About observer synthesis for a system whose solutions converge to a set of indistinguishable states

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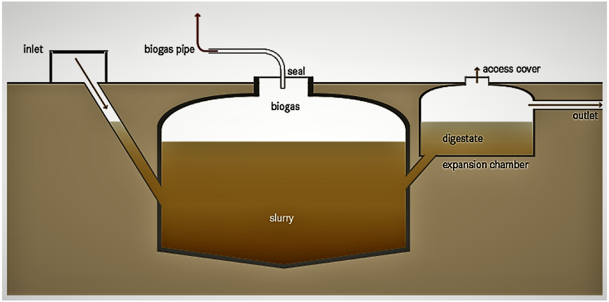
Motivation and context

Biogas reactor:



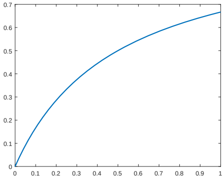
Principle: "substrate $\xrightarrow{biomass}$ biomass + biogas"

research conducted at LBE (Laboratory of Environmental Biotechnology), INRAE, Narbonne

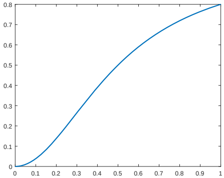
Batch process with single biogas measurement

$$(\mathcal{S}) : \begin{cases} \dot{b} &= \mu(s)b \\ \dot{s} &= -\mu(s)b \end{cases} \qquad y = \mu(s)b$$

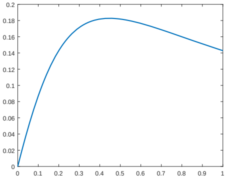
Assumption. μ is C^∞ with $\mu(0) = 0$ and $\mu(s) > 0$ for $s > 0$.



Monod: $\mu(s) = \frac{\bar{\mu}s}{K+s}$



Hill: $\mu(s) = \frac{\bar{\mu}s^\alpha}{K^\alpha+s^\alpha}$



Haldane: $\mu(s) = \frac{\bar{\mu}s}{K+s+\frac{s^2}{K_i}}$

Batch process with single biogas measurement

In coordinates $\begin{bmatrix} z \\ s \end{bmatrix} = \begin{bmatrix} b+s \\ s \end{bmatrix}$,

$$(\mathcal{S}) : \left\{ \begin{array}{rcl} \dot{z} & = & 0 \\ \dot{s} & = & -\mu(s)(z-s) \end{array} \right. \qquad y = \mu(s)(z-s)$$

defined on $\mathcal{C} := \{(z,s) \in \mathbb{R}^2 \mid z > s > 0\}$

Property. Any solution of $(\mathcal{S})$ on $\mathcal{C}$ is such that $\lim_{t \to +\infty} s(t) = 0$ and $\mathbb{R}_+ \times \{0\}$ is a set of indistinguishable states...

Observability analysis

$$\frac{d}{dt}\begin{bmatrix} z \\ s \end{bmatrix} = f(z,s) := \begin{bmatrix} 0 \\ -\mu(s)(z-s) \end{bmatrix}, \; y := h(z,s) = \mu(s)(z-s)$$

Definition. (Gauthier-Kupka 98) (f, h) is differentially observable on $\mathcal{D}$ (of order n) when it exists $n \in \mathbb{N}_\star$ such that

$$x \mapsto (h(x), L_f(x), \cdots f_f^{n-1} h(x)) \text{ is injective on } \mathcal{D}$$

where $L_f h(x) := \partial_x h(x).f(x)$, $L_f^i h(x) := L_f(L_f^{i-1} h)(x)$.

Proposition. If μ is

- concave increasing, or
- increasing with $\log \mu$ decreasing,

then (f, g) is differentially observable of order 2 on $\mathcal{C}$ but not on $\partial\mathcal{C}$.

