A singular observation problem and its applications in automotive control

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Outline of the talk

- The considered class of systems
- Interest of this class for anti-lock braking systems design
- Available stability results
- Pros and cons of current results
- Some perspectives

Part I – In which systems are we interested?

The class of systems

We are interested in systems of the form

$$\dot{x} = f(x) + g(x)u, \quad y = h(x),$$

where $x(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}$ is the measurement. And, among them, we consider those that can be **transformed**, via $z = \varphi(x)$, into

$$\frac{dz}{dt} = s(y)\big(Az + d(y)\big) + b(y)u, \quad y = Cz,$$

where the pair (A, C) is **observable** and the vector fields b and d depend on the output only. The function $s(\cdot)$ is assumed to be **strictly increasing**.

We consider the **observer**

$$\frac{d\hat{z}}{dt} = s(y)\big(A\hat{z} + d(y) + K(s(y))(y - C\hat{z})\big) + b(y)u,$$

where the **gain** $K(\cdot)$ depends on the system's output only.

The regular case is a classical result...

The **observation error** $e = z - \hat{z}$ has the following dynamics

$$\frac{de}{dt} = s(y)(A - K(s(y))C)e.$$

When $s > 0$ (**regular case**), stability can be analyzed in a new **time-scale**

$$\tau(t) = \int_0^t s(y(\nu))d\nu.$$

If s *does not* converge to zero, the error dynamics

$$\frac{de}{d\tau} = (A - KC)e$$

is **asymptotically stable** provided that $A - KC$ is **Hurwitz**[a].

[a]Krener & Respondek (1985), Guay (2002), and Respondek, Pogromsky & Nijmeijer (2004).

The singular case asks for extra work...

In the **singular** case, when the function $s(y(t))$ **changes its sign**, we can still consider a new **time-scale**

$$\tau(t) = \int_0^t |s(y(\nu))| d\nu,$$

and to choose the **observer gains** depending on the output's sign to obtain

$$\frac{de}{d\tau} = \begin{cases} (A - K^+ C)e & \text{if} : s(y(\tau)) > 0 \\ (-A + K^- C)e & \text{if} : s(y(\tau)) < 0, \end{cases}$$

which defines as a **switched system**

$$\frac{de}{d\tau} = A_{\sigma(\tau)} e.$$

Part II – Why are these systems interesting ?

An example coming from the ABS literature

The **wheel speed** ω has the following dynamics

$$I\frac{d\omega}{dt} = -RF_x - \gamma_b P_b,$$

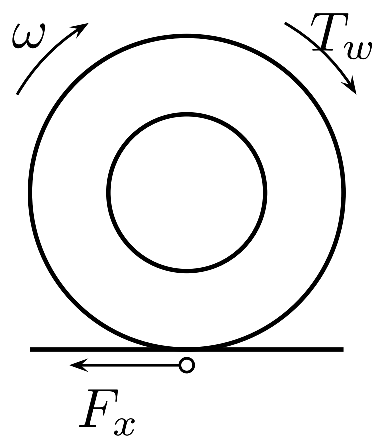
where I is the wheel's inertia, R its radius, F_x the longitudinal tyre force, P_b the brake pressure, and γ_b the brake efficiency.

