Coupled methods of nonlinear control and estimation applicable to Terrain-Aided Navigation

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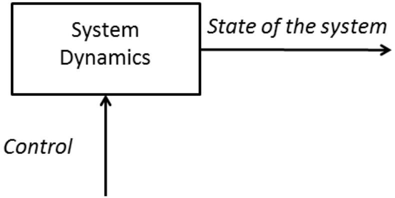
Motivation

Typical control and estimation problem



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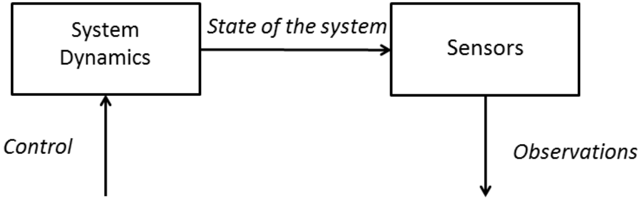

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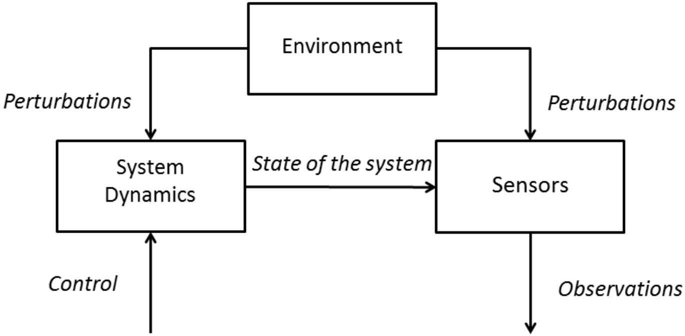

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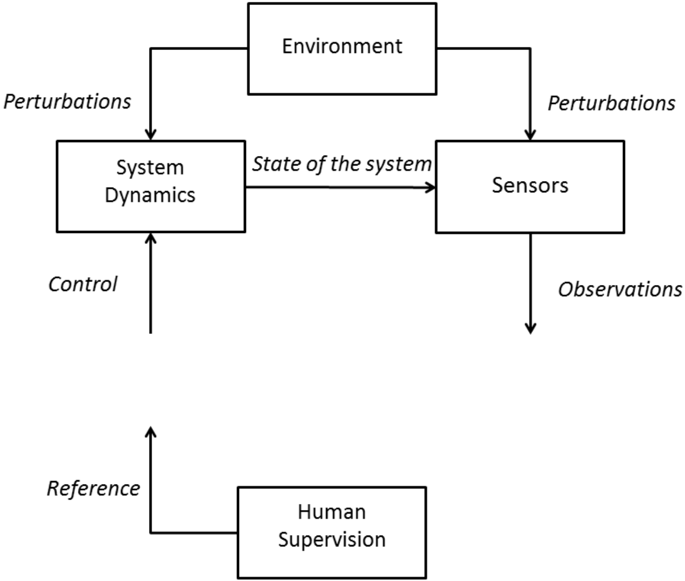
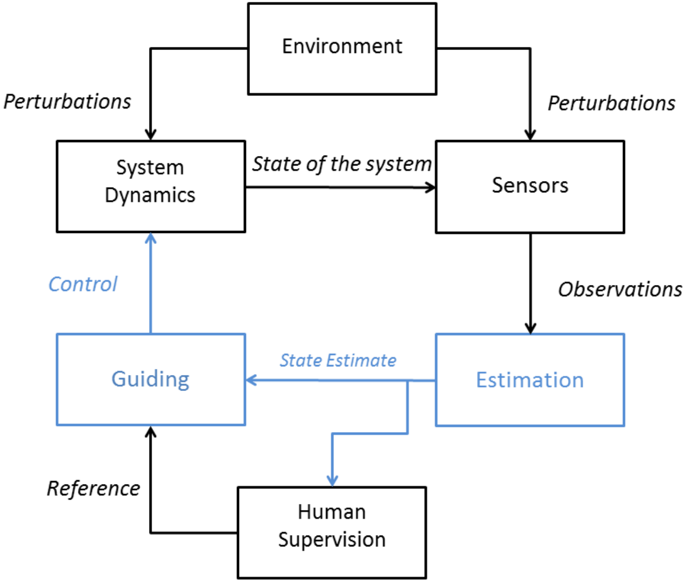
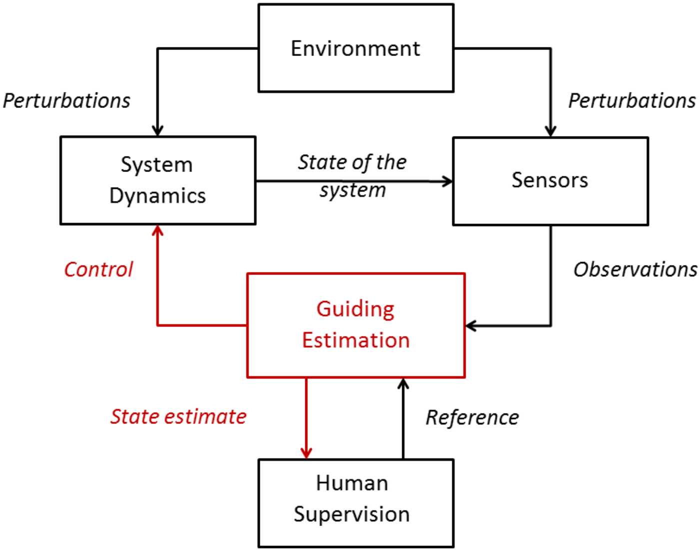