Observability analysis

$$
\begin{aligned}
L_f h(z,s) &= [\mu'(s)(z-s)-\mu(s)]\mu(s)(z-s) \\
&= -\left(\frac{\mu'(s)}{\mu(s)}h(z,s)-\mu(s)\right)h(z,s)
\end{aligned}
$$

For a given $y > 0$, define $\varphi_y(s) = \mu(s) - \frac{\mu'(s)}{\mu(s)}y$

$$
\varphi_y'(s) = \mu'(s) - \left(\frac{\mu''(s)}{\mu(s)} - \frac{\mu'(s)^2}{\mu(s)^2}\right)y > 0 \;\Rightarrow \varphi_y \text{ invertible}
$$

$$
\Rightarrow \begin{cases} s = \varphi_{h(z,s)}^{-1}\left(\dfrac{L_f h(z,s)}{h(z,s)}\right) \\[2ex] z = \dfrac{h(z,s)}{\mu(s)} + s \end{cases} \quad \text{if } h(z,s) \neq 0
$$

Consideration of Luenberger observers

In (s, y) coordinates:

Observer:

$$\left\{ \begin{array}{rcl} \dot{\hat{s}} & = & -y + G_1(\hat{y} - y) \\ \dot{\hat{y}} & = & -\dfrac{\mu'(\hat{s})}{\mu(\hat{s})} y^2 + \mu(\hat{s}) y + G_2(\hat{y} - y) \end{array} \right.$$

has a singularity at $\hat{s} = 0$...

Consideration of Luenberger observers

In $(y, \dot{y})$ coordinates:

Assume that μ is increasing and μ' is bounded.

$$\begin{cases} \dot{x} = y \\ \dot{y} = \phi(x,y)y \end{cases} \quad \text{with } \phi(x,y) = -\mu' \circ \mu^{-1}\left(\frac{y}{x}\right)x + \frac{y}{x}$$

Observer:

$$\begin{cases} \dot{\hat{x}} = y(t) + G_1(\hat{y} - y(t)) \\ \dot{\hat{y}} = \tilde{\phi}(\hat{x}, y(t))y(t) + G_2(\hat{y} - y(t)) \end{cases}$$

with $\tilde{\phi}(x,y) = \phi(\max(x, \epsilon, y/\bar{\mu}), y)$, $\bar{\mu} = \sup_{s>0} \mu(s)$ and $\epsilon > 0$ such that $x_0 > \epsilon$.

Convergence of the Luenberger observer

$$\frac{d}{dt}\begin{bmatrix} e_x \\ e_y \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & G_1 \\ \delta(t)y(t) & G_2 \end{bmatrix}}_{A(t)} \begin{bmatrix} e_x \\ e_y \end{bmatrix}$$

$$\text{with } \delta(t) = \left| \begin{array}{ll} \dfrac{\tilde{\phi}(\hat{x}(t), y(t)) - \phi(x(t), y(t))}{\hat{x}(t) - x(t)} & \text{if } \hat{x}(t) \neq x(t), \\ \partial_x \phi(x(t), y(t)) & \text{if } \hat{x}(t) = x(t). \end{array} \right.$$

$$\implies \lim_{t \to +\infty} A(t) = \begin{bmatrix} 0 & G_1 \\ 0 & G_2 \end{bmatrix}$$

Luenberger observer with log measurement

$$(x, y) \mapsto (x, w) := (x, \log y)$$

Observer:

$$\begin{cases} \dot{\hat{x}} = y(t) + G_1(\hat{w} - \log y(t)) \\ \dot{\hat{w}} = \tilde{\phi}(\hat{x}, y(t)) + G_2(\hat{w} - \log y(t)) \end{cases}$$

$$\Rightarrow \quad \frac{d}{dt}\begin{bmatrix} e_x \\ e_w \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & G_1 \\ \delta(t) & G_2 \end{bmatrix}}_{A(t)} \begin{bmatrix} e_x \\ e_w \end{bmatrix}$$

$$\text{with} \quad \lim_{t\to+\infty} A(t) = \begin{bmatrix} 0 & G_1 \\ -\mu'(0) & G_2 \end{bmatrix} \text{ (for } \epsilon \text{ small enough)}$$

Note that $w(t) \to +\infty$ when $t \to +\infty$.