The **tyre force** F_x is a nonlinear function

$$F_x(\lambda) = \mu(\lambda)F_z$$

of the **wheel slip**

$$\lambda = \frac{R\omega - v_x}{v_x},$$



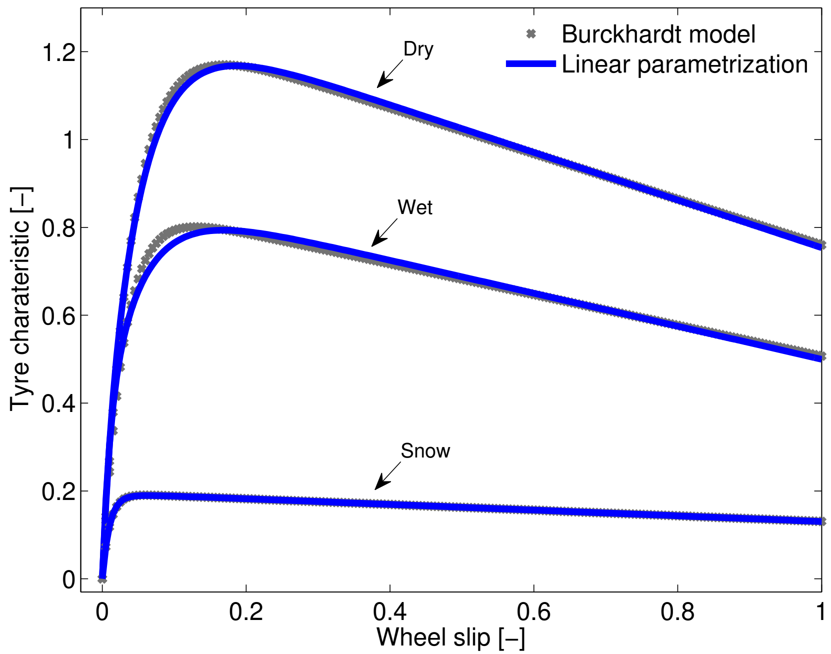
where F_z is the vertical load, v_x the vehicle speed, and $\mu(\lambda)$ the **tyre characteristic**.

Burckhardt's tyre model

The **tyre characteristic** is described[a] by a function

$$\mu(\lambda) = c_1(1 - e^{-c_2\lambda}) - c_3\lambda,$$

where the constants c_1, c_2, and c_3 depend on the **road conditions**.



[a] Burckhardt, M. (1993). *Fahrwerktechnik : Radschlupf-Regelsysteme*. Vogel-Verlag.

Wheel acceleration and XBS dynamics

The **wheel-acceleration** dynamics can be written[a]

$$\frac{dz_1}{dt} = \frac{-a}{v_x(t)} z_1 z_2 + bu$$
$$\frac{dz_2}{dt} = (cz_2 + z_3)\frac{z_1}{v_x(t)} \qquad \text{and} \qquad y = z_1,$$
$$\frac{dz_3}{dt} = 0$$

where a, b, and c are known constants, v_x the **longitudinal speed**, and u the time-derivative of the **brake pressure**.

The only measurable variable is $z_1 = R\dot{\omega} - a_x$, the **wheel acceleration** offset. The unmeasurable state $z_2 = \mu'(\lambda)$ is the **extended braking stiffness**. The constant z_3 is *unknown* and depends on the parameters of Burckhardt's model.

[a] Hoang, T. B. *et al*. (2014). Extended braking stiffness estimation based on a switched observer, with an application to wheel-acceleration control. *IEEE Trans. on Control Systems Technology*, 22(6), 2384-2392.