Typical guiding and estimation scheme



Desired solution



Localisation and guidance by terrain-aided navigation

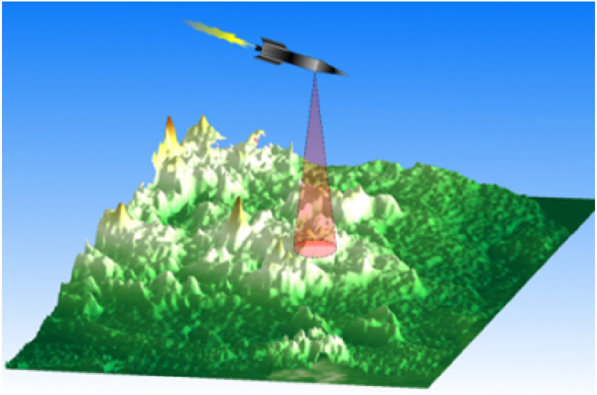


Figure: Example of a UAV localised by terrain-aided navigation

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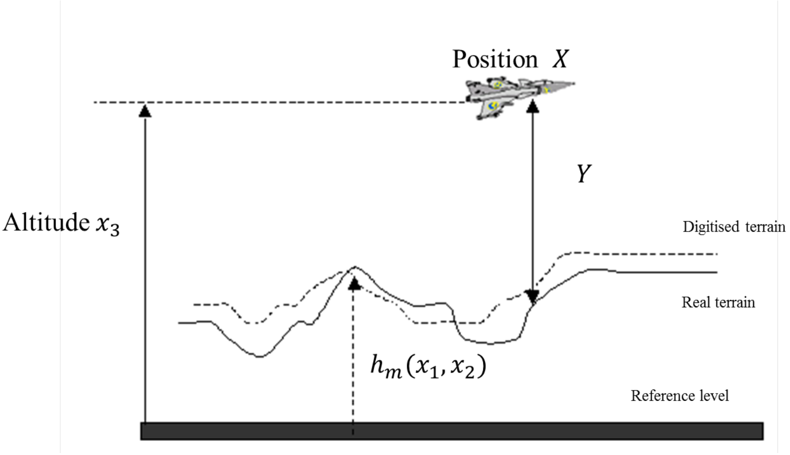

Figure: Representation of terrain-aided navigation

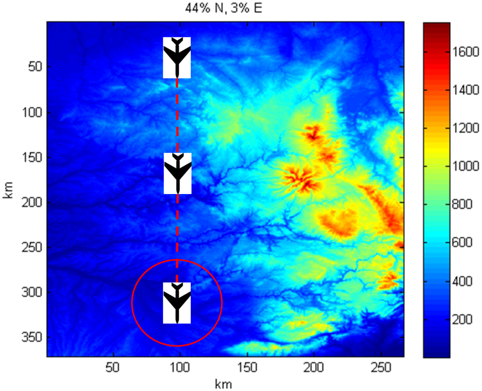

Figure: Trajectory where the UAV goes straight to the target over a flat area

⇒ guiding and estimation must be coupled: **dual effect**.