Contruction of an asymptotic estimator

A forward integral invariant:

$$\lim_{t\to+\infty} s(t) = 0 \Rightarrow s(t) = \int_t^{+\infty} y(\tau)d\tau$$
$$\Rightarrow x(t) = \frac{y(t)}{\mu\left(\int_t^{+\infty} y(\tau)d\tau\right)}$$

$$\Rightarrow z = x(t) + s(t) = \boxed{\frac{y(t)}{\mu\left(\int_t^{+\infty} y(\tau)d\tau\right)} + \int_t^{+\infty} y(\tau)d\tau = Cst}$$

Contruction of an asymptotic estimator

$$z = \frac{y(t)}{\mu\left(\int_t^{+\infty} y(\tau)d\tau\right)} + \int_t^{+\infty} y(\tau)d\tau \ , \ \forall t > 0$$

$$\Rightarrow \hat{z}(t) = \frac{y(t_0)}{\mu\left(\int_{t_0}^{t} y(\tau)d\tau\right)} + \int_{t_0}^{t} y(\tau)d\tau \text{ is an estimator of } z$$

Estimator:

$$\hat{x}(t) = \frac{y(t_0)}{\mu\left(\int_{t_0}^{t} y(\tau)d\tau\right)} + \int_{t_0}^{t} y(\tau)d\tau, \quad \hat{s}(t) = \mu^{-1}\left(\frac{y(t)}{\hat{x}(t)}\right)$$

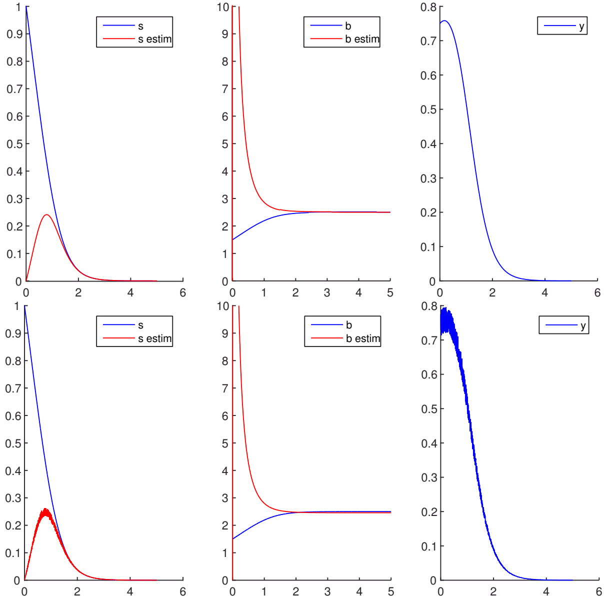
Asymptotic observer

$$\begin{cases} \dot{v}(t) = y(t), \quad v(0) = 0 \\ \hat{x}(t) = \dfrac{y(0)}{\mu(v(t))} + v(t) \quad (t > 0) \\ \hat{s}(t) = \mu^{-1}\left(\frac{y(t)}{\hat{x}(t)}\right) \quad (t > 0) \end{cases}$$

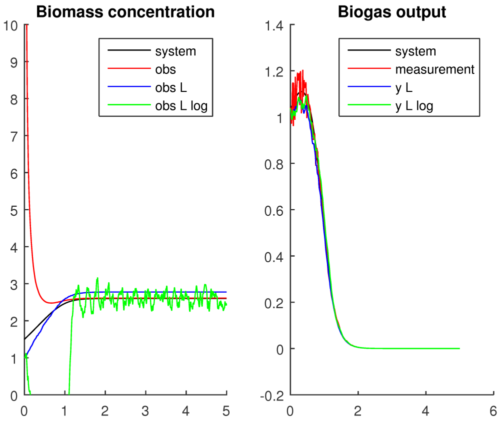
Property. When μ is increasing, $\hat{x} - x$ is decreasing with time:

$$\frac{d}{dt}(\hat{x} - x)(t) = -y(0)\frac{\mu'\left(\int_0^t y(\tau)d\tau\right)}{\left[\mu\left(\int_0^t y(\tau)d\tau\right)\right]^2} y(t)$$