The proposed observer

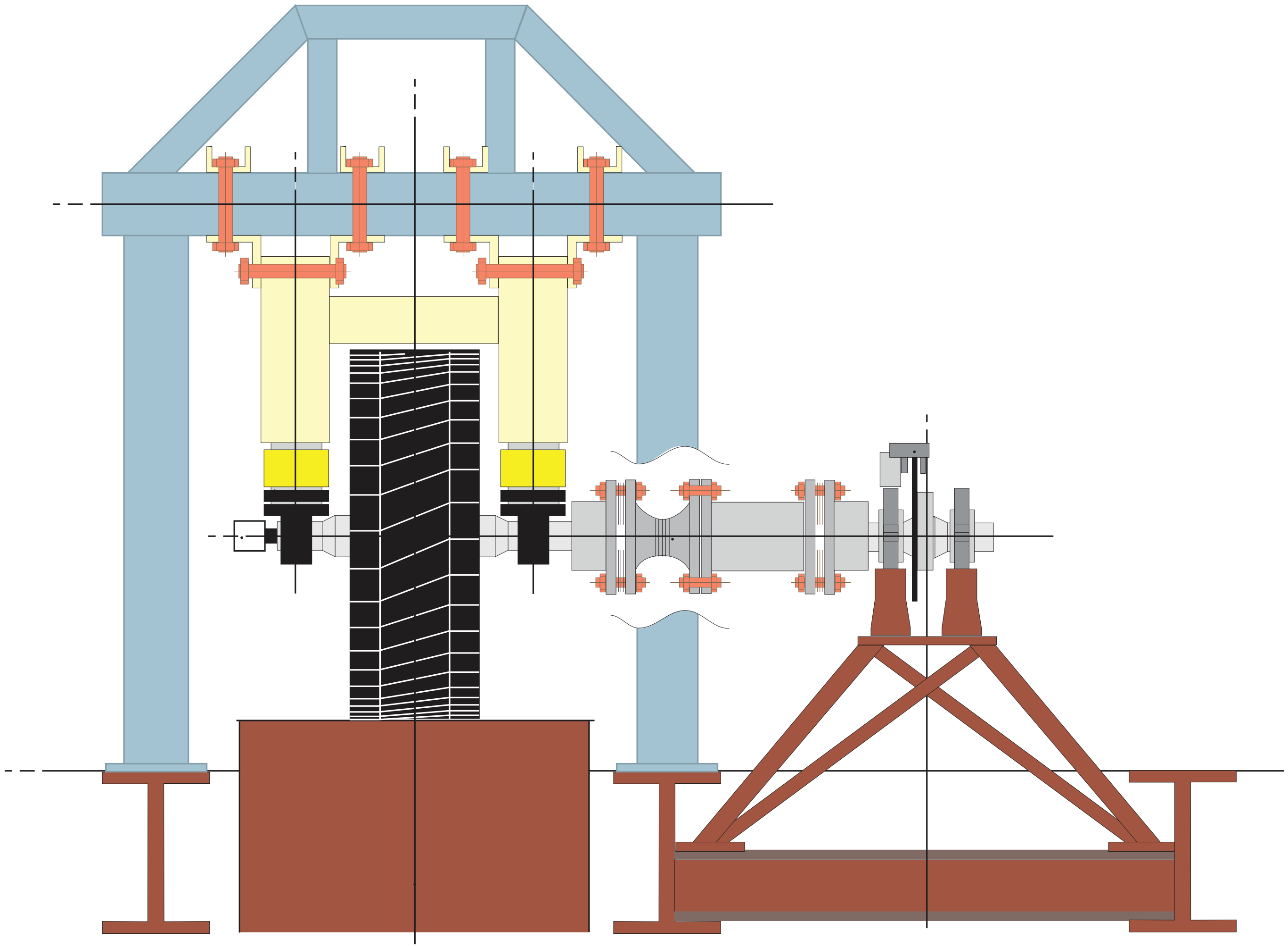
Our approach leads to the following **observer** :

$$\frac{d\widehat{z}_1}{dt} = \frac{-a}{v_x} z_1 \widehat{z}_2 + bu + \frac{k_1(z_1)}{v_x} z_1 (z_1 - \widehat{z}_1)$$
$$\frac{d\widehat{z}_2}{dt} = (c\widehat{z}_2 + \widehat{z}_3)\frac{z_1}{v_x} + \frac{k_2(z_1)}{v_x} z_1 (z_1 - \widehat{z}_1)$$
$$\frac{d\widehat{z}_3}{dt} = \frac{k_3(z_1)}{v_x} z_1 (z_1 - \widehat{z}_1).$$

