/2}(P,\mathbb{R}^3)} \leq C|g - \tilde{g}|_{L^2(\partial P)}$ where $\tilde{g}$ is the normal component of $\tilde{B}|_{\partial P}$.

Idea : consider the cochain complex

$$\mathscr{C}^\infty(P) \xrightarrow{\text{grad}} \mathscr{C}^\infty(P, \mathbb{R}^3) \xrightarrow{\text{curl}} \mathscr{C}^\infty(P, \mathbb{R}^3) \xrightarrow{\text{div}} \mathscr{C}^\infty(P).$$