Simulations of the asymptotic estimator



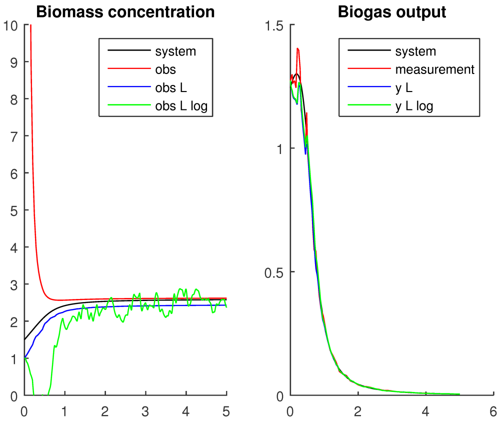
Numerical comparisons - the Monod case



small noise on y

Luenberger: $G_1 = G_2 = -20$;
Luenberger with log: $G_1 = \lambda_1\lambda_2/\mu'(0)$, $G_2 = \lambda_1 + \lambda_2$

Numerical comparisons - the Hill case



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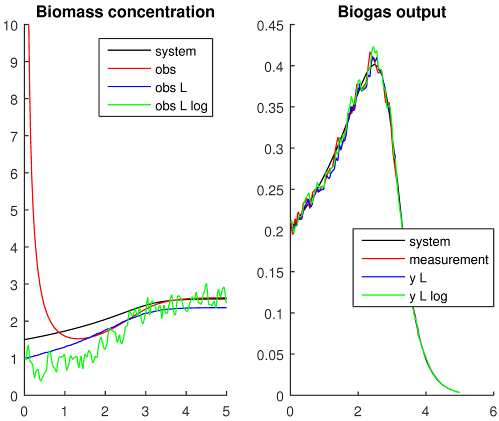
Numerical comparisons - the Haldane case



small noise on y

Luenberger: $G_1 = G_2 = -20$;
Luenberger with log: $G_1 = \lambda_1\lambda_2/\mu'(0)$, $G_2 = \lambda_1 + \lambda_2$

Robustness issue

Consider corrupted measurements: $y_{obs}(t) = y(t) + p(t)$,
(assumed to be unbiased i.e. with $\int_0^{+\infty} p(t)dt = 0$)

$$\Rightarrow \lim_{t \to +\infty} \hat{x}(t) - x(t) = \frac{p(0)}{\mu\left(\int_0^{+\infty} y(\tau)d\tau\right)}.$$

Assumption. There exists $T < +\infty$ such that

$$\frac{1}{T}\int_t^{t+T} p(\tau)d\tau = 0, \quad \forall t > 0.$$

Robust asymptotic estimator

Proposition. The estimator

$$\hat{x}(t) = \frac{\displaystyle\int_{t_0=0}^{t_0=T} \left\{ y_{obs}(t_0) + \int_{t_0}^{t} y_{obs}(\tau) d\tau . \mu \left(\int_{t_0}^{t} y_{obs}(\tau) d\tau \right) \right\} dt_0}{\displaystyle\int_{t_0=0}^{t_0=T} \mu \left(\int_{t_0}^{t} y_{obs}(\tau) d\tau \right) dt_0}, \quad t > T$$

is unbiased.

Remark. $\hat{x}(\cdot)$ IS NOT the averaging of the asymptotic observer.

Robust asymptotic observer

Define $I(t_0, t) = \int_{t_0}^{t} y_{obs}(\tau) d\tau, \quad t > t_0$

Let $\dot{v}_i = y_{obs(t)} \mathbb{1}_{\{t>(i-1)h\}}(t), \; v_i(0) = 0 \quad \text{for } i = 1 \cdots N$

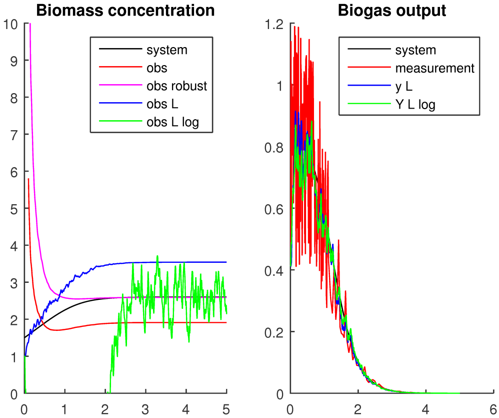
Then for any integrable function γ, one has the approximation

$$\int_{t_0=0}^{t_0=T} \gamma(I(t_0, t)) \simeq h \sum_{i=1}^{N} \gamma(v_i(t)), \quad t > T$$