The **observer gains** $k_i(z_1)$, for $1 \leq i \leq 3$, are

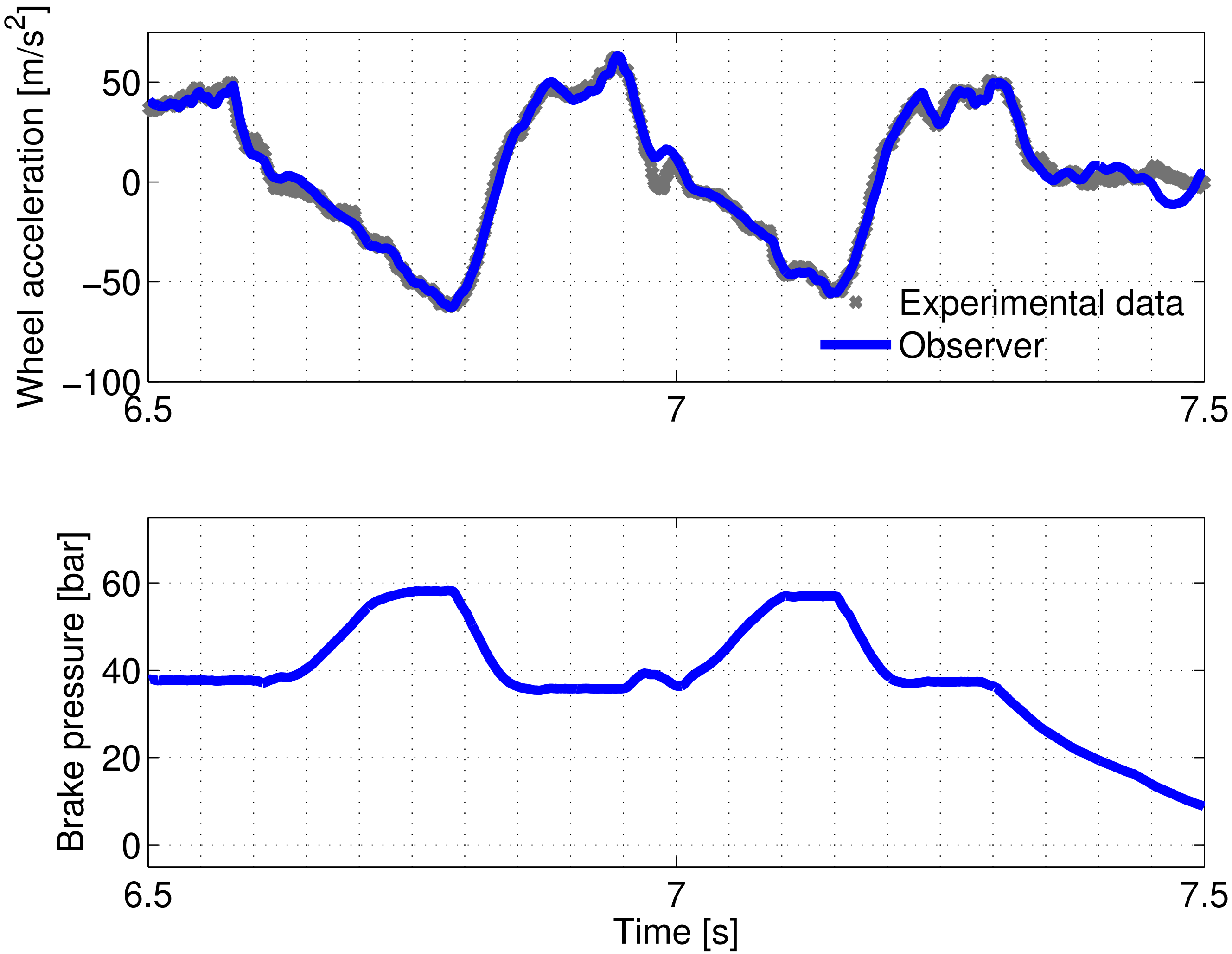
$$k_i(z_1) = \begin{cases} k_i^+ & \text{if } z_1 > 0 \\ k_i^- & \text{if } z_1 < 0. \end{cases}$$

TU-Delft's tyre setup[a]

