On the decay rate for degenerate gradient flows subject to persistent excitation

Yacine Chitour
Université Paris-Saclay, CNRS, CentraleSupélec,
Laboratoire des signaux et des systèmes

Joint work with Dario Prandi and Paolo Mason

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Degenerate gradient flows

$$\dot{x}(t) = -c(t)c(t)^\top x(t), \qquad x \in \mathbb{R}^n,\ c : [0, +\infty) \to \mathbb{R}^n. \tag{DGF}$$

These systems appear in algorithms for, e.g.,

1. Gradient descent with incomplete knowledge of the gradient
2. Identification and model reference adaptive control
3. Consensus in multi-agent systems

Objectives

(a) Guarantee convergence and stability of (DGF) at the origin
(b) extract information on the decay rate (in case of exponential convergence).

Motivation: Adaptive filters

Problem

$z(t) = h^\top c(t)$ scalar output system, estimate parameter $h \in \mathbb{R}^n$, knowing input $c : [0, +\infty) \to \mathbb{R}^n$ and output $z : [0, +\infty) \to \mathbb{R}$.

Let $\hat{h} : [0, +\infty) \to \mathbb{R}^n$ estimate of h so that

$$\frac{d}{dt}\hat{h}(t) = (z(t) - \hat{z}(t))c(t), \qquad \hat{z}(t) := \hat{h}(t)^\top c(t).$$

Error vector $x(t) = h - \hat{h}(t)$ satisfies (DGF):

$$\dot{x}(t) = -\left(z(t) - \hat{z}(t)\right) c(t) = -\left(x(t)^\top c(t)\right) c(t)$$

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Convergence to 0 of (DGF) $\iff$ Quality of the estimator $\hat{h}$

Obvious fact: $c : [0, +\infty) \to \mathbb{R}^n$ must "visit regularly" all directions!!
$\Rightarrow$ Need for such condition to have convergence.

Persistent excitation

We say that c verifies the *persistent excitation* condition if

$$\exists\ a, b, T > 0, \quad \forall t \geq 0, \qquad a\,\mathrm{Id}_n \leq \int_t^{t+T} c(s)c(s)^\top\, ds \leq b\,\mathrm{Id}_n\,. \quad \text{(PE)}$$

Theorem (cf., Anderson, Narendra, et al. 80s)

Signal c verifies (PE) *if and only if* (DGF) *is uniformly globally exponentially stable at* 0, *i.e.,*

$$\exists\ C, \alpha > 0, \quad \|x(t)\| \leq Ce^{-\alpha(t-s)}\|x(s)\|, \qquad \forall t > s \geq 0,$$

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- (PE) says that c "visits all directions of $\mathbb{R}^n$ during a time window".
- Upper bound b is *essential*: (cf. Barabanov *et al.* 2005), if $b = +\infty$ it can happen that

$$x(t) \longrightarrow \bar{x} \neq 0 \quad \text{as } t \to +\infty$$

Decay rate

Under (PE), system $\dot{x} = -cc^\top x$ is *globally exponentially stable*:

$$\|x(t)\| \leq Ce^{-\alpha t}\|x(0)\|, \qquad \forall t \geq 0, \tag{GES}$$

with C, α only depending on a, b, T.

Decay rate for a signal c verifying (DGF)

$$R(c) := \sup\{\alpha > 0 \mid \text{(GES) holds}\} = -\limsup_{t \to +\infty} \frac{\log \|\Phi_c(t, 0)\|}{t},$$

$\Phi_c(t, 0) =$ fundamental matrix of (DGF) from 0 to t.

Definition

The *worst decay rate* is

$$R(a, b, T, n) := \inf\{R(c) \mid c \text{ satisfies (PE) with parameters } a, b, T\}.$$

$\rightsquigarrow$ Yields *guaranteed* convergence rate for ANY signal c verifying (DGF).

Main result

Many lower bounds for $R(a, b, T, n)$ exist in the literature, of the type:

Theorem (cf., Andersson and Krishnaprasad (2002))

There exists $C_1 > 0$ such that

$$R(a, b, T, n) \geq \frac{C_1 a}{(1 + nb^2)T}, \qquad \forall T > 0,\ a < b,\ n \in \mathbb{N}.$$

Problem: Are these bounds optimal ?

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Theorem (C.-Mason-Prandi)

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$\rightsquigarrow$ We recover the result by Barabanov *et al.* (2005) if $b = +\infty$.