Finite dimensional observer.

$$\hat{x}(t) = \frac{\frac{N}{T} v_1(T) + \sum_{i=1}^{N} v_i(t) \mu(v_i(t))}{\sum_{i=1}^{N} \mu(v_i(t))} \quad t > T,$$

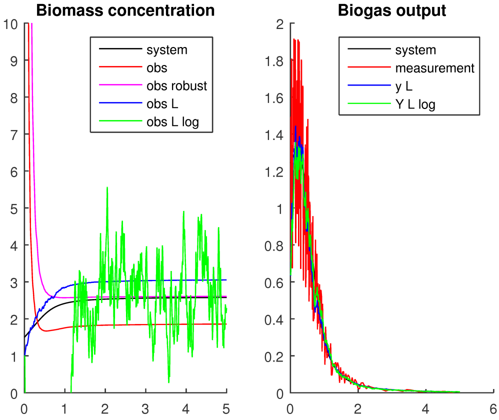
Numerical comparisons - the Monod case



$T = 0.1$
$N = 10$

Luenberger: $G_1 = G_2 = -20$;
Luenberger with log: $G_1 = \lambda_1 \lambda_2 / \mu'(0)$, $G_2 = \lambda_1 + \lambda_2$

Numerical comparisons - the Hill case



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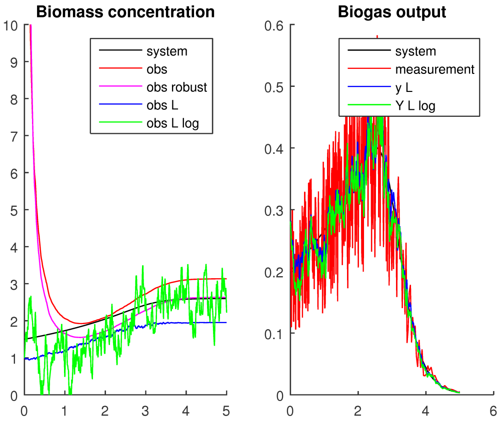
Numerical comparisons - the Haldane case



$T = 0.1$
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Luenberger: $G_1 = G_2 = -20$;
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Other attempts...

The Kazantis-Kravaris approach

$$(\mathcal{S}) : \begin{cases} \dot{x} & = & f(x) \\ \dot{y} & = & h(x) \end{cases}$$

Proposition (Kazantis-Kravaris 98): If there exists a diffeomorphism $T : \mathbb{R}^n \mapsto \mathbb{R}^n$ such that

$$\frac{d}{dt} T(x) = AT(x) + Bh(x) \quad \text{with} \quad A \text{ Hurwitz}$$

then $\dot{z} = Az + By(t)$, $\hat{x} = T^{-1}(z)$ is an observer for $(\mathcal{S})$.

Remark. Such a transformation T is solution of the p.d.e.

$$\partial_x T(x) f(x) = AT(x) + Bh(x)$$

(and one can try $A = \mathrm{diag}(-\lambda_1, \cdots, -\lambda_n)$, $B = [1, \cdots, 1]^t$).

Observer construction

Let $T_\lambda : (z,s) \mapsto T_\lambda(z,s)$ be a solution on of the o.d.e.

$$\mu(s)(z-s)\frac{\partial T_\lambda}{\partial s} = \lambda T_\lambda - \mu(s)(z-s) \quad \text{where } z \text{ is a parameter}$$

for $(z,s) \in \mathcal{T} = \{(z,s) \in \mathbb{R}^2 \,|\, s \in (0, z - z_{\min}),\ z \in [z_{\min}, z_{\max}]\}$.

$$\text{Define } \phi_{\lambda_1,\lambda_2} : \begin{bmatrix} z \\ s \end{bmatrix} \mapsto \begin{bmatrix} T_{\lambda_1}(z,s) \\ T_{\lambda_2}(z,s) \end{bmatrix} \quad (\lambda_1 \neq \lambda_2)$$