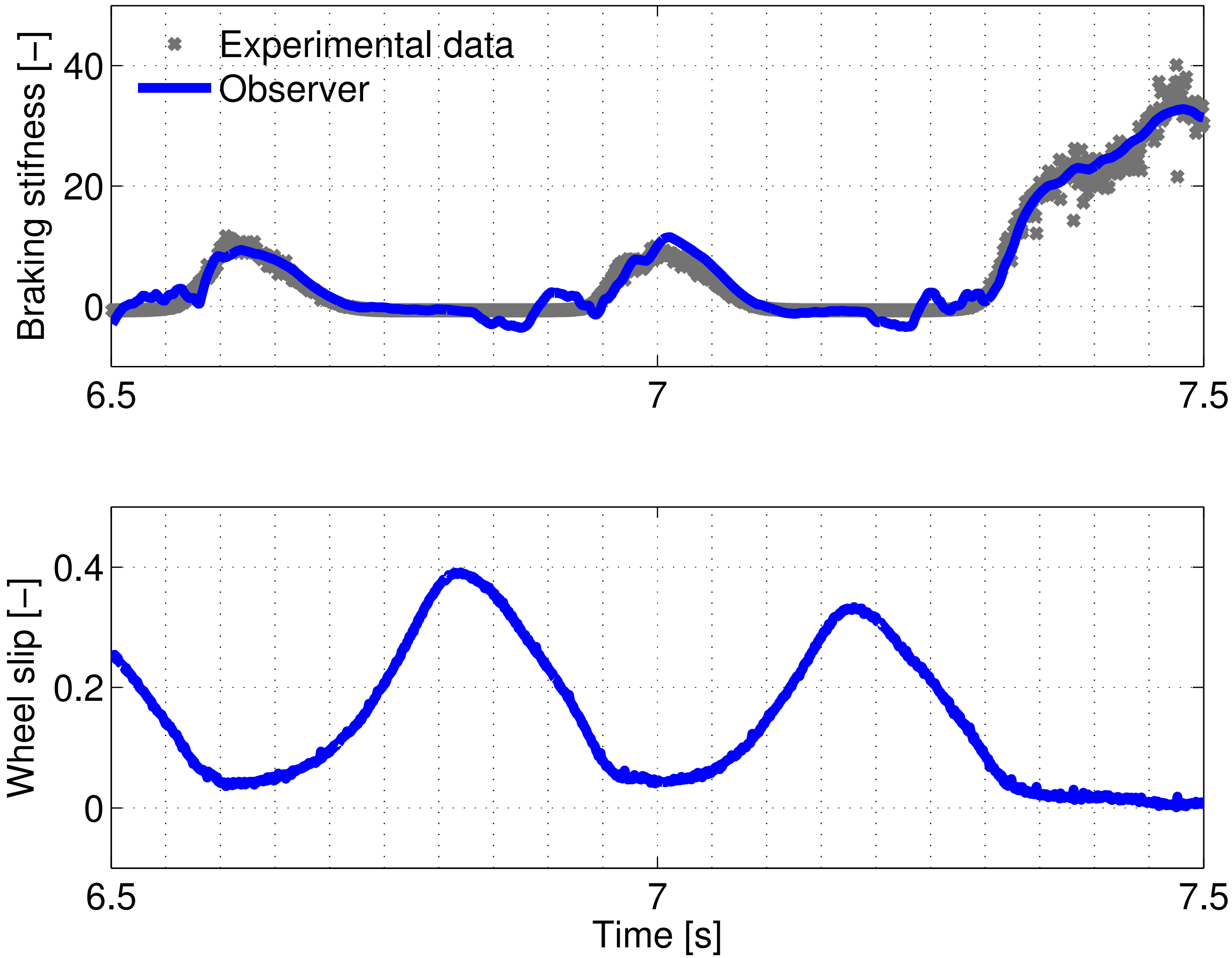


[a] H. Pacejka. Tyre and Vehicle Dynamics. *Butterworth-Heinemann*, 2005.

Wheel acceleration & brake pressure



Extended braking stiffness & wheel slip



Part III – What can we prove for these system ?

The class of systems

We are interested in systems of the form

$$\dot{x} = f(x) + g(x)u, \quad y = h(x),$$

where $x(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}$ is the measurement, and both f and g are smooth.

We look for those that can be **transformed** into

$$\frac{dz}{dt} = s(y)\big(Az + d(y)\big) + b(y)u, \quad y = Cz, \tag{1}$$

where the pair (A, C) is **observable** and the vector fields b and d depend on the output only. The function $s(\cdot)$ is assumed to be **strictly increasing**.

We consider the **observer**

$$\frac{d\hat{z}}{dt} = s(y)\big(A\hat{z} + d(y) + K(s(y))(y - C\hat{z})\big) + b(y)u, \tag{2}$$

where the **gain** $K(\cdot)$ depends on the system's output only.