Localisation and guidance by terrain-aided navigation

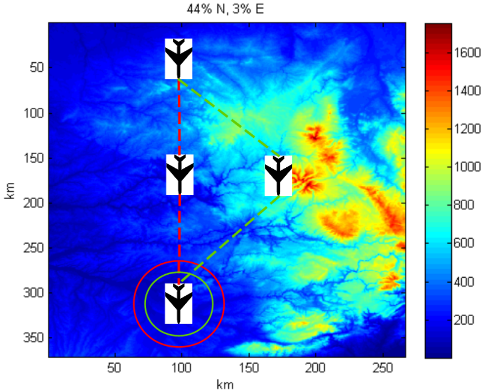

Figure: Compromise between going toward the target and getting information

Two main approaches

- An analytical approach in a deterministic framework (analytical terrain maps)

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- A non-analytical approach in a discrete-time stochastic framework (arbitrary terrain maps)

Analytical approach of TAN

1. Modelling
2. Estimation
3. Output-feedback control

Plan

1. Modelling

2. Estimation

3. Output-feedback control

System's dynamics

The system is a mass point in 3D controlled in acceleration with a speed and a height measurement leading to:

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The only information on the position $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ comes from the height measurement:

$$Y = x_3 - h_m(x_1, x_2)$$

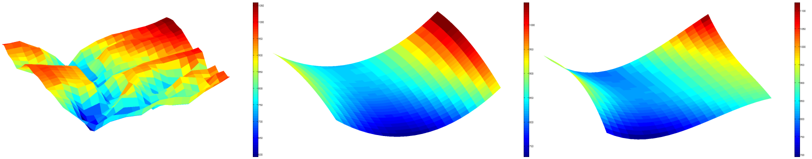
where h_m is the terrain map.

Terrain modelling

Several terrain profiles, h_m, are considered:

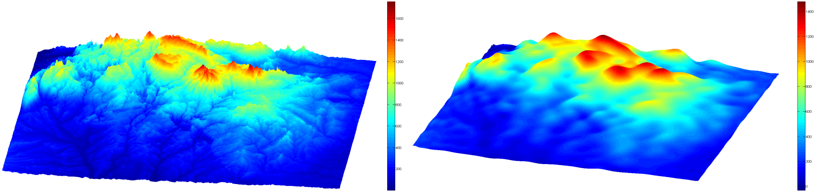
- Quadratic functions
- Cubic functions or higher order polynomials
- Gaussian functions or sum of Gaussian functions
- Spatial sine waves

Terrain modelling: examples



(a) Real terrain (b) Quadratic approximation (c) Cubic approximation

Figure: Small scale real map with approximations



(a) Real terrain (b) Fourier approximation

Figure: Large scale real map with an approximation

Plan

Locally weak observability conditions

The local weak observability conditions (Lie derivatives) can be interpreted depending on their order as follows:

- Order 0: linked to the absolute altitude x_3
- Order 1: colinear to the horizontal speed V_{hor}
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$\Rightarrow$ Local inversion of the observations possible if V_{hor} and U_{hor} are **not colinear**

$\Rightarrow$ First occurrence of the dual effect in our model

What about explicit reconstruction of the state ?

Observer design by I&I: general principle

1. By defining the unmeasured state η and the measured state y:

$$\dot{\eta} = f_1(\eta, y, u)$$
$$\dot{y} = f_2(\eta, y, u)$$

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3. Choose β, ϕ and ξ such that:

$$\mathcal{M} = \{(\eta, y, \xi, t) \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R} : \beta(\xi, y, t) = \phi(\eta, y, t)\}$$

is *Invariant* and *Globally Attractive*

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4. Define the estimator $\hat{\eta} = \phi^L(\beta(\xi, y, t))$ and make sure that $\|\hat{\eta} - \eta\|$ tends to 0.

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In the following two method are considered, with $\eta = X$:

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- An indirect one: $e := \xi - \phi(X)$ with $\phi : \mathbb{R}^3 \longrightarrow \mathbb{R}^q$ and $q > 3$

Observer design by I&I: direct method

- For h_m Gaussian or quadratic:

$$\dot{e} = -\kappa V_{hor} V_{hor}^T e$$

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Proposition: convergence of e under PE

If there exist $T > 0$ and $\mu > 0$, such that for any $t \geq t_0$:

$$\frac{1}{T}\left(\int_t^{t+T} V_{hor}(\tau) V_{hor}^T(\tau) d\tau\right) \succeq \mu I_2$$

then e converges exponentially to 0.

Proof using a new time-dependent strict Lyapunov function (*in collaboration with Ioannis Sarras*)

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$\Rightarrow$ condition on V_{hor} and implicitly on U_{hor}

Observer design by I&I: indirect method

- For h_m sinusoidal or cubic, one gets a LTV system in $\chi = \phi(X)$ s.t.

$$\dot{\chi} = A(V)\chi + BV$$
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A Kalman observer + a persistence condition lead to vanishing

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$\Rightarrow$ the persistence conditions are a second occurrence of dual effect

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Output feedback Control

We consider the horizontal output-feedback system in the augmented state position/speed/error.