- Can one extend $\phi_{\lambda_1,\lambda_2}$ outside $\mathcal{T}$ to be globally Lipschitz on $\mathbb{R}^2$ with a Lipschitz inverse?

$$\text{Then } \left\{ \begin{array}{rcl} \dot{\hat{T}}_{\lambda_1} & = & -\lambda_1 \hat{T}_{\lambda_1} + y(t) \\ \dot{\hat{T}}_{\lambda_2} & = & -\lambda_2 \hat{T}_{\lambda_2} + y(t) \end{array} \right. , \quad \begin{bmatrix} \hat{z} \\ \hat{s} \end{bmatrix} = \phi^{-1}_{\lambda_1,\lambda_2}\left(\begin{bmatrix} \hat{T}_{\lambda_1} \\ \hat{T}_{\lambda_2} \end{bmatrix} \right)$$

is an observer of $(\mathcal{S})$.

Transformation construction

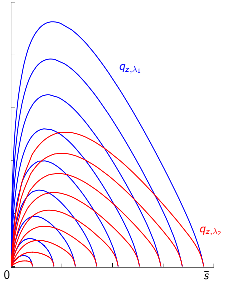
$$\text{Define } \gamma_z(s) = \mu(s)(z-s),\ s \in [0,z] \ \text{ for } z > 0$$

Lemma. Fix $\lambda > 0$, $z > 0$ and $\bar{s} \in]0, z[$

1. There exists an unique $q_{z,\lambda} \in C^1([0,\bar{z}], \mathbb{R}^+)$ s.t.

$$\gamma_z(s) q'_{z,\lambda}(s) = \lambda q_{z,\lambda}(s) - \gamma_z(s), \quad \forall s \in [0, \bar{s}]$$

   with $q_{z,\lambda}(0) = q_{z,\lambda}(\bar{s}) = 0$.
2. $\lambda_2 > \lambda_1 > 0 \Rightarrow q_{z,\lambda_1}(s) > q_{z,\lambda_2}(s) > 0, \quad \forall s \in]0, \bar{s}[$
3. γ_z concave $\Rightarrow q_{z,\lambda}$ (strictly) concave function on $[0, \bar{s}]$.

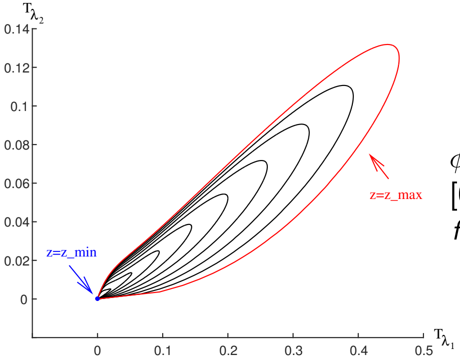


Functions $q_{\lambda,z}$ with $\lambda_1 < \lambda_2$ for various z

Transformation construction

$$\phi_{\lambda_1,\lambda_2}(z,s) = \begin{bmatrix} q_{z,\lambda_1}(s) \\ q_{z,\lambda_2}(s) \end{bmatrix} \text{ with } \phi_{\lambda_1,\lambda_2}(z, z - z_{\min}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

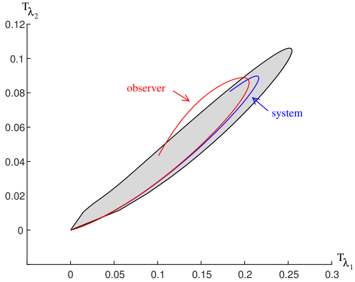
Proposition For $\lambda_2 > \lambda_1 > 0$, the map $\phi_{\lambda_1,\lambda_2}$ is injective on $\phi_{\lambda_1,\lambda_2}(\bar{\mathcal{T}}) \setminus \{(0,0)\}$.