Ingredients of the stability proof

In the **singular** case, when the function $s(y(t))$ **changes its sign**, we must deal with three difficulties.

(1) When $s(y(t)) < 0$, the time-scale is **inverted** :

The error must **diverge** in the new time-scale τ in order to *converge* in t.

(2) The instants t_i at which $s(y(t_i)) = 0$ should not *accumulate* :

This can be avoided by imposing a **dwell-time** condition.

(3) The system is not **uniformly observable**. We must exclude the controls that generate *indistinguishable* states. We must also guarantee the time-scale change is **well-posed**, *i.e.*, that $\tau \in \mathcal{K}_\infty$:

This can be ensured by imposing a **persistency of excitation** condition.

The first ingredient : The time scale

The key step is to consider a **new time-scale**

$$\tau(t) = \int_0^t |s(y(\nu))| d\nu,$$

and to choose the **observer gains** depending on the output's sign

$$K(s(y(\tau))) = \begin{cases} K^+ & \text{if : } s(y(\tau)) > 0 \\ K^- & \text{if : } s(y(\tau)) < 0. \end{cases} \tag{3}$$

This gives

$$\frac{de}{d\tau} = \begin{cases} (A - K^+C)e & \text{if : } s(y(\tau)) > 0 \\ (-A + K^-C)e & \text{if : } s(y(\tau)) < 0, \end{cases}$$

which gives as a **switched system**

$$\frac{de}{d\tau} = A_{\sigma(\tau)}e. \tag{4}$$

Uniform exponential stability & dwell-time

The origin of a switched system is said to be **uniformly exponentially stable** if there exists constants $a_0 > 0$ and $\lambda_0 > 0$ such that, for each $\tau \geq 0$, we have

$$\|e(\tau)\| \leq a_0 \exp(-\lambda_0 \tau)\|e(0)\|.$$

Here, the word **uniform** refers to the fact that a_0 and λ_0 do not depend on the switching signal[a].

The observer's switching signal $\sigma(\cdot)$ admits a **dwell-time**[b] if $\exists \tau_D > 0$ such that any two **consecutive discontinuities** of σ are separated by no less than τ_D.

[a] D. Angeli. A note on stability of arbitrarily switched homogeneous systems. *Nonlinear Control Abstracts*, 1(13), 1999.

[b] Hespanha, J. P. (2004). Uniform stability of switched linear systems : extensions of LaSalle's invariance principle. *IEEE Transactions on Automatic Control*, 49(4), 470-482.

Why asking for dwell-time ?

The following switched system is a **two-dimensional example** of the error dynamics we can have for our system :

$$\frac{de(s)}{ds} = \begin{cases} A_1 e(s) = \begin{pmatrix} -k_1^+ & -a \\ -k_2^+ & c \end{pmatrix} e(s) & \text{lorsque } z_1 > 0 \\ A_2 e(s) = \begin{pmatrix} k_1^- & a \\ k_2^- & -c \end{pmatrix} e(s) & \text{lorsque } z_1 < 0. \end{cases}$$

The non-existence of a of **strict** common Lyapunov function for this system results from the classification of Balde, Boscain and Mason (2009).

Classification of planar switched systems

Consider a **planar switched system**

$$\frac{dx(s)}{ds} = A_\sigma x(s), \tag{5}$$

where $\sigma : [0, \infty) \to \mathcal{P}$, with $\mathcal{P} := \{1, 2\}$ denotes a signal that selects, at each instant, a matrix in $\{A_1, A_2\}$. Here $A_i \in \mathbb{R}^{2\times 2}$.

Balde, Boscain and Mason introduced the **invariant** :

$$\Gamma(A, B) = \frac{1}{2}\left(\mathrm{tr}(A)\mathrm{tr}(B) - \mathrm{tr}(AB)\right),$$

as well as an other invariant R, more complicated to compute (see their paper[a]).

[a] Balde, M., Boscain, U., and Mason, P. (2009). A note on stability conditions for planar switched systems. *International Journal of Control*, 82(10), 1882-1888.