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The objective is to choose U_{hor} to ensure the convergence of the full system:

$$\begin{aligned} \dot{X}_{hor} &= V_{hor} \\ \dot{V}_{hor} &= U_{hor} \\ \dot{e} &= -\kappa V_{hor} V_{hor}^T e \end{aligned}$$

δ-persistence of V_{hor}

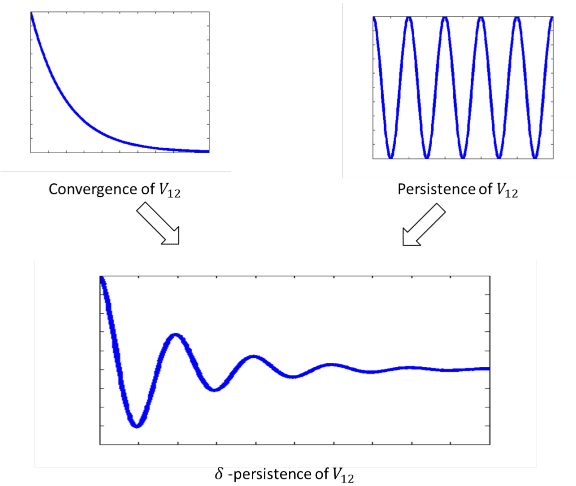

Figure: Principle of the δ-persistence

Output feedback Control: δ-persistence control law

U_{hor} is chosen s.t.

$$U_{hor} = -K_x\hat{\eta} - K_v V_{hor} + \alpha\left(\left\|\begin{bmatrix}\hat{\eta}\\ V_{hor}\end{bmatrix}\right\|\right)\phi_{per}(t),$$

where

- K_x and K_v are gains to tune.
- ϕ_{per} is a persistent signal.
- α is increasing.

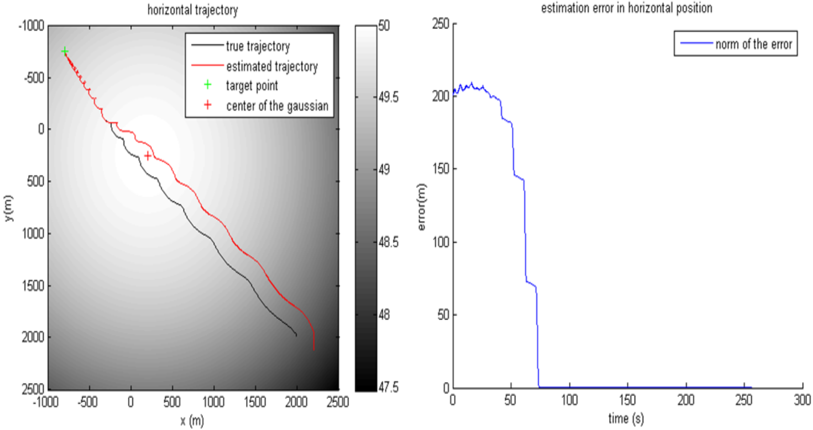

Figure: Example of trajectory of the output-feedback system and of the estimation errors

Conclusion on the analytical part

- Informative Observability conditions
- Several Observers and and a controller for simple maps h_m
- Potential extensions to arbitrary maps through approximations

$\Rightarrow$ Methods limited to simple maps and not very robust but in a simple framework

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$\Rightarrow$ Methods limited to simple maps and not very robust but in a simple framework

$\Rightarrow$ Solution to deal with arbitrary maps: use a more flexible framework

Stochastic approach of TAN for arbitrary maps

State of the art

- If the system is linear Gaussian: **separation principle**

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In the literature:

- Some dual controllers lack of formal justification
- No dual controllers using Particle Filters

Plan

Partially Observable Markov Decision Process (POMDP)

A POMDP is composed of:

- A discrete-time stochastic dynamical system:

$$X_{k+1} = f(X_k, U_k, \xi_k) \qquad\qquad X_0 \sim p_0$$

  where:

  - X_k is a state variable
  - U_k is a control variable
  - ξ_k is a noise on the dynamics

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  where:
  - X_k is a state variable
  - U_k is a control variable
  - ξ_k is a noise on the dynamics
- An observation equation:

$$Y_k = h(X_k, \eta_k)$$

  where:
  - Y_k is an observation
  - η_k a noise on the observations

Partially Observable Markov Decision Process (POMDP)