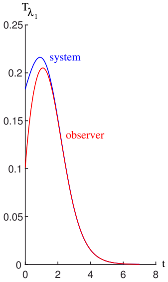


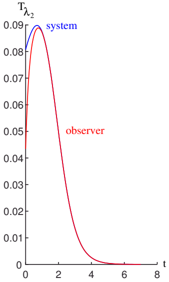
Sets $\phi_{\lambda_1,\lambda_2}(\{z\} \times [0, z - z_{\min}])$ *for various z*

Simulations in $(T_{\lambda_1}, T_{\lambda_2})$ coordinates

$$\mu(s) = \frac{s}{1+s}, \quad \lambda_1 = 2, \quad \lambda_2 = 5$$

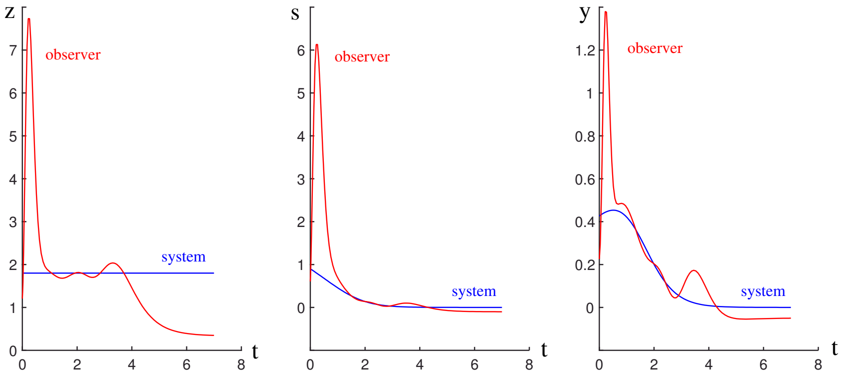






Come back in (z, s) coordinates

- Open problem: how to write **explictly** $\phi^{-1}_{\lambda_1,\lambda_2}$?
- numerical approximation with spline functions:



A particular case: $\mu(s) = s$

$$Z = \left[\begin{array}{c} \log(xs) \\ s - x \end{array} \right] \quad \Rightarrow \quad \dot{Z} = \left[\begin{array}{c} Z_2 \\ -2y(t) \end{array} \right]$$

We consider then the observer

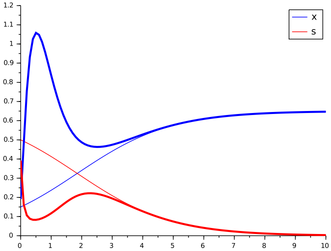
$$\left\{ \begin{array}{rcl} \dot{\hat{Z}}_1 & = & \hat{Z}_2 + G_1(\log(y(t)) - \hat{Z}_1) \\ \dot{\hat{Z}}_2 & = & -2y(t) + G_2(\log(y(t)) - \hat{Z}_1) \end{array} \right. \quad \text{defined for } y(t) > 0$$

$$\text{with} \left\{ \begin{array}{rcl} \hat{x} & = & \phi_x(\hat{Z}_2, y(t)) = \dfrac{-\hat{Z}_2 + \sqrt{\hat{Z}_2^2 + 4y(t)}}{2} \\ \hat{s} & = & \phi_s(\hat{Z}_2, y(t)) = \dfrac{\hat{Z}_2 + \sqrt{\hat{Z}_2^2 + 4y(t)}}{2} \end{array} \right.$$

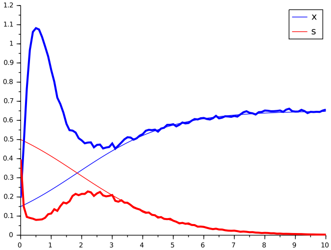
$$\Rightarrow \dot{E} = \left(\begin{array}{cc} G_1 & 1 \\ G_2 & 0 \end{array} \right) E \text{ in } Z \text{ coordinates,}$$

$|\partial_{Z_2}\phi_x| \leq 1$, $|\partial_{Z_2}\phi_s| \leq 1$ preserve the convergence speed in (x, s)

A particular case: $\mu(s) = s$



without noise



with noise

Conclusion and perspectives

- The asymptotic observer converges to the exact unobservable state in $\partial\mathcal{C}$
- Can we improve the convergence speed without asymptotic bias?
- Extension of the method to a larger class of problems?

Reference: Rapaport, A. and Dochain, D. (2020). A robust asymptotic observer for systems that converge to unobservable states. A batch reactor case study. IEEE Transactions on Automatic Control, 65 (6), 2693–2699.