Application I: L^2-gain for (DGF) systems with linear input

Consider the controlled (DGF) system:

$$\dot{x}(t) = -c(t)c(t)^\top x(t) + u(t), \qquad u \in L^2([0,+\infty),\mathbb{R}^n).$$

Let $\gamma(c)$ be the L^2-gain of the input/output map $u \mapsto x$:

$$\gamma(c) = \sup_{u \in L^2 \setminus \{0\}} \frac{\|x_u\|_2}{\|u\|_2}$$

Rantzer (1999) posed the problem of determining the worst L^2 gain:

$$\gamma(a,b,T,n) = \sup\{\gamma(c) \mid c \text{ satisfies (PE) }\}.$$

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Theorem (C.-Mason-Prandi)

There exists $\kappa_0, \kappa_1 > 0$ such that for all $T > 0$, $a \leq b$, $n \geq 2$, it holds

$$\kappa_0 \frac{(1+b^2)T}{a} \leq \gamma(a, b, T, n) \leq \kappa_1 \frac{(1+nb^2)T}{a}.$$

Sketch of the proof

Idea

Connect $R(a, b, T, n) = \inf R(c)$ with an optimal control problem.

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$$R(c) = -\limsup_{t \to +\infty} \frac{1}{t} \underbrace{\sup \left\{ \log \frac{\|x(t)\|}{\|x(0)\|} \mid x(0) \in \mathbb{R}^n \right\}}_{\log \|\Phi_c(t,0)\|}$$

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Polar coordinates: Letting $x = r\omega$ for $r > 0$ and $\omega \in \mathbb{S}^{n-1}$, (DGF) reads

$$\begin{cases} \dot{r} &= -(c^\top \omega)^2 r, \\ \dot{\omega} &= -c^\top \omega \left(c - (c^\top \omega)\omega \right), \end{cases} (Pol) \qquad r_0 = \|x(0)\|, \quad \omega_0 = \frac{x(0)}{\|x(0)\|}.$$

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- (Pol) = Dynamics of ω independent of r.
- The dynamics of r yield:

$$-\log \frac{\|x(T)\|}{\|x(0)\|} = -\log \frac{r(T)}{r(0)} = \int_0^T (c^\top \omega)^2 \, ds =: J_T(c, \omega_0).$$

Sketch of the proof II

Optimal control problem:

$$\mu(a, b, T, n) := \min J_T(c, \omega_0) = \min \int_0^T (c^\top \omega)^2 \, ds$$

Here, $c : [0, T] \to \mathbb{R}^n$ runs over all signals satisfying

$$a \, \mathrm{Id}_n \leq \int_0^T c(s)c(s)^\top \, ds \leq b \, \mathrm{Id}_n,$$

and ω is a solution to (Pol) with $\omega(0) = \omega_0 \in \mathbb{S}^{n-1}$.

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Steps:

1. Prove that

$$R(a, b, T, n) \leq 2 \frac{\mu(a/2, b/2, T, n)}{T}$$

   $\rightsquigarrow$ Show that $\mu(a/2, b/2, T, n)$ is realized by an optimal control $c_\star : [0, T] \to \mathbb{R}^n$, which extends to a $2T$-periodic (PE) signal $c_\star : \mathbb{R}_+ \to \mathbb{R}^n$

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3. Precisely estimate $\mu(a, b, T, 2)$.

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PMP

More general systems

We obtain the same result for the worst rate of decay for the more general system

$$\dot{x}(t) = -S(t)x(t)$$

were $S(t) \in \mathbb{R}^{n\times n}$ is such that $S(t) \geq 0$ and for $a, b, T > 0$

$$a\,\mathrm{Id}_n \leq \int_t^{t+T} S(s)\,ds \leq b\,\mathrm{Id}_n$$

$\rightsquigarrow$ The family of signals S is obtained as the convexification of the family $cc^\top$ where $c : [0, T] \to \mathbb{R}^n$ satisfies (PE)

$\rightsquigarrow$ the worst rate of decay is realized by (DGF), e.g., $S = cc^\top$

Perspectives

Open question

For a, b, T fixed, what dependence on the dimension?

$$\frac{C_1}{n} \leq \lim_{b \to \infty} R(a, b, T, n) \frac{(1+b^2)T}{a} \leq C_0.$$

- The technique used in the proof yields also the lower bound

$$R(a, b, T, n) \geq \frac{\mu(a, b, T, n)}{T}.$$

- At the moment we cannot directly access $\mu(a, b, T, n)$ for $n \neq 2$.

Thank you for your attention!

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