- *Available information* I_k defined by:

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The two main issues are:

- Finding an estimator $\widehat{X_k} = \boldsymbol{\pi}^{\boldsymbol{e}}(I_k)$ or $\boldsymbol{\pi}^{\boldsymbol{e}}(\mu_k)$
- Finding a control $U_k = \boldsymbol{\pi}^{\boldsymbol{c}}(I_k)$ or $\boldsymbol{\pi}^{\boldsymbol{c}}(\mu_k)$

both as a function of I_k or μ_k

Plan

Optimal Estimation

For some measure of the estimation error g^e:

$$\begin{aligned} (P_E) : \min_{\pi^e} \quad & E\left[g^e(X_k, \widehat{X}_k) | I_k\right] \\ \text{s.t.} \quad & \widehat{X}_k = \pi^e(I_k). \end{aligned}$$

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If $g^e(x, \hat{x}) = \|\hat{x} - x\|^2$ (**Mean Square Error**), then:

$$\widehat{X}_k^* = E[X_k|I_k] = \langle \mu_k, Id \rangle$$

which is hard to compute directly.

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which is hard to compute directly.
$\Rightarrow$ One is rather looking for $\boldsymbol{\mu_k}$

- An instantaneous cost g^c

Stochastic Optimal Control with perfect state information

- An instantaneous cost g^c
- A stochastic optimal control problem:

$$
\begin{array}{rrcl}
(P_C) : \min\limits_{\pi_0} & E\left[\sum\limits_{k=0}^{+\infty} g^c(X_k, U_k, \xi_k) | X_0\right] & & \\
\text{s.t.} & X_{k+1} & = & f(X_k, U_k, \xi_k) \\
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- Dynamic Programming Principle: V value function

$$
\begin{array}{rl}
V(x) = \min\limits_{u \in \mathcal{U}} & E\left[g^c(X_k, u) + V(X_{k+1}) | X_k = x\right] \\
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- A reformulation using μ_k leads to

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\begin{array}{rrcl}
V(\mu) = \min\limits_{\pi_0} & E\left[\sum\limits_{k=0}^{+\infty} \tilde{g}^c(\mu_k, U_k) | \mu_0 = \mu\right] & & \\
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\end{array}
$$

$\Rightarrow \pi_0^*$ has the **implicit** dual effect property.

A Coupled Stochastic Optimisation problem

$$
\begin{array}{rrcl}
(P_{CE}) : \min\limits_{\pi_0^c, \pi_0^e} & E\left[\sum\limits_{k=0}^{+\infty} g^c(X_k, U_k, \xi_k) + g^e(X_k, \widehat{X}_k) | \tilde{I}_0\right] & & \\
\text{s.t.} & X_{k+1} & = & f(X_k, U_k, \xi_k) \\
 & Y_k & = & h(X_k, \eta_k) \\
 & \tilde{I}_{k+1} & = & (\tilde{I}_k, U_k, \widehat{X}_k, Y_{k+1}) \\
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\end{array}
$$

On the space of probability measures:

$$
\begin{array}{rrcl}
\widetilde{V}(\mu) = \min\limits_{\pi_0^c, \pi_0^e} & E\left[\sum\limits_{k=0}^{+\infty} \tilde{g}^c(\mu_k, U_k) + \tilde{g}^e(\mu_k, \widehat{X}_k) | \mu_0 = \mu\right] & & \\
\text{s.t.} & \mu_{k+1} & = & F(\mu_k, Y_{k+1}, U_k) \\
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Decomposition in two steps

As F does not depend on $\widehat{X}_k$, two steps naturally appear:

- An inner optimal estimation problem:

$$\tilde{g}^e_*(\mu_k) = \min_{\hat{x}\in\mathbb{R}^{n_x}} \tilde{g}^e(\mu_k, \hat{x}) = \min_{\hat{x}\in\mathbb{R}^{n_x}} E[g^e(X_k, \hat{x})|I_k]$$

- An outer stochastic control problem with an additional estimation-based cost:

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- An outer stochastic control problem with an additional estimation-based cost:

$$\begin{array}{rrcl} \widetilde{V}(\mu) = \min\limits_{\pi^c_0} & E\left[\sum\limits_{k=0}^{+\infty} \tilde{g}^c(\mu_k, U_k) + \tilde{g}^e_*(\mu_k) | \mu_0 = \mu\right] & & \\ \text{s.t.} & \mu_{k+1} & = & F\left(\mu_k, Y_{k+1}, U_k\right) \\ & U_k & = & \pi^c_0(\mu_k) \end{array}$$

A **hierarchy** and not a separation !