The Theorem of Balde, Boscain, and Mason (2009)

For **planar switched systems**, we have four possible cases :

(i) If $\Gamma(A_1, A_2) < -\sqrt{\det A_1 \det A_2}$ then there exits an initial condition and a switching signal that generates and **unbounded trajectory**.

(ii) If $\Gamma(A_1, A_2) = -\sqrt{\det A_1 \det A_2}$ then the origin of the system is **UGS but not UGES**.

(iii) If $\Gamma(A_1, A_2) > -\sqrt{\det A_1 \det A_2}$ and $\mathrm{tr}(A_1 A_2) > -2\sqrt{\det A_1 \det A_2}$ then the system admits a (quadratic) **strict common Lyapunov function** (it is thus UGES).

(iv) If $\Gamma(A_1, A_2) > -\sqrt{\det A_1 \det A_2}$ and $\mathrm{tr}(A_1 A_2) \leq -2\sqrt{\det A_1 \det A_2}$ then the origin is UGES if $R < 1$, UGS if $R = 1$, and admits an initial condition and a switching signal that generates and unbounded trajectory if $R > 1$.

Hespanha's Theorem (2004)

Assume that there exist a finite family $\{P_p : p \in \mathcal{P}\}$ of **symmetric positive definite** matrices such that, for each $(x, \sigma) \in \mathcal{S}$,

$$x^T(t)P_{\sigma(t)}x(t) \leq x^T(t)P_{\sigma(t^-)}x(t), \quad \forall t \geq 0,$$

et

$$A_p^T P_p + P_p A_p \leq -C_p^T C_p, \quad \forall p \in \mathcal{P},$$

for an appropriate family of matrices $\{C_p : p \in \mathcal{P}\}$ such that all $C_p^T C_p$ are **symmetric** and **positive**. Then, the origin of the switched system $\dot{x}(t) = A_{\sigma(t)}x(t)$, with $\sigma \in \mathcal{S}$, is **stable**.

If, additionally, each pair (A_p, C_p) is **observable** ($\forall p \in \mathcal{P}$) : (i) When $\mathcal{S} \subset \mathcal{S}_{\text{weak-dwell}}$ it is asymptotically stable ; (ii) When there exist $\tau_D > 0$ et $T < \infty$ such that $\mathcal{S} \subset \mathcal{S}_{\text{p-dwell}}[\tau_D, T]$ it is (uniformly) **exponentially stable**.

The second ingredient : A combined Lyapunov equation

Theorem 1 *Consider an observable pair* (A, C)*. Define* $Q = C^T C$*. For any given pair of gains* K^+ *and* K^-*, define*

$$A_+ = A - K^+ C \quad \text{and} \quad A_- = -A + K^- C.$$

If K^+ *is such that* A_+ *is Hurwitz, then there exists a unique* K^- *such that the two following Lyapunov equations*

$$A_+^T P + P A_+ = -Q \quad \text{and} \quad A_-^T P + P A_- = -Q$$

admit a common solution P *that is symmetric and positive definite.*

Moreover, if (A, C) *is in observer normal form, then the components* k_i^- *of* K^- *are expressed in terms of the components* k_i^+ *of* K^+ *by*

$$k_i^- = (-1)^i k_i^+ + \big(1 - (-1)^i\big) a_i,$$

where the constants a_i *are the coefficients of the characteristic polynomial of* A*.*

Observer normal form

Recall that for any observable pair (A, C), there exist linear coordinates in which

$$A = \begin{pmatrix} a_1 & 1 & 0 & \dots & 0 \\ \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & 0 & \dots & 0 & 1 \\ a_n & 0 & \dots & \dots & 0 \end{pmatrix}$$

and

$$C = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}.$$

In these coordinates, the system is in *observer normal form*.