How to solve these steps in theory and in practice

- Inner optimal estimation step $\Rightarrow$ Particle filtering
- Outer optimal control step $\Rightarrow$ Dual Stochastic MPC

Plan

Classical nonlinear filtering methods

Methods for estimating μ_k:

- Set-membership estimation: good only for bounded uncertainties and very pessimistic
- Kalman filtering: low cost but good only for unimodal uncertainties
- **Particle filtering**: rather high cost but good for **multimodal uncertainties**

A **particle filter** is an Monte Carlo approximation of μ_k defined by:

$$\mu_k^N = \sum_{i=1}^{N} \omega_k^i \delta_{x_k^i}$$

where $\left(\omega_k^i\right)_{i=1..N}$ are weights and $\left(x_k^i\right)_{i=1..N}$ are interacting random variables in $\mathbb{R}^{n_x}$.

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The **empirical mean** reads:

$$\widehat{X}_k^N = \sum_{i=1}^{N} \omega_k^i x_k^i = \langle \mu_k^N, Id \rangle$$

and is used to approach $\widehat{X}_k$.

Inner near-optimal estimation

Consider MSE minimisation and set:

$$e_{k,*}^{cond} = E\left[\|X_k - \widehat{X}_k^*\|^2 | I_k\right] = \tilde{g}_*^e(\mu_k)$$
$$e_{k,N}^{cond} = E\left[\|X_k - \widehat{X}_k^N\|^2 | I_k\right]$$

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Considering the particle filter from [Hu et al]:

Theorem

For any $\epsilon > 0$ and $\forall k \geq 0$, there exists $C_k > 0$ s.t:

$$0 \leq e_{k,N}^{cond} - e_{k,*}^{cond} \leq \epsilon e_{k,*}^{cond} + \left(1 + \frac{1}{\epsilon}\right) C_k \frac{\sum_{j=1}^{n} \|\phi_j\|_{k,2}^2}{N}$$

for N sufficiently large.

Proof based on "weak" error bounds for unbounded functions

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Proof based on "weak" error bounds for unbounded functions
$\Rightarrow \widehat{X}_k^N$ is near optimal for N sufficiently large.

Plan

Outer minimisation and Explicit dual control

The optimal estimation error $g_*^e \simeq$ an information measure

$$\begin{array}{rrcl} \widetilde{V}(\mu) = \displaystyle\min_{\pi_0^c} & E\left[\displaystyle\sum_{k=0}^{+\infty} \tilde{g}^c(\mu_k, U_k) + \tilde{g}_*^e(\mu_k) \middle| \mu_0 = \mu\right] & & \\ \text{s.t.} & \mu_{k+1} & = & F\left(\mu_k, Y_{k+1}, U_k\right) \\ & U_k & = & \pi_0^c(\mu_k) \end{array}$$

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In practice it is replaced by an empirical approximation g^{info} leading to:

$$\widetilde{V}_{EX}(\mu) = \min_{\pi_0^c} \quad E\left[\sum_{k=0}^{+\infty} \tilde{g}^c(\mu_k, U_k) + \tilde{g}^{info}(\mu_k) | \mu_0 = \mu\right]$$
$$\begin{aligned} \text{s.t.} \quad \mu_{k+1} &= F\left(\mu_k, Y_{k+1}, U_k\right) \\ U_k &= \pi_0^c(\mu_k) \end{aligned}$$

$\Rightarrow$ We recover an Explicit Dual control problem.

Explicit dual MPC

Solving a modified Open-Loop problem instead

$$V_{EX}^{T}(\mu) = \min_{u_0,\dots,u_{T-1}\in\mathcal{U}} \quad E\left[\sum_{k=0}^{T-1} \tilde{g}_k^{ex}(\mu_{k|0}, u_k) + \tilde{g}_F^{ex}(\mu_{T|0}) | \mu_0 = \mu\right]$$
$$\text{s.t.} \quad \mu_{k+1|0} \quad = \quad G\left(\mu_{k|0}, u_k\right)$$

where $\tilde{g}_k^{ex} = \tilde{g}_k + \tilde{g}_k^{info}$ and $\tilde{g}_k^{info} = \langle \mu_k, g_k^{info} \rangle$

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2 remaining issues in order to design an output-feedback law:

- Choice of the explicit dual problem
  - How to deal with the guiding goal ?
  - How to impose the explicit dual effect ?
- Numerical resolution

Plan

Localisation and guidance by terrain-aided navigation

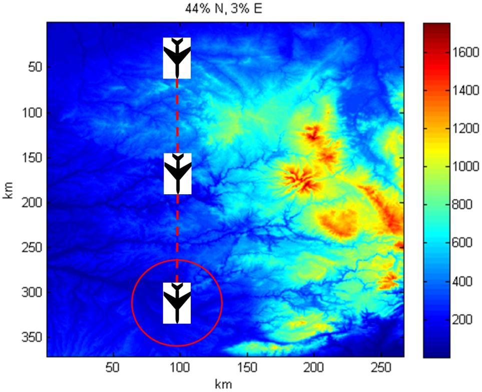

Figure: Trajectory where the UAV goes straight to the target over a flat area

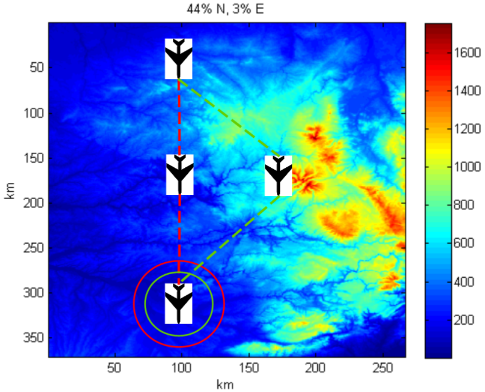

Figure: Compromise between going toward the target and getting information

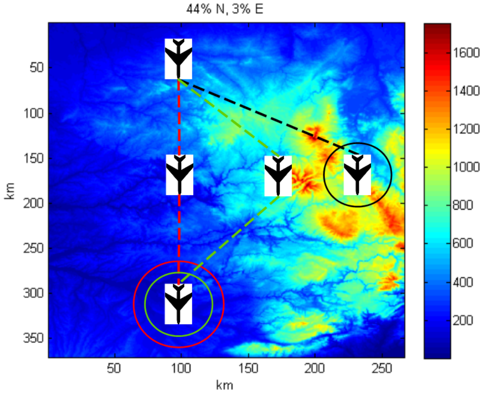

Figure: Trajectory where the UAV is stuck over a rough area

⇒ idea: prioritise the guiding objective with a stabilising constraint.

How to deal with the guiding goal ?

- With a stabilising cost:

$$g_i = g_i^{stab} + g_i^{eco}$$

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a negative drift condition on u_0

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J_i: Fisher Information Matrix

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$\Rightarrow$ very flexible but hard to tune

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- With a stabilising constraint:

$$g_i = g_i^{eco} + g_i^{info}(J_i),$$

a negative drift condition on u_0.

$\Rightarrow$ easier to tune but requires the knowledge of a drift

Monte Carlo approximation

The practical resolution is done by combining a Monte Carlo approximation and a particle filter:

$$
\begin{array}{rrcl}
(P_C^N): \min\limits_{u_0 \cdots u_{T-1}} & \multicolumn{3}{l}{\sum\limits_{i=1}^{N_s} \omega_\ell^i \left(\sum\limits_{k=0}^{T-1} g_k\left(X_k^i, u_k, \xi_k^i\right) + g_T\left(X_T^i\right) \right)} \\
\text{s.t.} & X_{k+1}^i & = & f(X_k^i, u_k, \xi_k^i) \\
& u_k & \in & \mathcal{U} \\
& X_0^i & = & x_\ell^i,
\end{array}
$$

where x_ℓ^i comes from a particle filter

Application in terrain-aided navigation

- state representation: 3 positions (x, y, z) , 3 speeds (v_x, v_y, v_z) and 3 accelerations (u_x, u_y, u_z)

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$$f_k(X_k, U_k, \xi_k) = AX_k + BU_k + \xi_k$$
$$\|U_k\| \leq U_{max}$$

where A and B from a double integrator with damping on the speed, and ξ_k a white Gaussian noise