The third ingredient : Technical assumptions (Option I)

Assumption 1 *The function* s *and the output trajectories,* $y(t, t_0, z_0)$*, satisfy the following conditions :*

(*persistency of excitation*) *there exist* $\mu_0 > 0$ *and* $T_0 > 0$ *such that, for all* $(t_0, z_0) \in \mathbb{R} \times \mathbb{R}^n$,

$$\int_t^{t+T_0} s(y(s, t_0, z_0))^2 ds \geq \mu_0, \quad \forall\, t \geq t_0; \tag{6}$$

(*dwell-time*) *there exists* $T_D > 0$ *such that, for any* $(t_0, z_0) \in \mathbb{R} \times \mathbb{R}^n$ *and any two instants* $t_k \neq t_l$ *satisfying* $s(y(t_k, t_0, z_0)) = s(y(t_l, t_0, z_0)) = 0$,

$$\left| t_k - t_l \right| \geq T_D. \tag{7}$$

Stability of the observer error dynamics

Theorem 2 *Consider system* (1)*, with* (A, C) *observable and under Assumption 1.*

Consider also the observer given by (2) *and* (3)*, with* K_+ *such that* $A - K_+ C$ *is Hurwitz.*

Then, there exists K_- *such that, for the estimation error dynamics* (4)*, the origin is globally asymptotically stable, uniformly in the output trajectories.*

Furthermore, for a pair (A, C) *in observable companion form, the elements of* K_- *may be taken according to*

$$k_i^- = (-1)^i k_i^+ + \big(1 - (-1)^i\big) a_i.$$

The third ingredient : Technical assumptions (Option II)

Fix two **compact** sets Ω_0 and Ω, with non-empty interiors, such that $0 \in \Omega_0$ and $\Omega_0 \subset \Omega$. Fix $\epsilon > 0$. Construct the compact set $\Sigma = \Omega \cap \{z \in \mathbb{R}^n : |Cz| \leq \epsilon\}$.

Define a_0, d_0, and s_0 as the **maxima** of the functions $|CAz|$, $|Cd(y)|$, and $|s(y)|$, respectively, on Σ. Define b_0 as the **minimum** of $Cb(y)$ on Σ. Introduce the constant $\alpha = s_0(a_0 + d_0)/b_0$. And, finally, fix $\beta > \alpha$.

We say that a controller is **admissible** if it generates a control signal $u(\cdot)$ that satisfies together with its state trajectory $x(\cdot)$ the following properties :

(i) The control signal $u(t)$ is a piecewise continuous function of time such that $|u(t)| < \beta$, for $t \geq 0$;

(ii) On any given time interval $[a, b]$ such that the output satisfies $|y(t)| < \epsilon$, for $t \in [a, b]$, the sign of $u(t)$ is constant and $|u(t)| > \alpha$.

(iii) If $z(0) \in \Omega_0$ then $z(t) \in \Omega$, for each $t \geq 0$.

Stability of the observer error dynamics

Theorem 3 *Consider system*

$$\dot{z} = s(y)\big(Az + d(y)\big) + b(y)u, \quad y = Cz,$$

and assume that the function $Cb(y)$ does not vanish at $y = 0$ and that the controller used to govern the system is ***admissible****. If $z_0 \in \Omega_0$, then the switching signals $\rho(t)$ and $\sigma(\tau)$ generated by the controller admit a strictly positive* ***dwell-time*** *in the original and in the new time-scale, respectively.*

If, additionally, the pair (A, C) is ***observable*** *and the gain matrices K^+ and K^- satisfy the conditions of Theorem 1, then the origin of the observer's error dynamics*

$$\frac{de}{d\tau} = \begin{cases} (A - K^+C)e & \text{if : } s(y(\tau)) > 0 \\ (-A + K^-C)e & \text{if : } s(y(\tau)) < 0 \end{cases}$$

is ***uniformly exponentially stable*** *in the new time-scale and* ***asymptotically stable*** *in the original time-scale.*

Part IV – Conclusion & Pespectives

Some final thoughts

- We proposed an observer for a class of non-uniformly observable systems ;
- We gave technical conditions that guarantee its convergence ;
- It can be used to monitor the wheel dynamics around and ABS limit cycle ;
- It cannot be used to stabilize the wheel slip around its optimal set-point ;
- In low dimension, we have an adaptive version of this observer...

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