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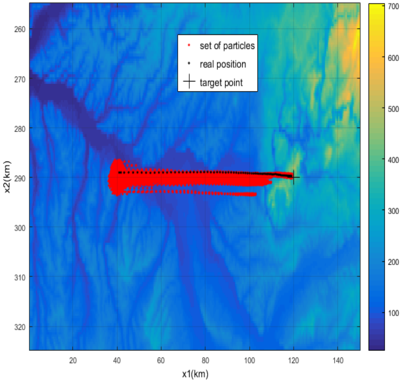
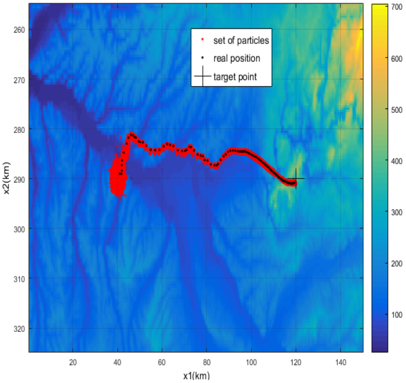
- observation equation:

$$h_k(X_k, \xi_k) = z_k - h_m(x_k, y_k) + \eta_k$$

where h_m is a map of the **height of the ground** and η_k a white Gaussian noise

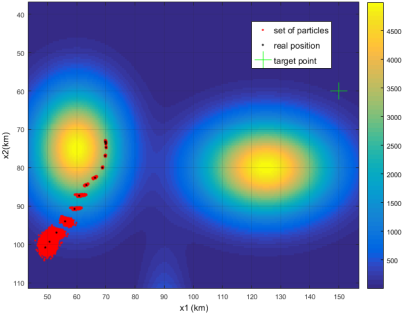


(a) Trajectory from a passive controller over a flat area

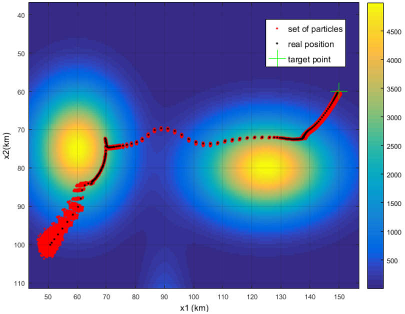


(b) Trajectory from active controller avoiding a flat area

Comparison of the two controllers



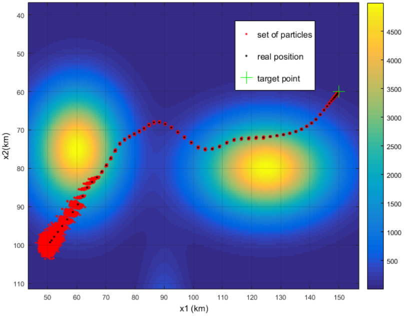
(a) Trajectory with a constant weight



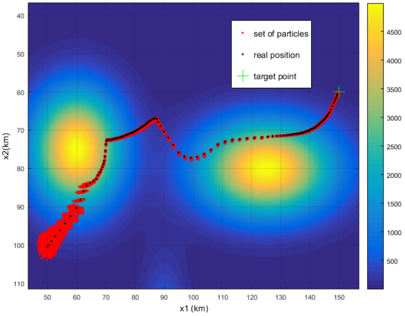
(b) Trajectory with a slowly decreasing weight

Figure: Realisation of a trajectory of the true state and the particles resulting from the Penalising Fisher controller on an artificial map with several tuning

Comparison of the two controllers



(a) Trajectory with a large weight on the FIM and rapidly converging



(b) Trajectory with a large weight on the FIM and slowly converging

Figure: Realisation of a trajectory of the true state and the particles resulting from the Lyapunov Fisher controller on an artificial map with with different tuning

Conclusion

In the analytic framework, we have:

- Studied observability and extracted conditions on the horizontal speed and acceleration
- Designed nonlinear observers using I&I for several terrain maps that converge under PE of the horizontal speed
- Designed a δ-persistent controller for output-feedback control

Conclusion

In the analytic framework, we have:

- Studied observability and extracted conditions on the horizontal speed and acceleration
- Designed nonlinear observers using I&I for several terrain maps that converge under PE of the horizontal speed
- Designed a δ-persistent controller for output-feedback control

In the non-analytic framework, we have:

- Proposed a joint modelling of optimal control and estimation which can be split sequentially into:
  - an optimal estimation step
  - an estimation-based stochastic optimal control step
- Showed that the Inner Estimation step can be solved by a PF
- The Outer Control problem can be relaxed into a Dual Explicit control one
- Proposed two Explicit Dual MPC schemes based on a PF and applied them to TAN for arbitrary maps

Future research axes

In the analytic framework:

- Design an observer for 3-Spline maps and deal with real maps
- Improve the convergence result of the output-feedback system

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- Design an observer for 3-Spline maps and deal with real maps
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In the non-analytic framework

- Show near optimality of the particle filter for a general cost g^e.
- Look for better approximation of g^e_* than the FIM.
- Show the formal closed-loop convergence of the dual MPC schemes
- Speed up the numerical resolution with a decomposition method

Thank you for your attention !